DOI: 10.15393/j3.art.2022.11350

UDC 517.53

Idrees Qasim

REFINEMENT OF SOME BERNSTEIN TYPE INEQUALITIES FOR RATIONAL FUNCTIONS

Abstract. In this paper, we establish some Bernstein-type inequalities for rational functions with prescribed poles. These results refine prior inequalities on rational functions and strengthen many well-known polynomial inequalities.

Key words: *Rational Functions, Polynomial Inequalities, Polar Derivative, Poles, Zeros*

2020 Mathematical Subject Classification: *30A10, 30C15, 30D15*

1. Introduction. Let $\mathcal{P}_n$ denote the space of complex polynomials $f(z) := \sum\limits_{j=0}^{n} a_j z^j$ of degree at-most $n \geqslant 1$. Let $T := \{z : |z| = 1\}$, D_- denote the region inside T and D_+ the region outside T. For $\alpha_j \in \mathbb{C}$ with $j = 1, 2, \dots, n$, let $w(z) = \prod\limits_{j=1}^{n}(z - \alpha_j)$ and let

$$B(z) := \prod_{j=1}^{n}\Big(\frac{1-\overline{\alpha_j}z}{z-\alpha_j}\Big),\ \mathcal{R}_n := \mathcal{R}_n(\alpha_1, \alpha_2, \dots, \alpha_n) := \left\{\frac{p(z)}{w(z)},\ p \in \mathcal{P}_n\right\}.$$

Then $\mathcal{R}_n$ is the set of rational functions with poles $\alpha_1, \alpha_2, \dots, \alpha_n$ at most and with finite limit at infinity. $B(z) \in \mathcal{R}_n$ is known as the Blaschke product. From now on, we shall assume that the poles $\alpha_1, \alpha_2, \dots, \alpha_n$ are in D_+. For the case when all the poles are in D_-, we can obtain analogous results with suitable modifications of our method.

For $r \in \mathcal{R}_n$, let $\|r\| = \max\limits_{z\in T} |r(z)|$ be the Chebyshev norm of r on T and $m = \min\limits_{z\in T} |r(z)|$.

Definitions and Notations:

1) For $p(z) := \sum_{j=0}^{n} a_j z^j$, the conjugate transpose (reciprocal) p^* of p is defined by

$$p^*(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}.$$

Therefore, if $p(z) = \prod_{j=1}^{n}(z - z_j)$, then $p^*(z) = \prod_{j=1}^{n}(1 - \overline{z_j}z)$.

2) For $r(z) = \dfrac{p(z)}{w(z)} \in \mathcal{R}_n$, the conjugate transpose r^* of r is defined by

$$r^*(z) = B(z)\overline{r\left(\frac{1}{\overline{z}}\right)}.$$

Note that if $r(z) = \dfrac{p(z)}{w(z)} \in \mathcal{R}_n$, then $r^*(z) = \dfrac{p^*(z)}{w(z)}$ and, hence, $r^*(z) \in \mathcal{R}_n$.

3) For $w(z) = \prod_{j=1}^{n}(z - \alpha_j)$, we denote by b the product of roots of $w(z)$, i. e., $b = \alpha_1 \times \alpha_2 \times \cdots \times \alpha_n$.

4) If $p(z) := \sum_{j=0}^{n} a_j z^j$, then $\overline{p}(z)$ is defined as

$$\overline{p}(z) = \overline{a_0} + \overline{a_1}z + \overline{a_2}z^2 + \cdots + \overline{a_n}z^n.$$

Note that $\overline{p(\overline{z})} = \overline{p}(z)$.

If $p \in \mathcal{P}_n$, then, concerning the estimate of $|p'(z)|$ on the unit circle T, we have the following well-known result due to Bernstein (see [6], p. 508, Theorem 14.1.1), which relates the norm of a polynomial to that of its derivative.

$$\|p'\| \leqslant n\|p\|. \tag{1}$$

The inequality (1) is sharp and equality holds for polynomials having all zeros at the origin.

Since equality in (1) holds if and only if $p(z) := cz^n$, one would except a relationship between the bound n and the distance to the zeros of the polynomial from the origin. This fact was observed by Erdös, who conjectured the following fact later proved by Lax [3]:

If $p \in \mathcal{P}_n$ and $p(z) \neq 0$ in D_-, then

$$\|p'\| \leqslant \frac{n}{2}\|p\|. \tag{2}$$

Turàn [7] considered the polynomial having all zeros in $T \cup D_-$ and proved the following reverse inequality:
If $p \in \mathcal{P}_n$ has all zeros in $T \cup D_-$, then

$$\|p'\| \geqslant \frac{n}{2}\|p\|. \tag{3}$$

Dubinin [2] proved the following strengthened version of inequality (3).
Theorem A. If $p(z) := \sum_{j=1}^{n} a_j z^j$ is such that $p(z) \neq 0$ in D_+, then

$$\|p'\| \geqslant \frac{1}{2}\left[n + \frac{\sqrt{|a_n|} - \sqrt{|a_0|}}{\sqrt{|a_n|}}\right]\|p\|. \tag{4}$$

In 1995, Li, Mohapatra, and Rodriguez [5] extended inequality (2) to rational functions with prescribed poles. Besides other things, they proved the following results:
Theorem B. If $r \in \mathcal{R}_n$, such that $r(z) \neq 0$ for $z \in D_-$, then, for $z \in T$:

$$|r'(z)| \leqslant \frac{|B'(z)|}{2} \max_{z \in T} |r(z)|.$$

Equality holds for $r(z) = \alpha B(z) + \beta$ with $|\alpha| = |\beta| = 1$.
Theorem C. Suppose $r \in \mathcal{R}_n$ and all the zeros of r lie in $T \cup D_-$. Then, for $z \in T$,

$$|r'(z)| \geqslant \frac{1}{2}\left[|B'(z)| - (n-t)\right] \max_{z \in T} |r(z)|$$

where t are the number of zeros of $r(z)$.

Recently, Wali and Shah [8] used the lemma of Dubinin [2] and proved the following result:
Theorem D. Suppose that $r(z) = \dfrac{p(z)}{w(z)} \in \mathcal{R}_n$, where $p(z) := \sum_{j=0}^{t} a_j z^j$, $t \leqslant n$, r has exactly n poles $\alpha_1, \alpha_2, \ldots, \alpha_n$ and all the zeros of r lie in $T \cup D_-$. Then, for $z \in T$,

$$|r'(z)| \geqslant \frac{1}{2}\left\{|B'(z)| - (n-t) + \frac{\sqrt{|a_t|} - \sqrt{|a_0|}}{\sqrt{|a_t|}}\right\}|r(z)|.$$

The result is sharp and the equality holds for $r(z) = B(z) + \lambda$ with $|\lambda| = 1$.

In this paper, we find some inequalities for rational functions, which, in particular, refine Theorem B and Theorem D for a particular class of rational functions. We also deduce some polynomial inequalities, which strengthens the prior inequalities, including inequality (4) and improves many other inequalities concerning the polar derivative of a polynomial.

Our first result gives a refinement of Theorem B for a particular class of rational functions.

Theorem 1. *If $r(z) = \dfrac{p(z)}{w(z)} \in \mathcal{R}_n$, where $p(z) := \sum\limits_{j=0}^{n} a_j z^j$, $r(z) \neq 0$ for all $z \in D_-$ and $|a_0| \leqslant |b| \cdot |a_n|$, then for $z \in T$,*

$$|r'(z)| \leqslant \frac{1}{2}\Big[|B'(z)| - \frac{\sqrt{|a_0|} - \sqrt{|a_n|}}{\sqrt{|a_0|}} \cdot \frac{(|r(z)| - m)^2}{(\|r\| - m)^2}\Big](\|r\| - m),$$

where $\|r\| = \max\limits_{z\in T} |r(z)|$.

Equality is obtained for $r(z) = B(z) + ke^{i\alpha}$, with $k \geqslant 1$ and real α. Since $r(z)$ does not vanish in D_-, $|a_0| \geqslant |a_n|$. Also, $m \geqslant 0$; hence, Theorem 1 is an improvement of Theorem B.

Remark 1. *Let $\alpha_j = \alpha > 1 \ \forall\ j = 1, 2, \ldots, n$; then $w(z) = (z - \alpha)^n$ and $r(z) = \dfrac{p(z)}{(z-\alpha)^n}$, so that $B(z) = \left[\dfrac{1 - \alpha z}{z - \alpha}\right]^n \to z^n$ as $\alpha \to \infty$. Also, $B'(z) \to nz^{n-1}$ as $\alpha \to \infty$.*
Further, let

$$\|r\| = \max_{z\in T} \left|\frac{p(z)}{(z-\alpha)^n}\right|$$

be obtained at $z = e^{i\zeta}$, $0 \leqslant \zeta < 2\pi$, and

$$m = \min_{z\in T} |r(z)| = \min_{z\in T} \left|\frac{p(z)}{(z-\alpha)^n}\right|$$

be obtained at $z = e^{i\beta}$, $0 \leqslant \beta < 2\pi$; then, clearly,

$$\|r\| = \max_{z\in T} \left|\frac{p(z)}{(z-\alpha)^n}\right| = \left|\frac{p(e^{i\zeta})}{(e^{i\zeta}-\alpha)^n}\right| \leqslant \frac{\max\limits_{z\in T}|p(z)|}{|(e^{i\zeta}-\alpha)^n|} = \frac{\|p\|}{|(e^{i\zeta}-\alpha)^n|}$$

and

$$m = \min_{z\in T}\left|\frac{p(z)}{(z-\alpha)^n}\right| = \left|\frac{p(e^{i\beta})}{(e^{i\beta}-\alpha)^n}\right| \geqslant \frac{\min_{z\in T}|p(z)|}{|(e^{i\beta}-\alpha)^n|} = \frac{m_p}{|(e^{i\beta}-\alpha)^n|},$$

where $m_p = \min_{z\in T}|p(z)|$.

Therefore, taking $\alpha_j = \alpha > 1$, for all $j = 1, 2, \ldots, n$ in Theorem 1, using the above observations, and letting $\alpha \to \infty$, we get the following result:

Corollary 1. *If* $p(z) := \sum_{j=0}^{n} a_j z^j \in \mathcal{P}_n$ *and* $p(z) \neq 0$ *in* D_-, *then, for* $z \in T$,

$$|p'(z)| \leqslant \frac{1}{2}\left[n - \frac{\sqrt{|a_0|}-\sqrt{|a_n|}}{\sqrt{|a_0|}}\right](\|p\| - m_p).$$

Equality is obtained for $p(z) = z^n + 1$.

Since $m_p \geqslant 0$ and $|a_0| \geqslant |a_n|$, it follows that Corollary 1 is an improvement of the result by Aziz and Dawood ([1], Theorem 2).

As a refinement of Theorem D, we present the following result:

Theorem 2. *If* $r(z) = \frac{p(z)}{w(z)} \in \mathcal{R}_n$, *where* $p(z) := \sum_{j=0}^{n} a_j z^j$, $|b|\cdot|a_n| \leqslant |a_0|$, *$r$ has exactly n poles at* $\alpha_1, \alpha_2, \ldots, \alpha_n$, *and* $r(z) \neq 0$ *for all* $z \in D_+$, *then, for* $z \in T$:

$$|r'(z)| \geqslant \frac{1}{2}\left[|B'(z)| + \frac{\sqrt{|a_n|}-\sqrt{|a_0|}}{\sqrt{|a_n|}}\right](|r(z)| + m).$$

Equality is obtained for $r(z) = B(z) + ke^{i\zeta}$, *with* $k \leqslant 1$ *and real* ζ.

For a complex number α and for $p \in \mathcal{P}_n$, let

$$D_\alpha p(z) := np(z) + (\alpha - z)p'(z),$$

where $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and is known as the polar derivative of $p(z)$ with respect to α. It generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha} = p'(z).$$

Assume that $\alpha_j = \alpha$ for all $j = 1, 2, \ldots, n$, with $|\alpha| > 1$ and $p(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n. Then it can be easily shown that $r'(z) = \dfrac{-D_\alpha p(z)}{(z-\alpha)^{n+1}}$ and $B'(z) = \dfrac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}$.

Using the above facts and those discussed in Remark 1, we get the following result from Theorem 2:

Corollary 1. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ with $|\alpha|^n|a_n| \leqslant |a_0|$ is a polynomial of degree n having all zeros in $T \cup D_-$, then, for every finite complex number α satisfying $|\alpha|^n|a_n| \leqslant |a_0|$ and $|\alpha| \geqslant 1$,*

$$|D_\alpha p(z)| \geqslant \frac{|\alpha|-1}{2}\left[n + \frac{\sqrt{|a_n|}-\sqrt{|a_0|}}{\sqrt{|a_n|}}\right](\|p\| + m_p).$$

2. Lemmas. To prove these theorems, we need the following lemmas. The first lemma is due to Li, Mohapatra, and Rodrigues [5]:

Lemma 1. *If $r \in \mathcal{R}_n$ and $z \in T$, then*

$$|r^{*\prime}(z)| + |r'(z)| \leqslant |B'(z)| \max_{z\in T} |r(z)|.$$

Equality holds for $r(z) = uB(z)$ with $u \in T$.

The next Lemma is due to Li [4]:

Lemma 2. *Let $r, s \in \mathcal{R}_n$ and assume that $s(z)$ has all zeros in $T \cup D_-$ and*

$$|r(z)| \leqslant |s(z)| \quad \text{for} \quad z \in T.$$

Then,

$$|r'(z)| \leqslant |s'(z)| \quad \text{for} \quad z \in T.$$

The next two lemmas are due to Wali and Shah [8]:

Lemma 3. *If $r(z) = \dfrac{p(z)}{w(z)} \in \mathcal{R}_n$, where $p(z) := \sum_{j=0}^{n} a_j z^j$ and all zeros of r lie in D_+, then, for $z \in T$,*

$$\operatorname{Re}\left(\frac{zr'(z)}{r(z)}\right) \leqslant \frac{1}{2}\left[|B'(z)| - \frac{\sqrt{|a_0|}-\sqrt{|a_n|}}{\sqrt{|a_0|}}\right].$$

Lemma 4. *If* $r(z) = \dfrac{p(z)}{w(z)} \in \mathcal{R}_n$*, where* $p(z) := \sum_{j=0}^{n} a_j z^j$*, and* r *has exactly* n *poles at* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *and all zeros of* r *lie in* D_-*, then for* $z \in T$*,*

$$\operatorname{Re}\left(\frac{zr'(z)}{r(z)}\right) \geqslant \frac{1}{2}\left[|B'(z)| + \frac{\sqrt{|a_n|} - \sqrt{|a_0|}}{\sqrt{|a_n|}}\right].$$

3. Proofs of the Theorems.

Proof of Theorem 1. Assume that all zeros of $r(z)$ lie in D_+; then $m > 0$ and

$$m \leqslant |r(z)|$$

for $z \in T$. Let α and β be two complex numbers, such that $|\alpha| < 1$ and $|\beta| < 1$; then

$$m|\alpha\beta| < |r(z)|$$

for $z \in T$. Since all the poles of $r(z)$ are in D_+, $r(z)$ is analytic in D_-. Also, $r(z)$ has no zero in $T \cup D_-$, therefore, by Rouche's Theorem, $R(z) = r(z) - \alpha\beta m$ has no zero in $T \cup D_-$. Therefore, by Lemma 3, for $z \in T$:

$$\operatorname{Re}\left(\frac{zR'(z)}{R(z)}\right) \leqslant \frac{1}{2}\left[|B'(z)| - \frac{\sqrt{|a_0 + (-1)^{n+1}\alpha\beta m b|} - \sqrt{|a_n - \alpha\beta m|}}{\sqrt{|a_0 + (-1)^{n+1}\alpha\beta m|}}\right]. \tag{5}$$

Since $|a_0| \leqslant |b| \cdot |a_n|$, then

$$|a_0| \cdot |\alpha| \cdot |\beta| \cdot m \leqslant |\alpha| \cdot |\beta| \cdot m \cdot |b| \cdot |a_n|,$$

$$|a_n| \cdot |a_0| + |a_0| \cdot |\alpha| \cdot |\beta| \cdot m \leqslant |a_n| \cdot |a_0| + |\alpha| \cdot |\beta| \cdot m \cdot |b| \cdot |a_n|,$$

$$\frac{|a_n|}{|a_0|} \geqslant \frac{|a_n| + |\alpha| \cdot |\beta| \cdot m}{|a_0| + |\alpha| \cdot |\beta| \cdot |b| \cdot m}.$$

Choosing argument of β in such a way that

$$|a_0 + (-1)^{n+1}\alpha \cdot \beta \cdot b \cdot m| = |a_0| + |\alpha| \cdot |\beta| \cdot |b| \cdot m,$$

we get

$$\frac{|a_n|}{|a_0|} \geqslant \frac{|a_n - \alpha \cdot \beta \cdot m|}{|a_0 + (-1)^{n+1}\alpha \cdot \beta \cdot b \cdot m|}.$$

Hence, it follows from inequality (5) that

$$\operatorname{Re}\left(\frac{zR'(z)}{R(z)}\right) \leqslant \frac{1}{2}\left[|B'(z)| - \frac{\sqrt{|a_0|} - \sqrt{|a_n|}}{\sqrt{|a_0|}}\right]. \tag{6}$$

Now, $R^*(z) = B(z)\overline{R\left(\frac{1}{\overline{z}}\right)} = B(z)\overline{R}\left(\frac{1}{z}\right)$, where $\overline{R}(z) = \dfrac{\overline{p}(z)}{\overline{w}(z)}$. Differentiating both sides gives

$$(R^*(z))' = B'(z)\overline{R}\left(\frac{1}{z}\right) - \frac{B(z)}{z^2}\overline{R}'\left(\frac{1}{z}\right).$$

Since $z \in T$, we have $\overline{z} = 1/z$, and so

$$|(R^*(z))'| = \left|(zB'(z)/B(z))\overline{R(z)} - \overline{zR'(z)}\right|. \tag{7}$$

By ([5], Lemma 1), we have

$$\frac{zB'(z)}{B(z)} = \left|\frac{zB'(z)}{B(z)}\right| = |B'(z)|.$$

Thus, from equation (7), we have

$$|(R^*(z))'| = ||B'(z)|R(z) - zR'(z)|.$$

Since $R(z) \neq 0$ on T, we have for $z \in T$ using inequality (6):

$$\begin{aligned}
\left|\frac{z(R^*(z))'}{R(z)}\right|^2 &= \left||B'(z)| - \frac{zR'(z)}{R(z)}\right|^2 = \\
&= \left|\frac{zR'(z)}{R(z)}\right|^2 + |B'(z)|^2 - 2|B'(z)| \operatorname{Re}\left(\frac{zR'(z)}{R(z)}\right) \geqslant \\
&\geqslant \left|\frac{zR'(z)}{R(z)}\right|^2 + |B'(z)|^2 - |B'(z)|\left[|B'(z)| - \frac{\sqrt{|a_0|} - \sqrt{|a_n|}}{\sqrt{|a_0|}}\right] = \\
&= \left|\frac{zR'(z)}{R(z)}\right|^2 + \frac{\sqrt{|a_0|} - \sqrt{|a_n|}}{\sqrt{|a_0|}}|B'(z)|.
\end{aligned}$$

This implies that for $z \in T$,

$$\left[|R'(z)|^2 + \frac{\sqrt{|a_0|} - \sqrt{|a_n|}}{\sqrt{|a_0|}}|B'(z)||R(z)|^2\right]^{\frac{1}{2}} \leqslant |(R^*(z))'|. \tag{8}$$

Now, $R^*(z) = r^*(z) - \overline{\alpha}\overline{\beta}mB(z)$, so that $R^{*\prime}(z) = r^{*\prime}(z) - \overline{\alpha}\overline{\beta}mB'(z)$.

Since all zeros of $r(z)$ lie in D_+, all zeros of $r^*(z)$ lie in D_-. Also, $|m\beta B(z)| \leqslant |r^*(z)|$ for $z \in T$. Hence, by Lemma 2, we have

$$|(r^*(z))'| \geqslant |m\beta B'(z)|$$

for $z \in T$. Therefore, we can choose argument of α such that

$$|(r^*(z))' - \overline{\alpha}\overline{\beta} m B'(z)| = |(r^*(z))'| - |\beta| \cdot |\alpha| \cdot m \cdot |B'(z)|.$$

Hence, from inequality (8), we have for $z \in T$:

$$\left[|r'(z)|^2 + \frac{\sqrt{|a_0|} - \sqrt{|a_n|}}{\sqrt{|a_0|}} |B'(z)| (|r(z)| - m \cdot |\alpha| \cdot |\beta|)^2\right]^{\frac{1}{2}} \leqslant$$
$$\leqslant |r^{*\prime}(z)| - |\beta| \cdot |\alpha| \cdot m \cdot |B'(z)|.$$

Letting $|\alpha| \to 1$ and $|\beta| \to 1$ gives

$$\left[|r'(z)|^2 + \frac{\sqrt{|a_0|} - \sqrt{|a_n|}}{\sqrt{|a_0|}} |B'(z)| (|r(z)| - m)^2\right]^{\frac{1}{2}} \leqslant |(r^*(z))'| - m|B'(z)|.$$

Applying Lemma 1, we have, for $z \in T$:

$$\left[|r'(z)|^2 + \frac{\sqrt{|a_0|} - \sqrt{|a_n|}}{\sqrt{|a_0|}} |B'(z)| (|r(z)| - m)^2\right]^{\frac{1}{2}} \leqslant$$
$$\leqslant |B'(z)| \cdot \|r\| - |r'(z)| - m|B'(z)|.$$

Equivalently, for $z \in T$:

$$|r'(z)|^2 + \frac{\sqrt{|a_0|} - \sqrt{|a_n|}}{\sqrt{|a_0|}} |B'(z)| (|r(z)| - m)^2 \leqslant$$
$$\leqslant \left[|B'(z)| \cdot \|r\| - |r'(z)| - m|B'(z)|\right]^2.$$

A simple manipulation gives, for $z \in T$:

$$|r'(z)| \leqslant \frac{1}{2}\left[|B'(z)| - \frac{\sqrt{|a_0|} - \sqrt{|a_n|}}{\sqrt{|a_0|}} . \frac{(|r(z)| - m)^2}{(\|r\| - m)^2}\right](\|r\| - m).$$

This proves the result when $r(z) \neq 0$ for $z \in T$; but if $r(z) = 0$ for $z \in T$, then the inequality is trivially true. This proves the result completely. $\square$

Proof of Theorem 2. Assume that $r \in \mathcal{R}_n$ has all zeros in D_-, so that $m > 0$. Hence, for every complex numbers α, β with $|\alpha| < 1$ and $|\beta| < 1$,

$$m|\alpha\beta| < |r(z)| \quad \text{for} \quad z \in T.$$

Therefore, by Rouche's Theorem, all zeros of $R(z) = r(z) + m\alpha\beta$ lie in D_-. Hence, using Lemma 4, we have for, $z \in T$:

$$\left|\frac{zR'(z)}{R(z)}\right| \geqslant \operatorname{Re}\left(\frac{zR'(z)}{R(z)}\right) \geqslant \\ \geqslant \frac{1}{2}\left[|B'(z)| + \frac{\sqrt{|a_n + \alpha\beta m|} - \sqrt{|a_0 + (-1)^n\alpha \cdot \beta \cdot b \cdot m|}}{\sqrt{|a_n + \alpha\beta m|}}\right]. \tag{9}$$

Since $|b| \cdot |a_n| \leqslant |a_0|$, it can we easily shown (as in Theorem 1) that

$$\frac{|a_0 + (-1)^n\alpha \cdot \beta \cdot m \cdot b|}{|a_n + \alpha\beta m|} \leqslant \frac{|a_0|}{|a_n|}.$$

Therefore, from inequality (9), we have

$$|R'(z)| \geqslant \frac{1}{2}\left[|B'(z)| + \frac{\sqrt{|a_n|} - \sqrt{|a_0|}}{\sqrt{|a_n|}}\right]|R(z)| \Rightarrow$$

$$\Rightarrow |r'(z)| \geqslant \frac{1}{2}\left[|B'(z)| + \frac{\sqrt{|a_n|} - \sqrt{|a_0|}}{\sqrt{|a_n|}}\right]|r(z) + \alpha\beta m|.$$

Choosing argument of α, such that $|r(z) + \alpha\beta m| = |r(z)| + |\alpha| \cdot |\beta| \cdot m$, we get

$$|r'(z)| \geqslant \frac{1}{2}\left[|B'(z)| + \frac{\sqrt{|a_n|} - \sqrt{|a_0|}}{\sqrt{|a_n|}}\right](|r(z)| + |\alpha| \cdot |\beta| \cdot m).$$

Letting $|\alpha| \to 1$ and $|\beta| \to 1$ gives, for $z \in T$:

$$|r'(z)| \geqslant \frac{1}{2}\left[|B'(z)| + \frac{\sqrt{|a_n|} - \sqrt{|a_0|}}{\sqrt{|a_n|}}\right](|r(z)| + m),$$

which proves the result when $r(z) \neq 0$ for $z \in T$. But the inequality above is trivially true if $r(z) = 0$ for $z \in T$: this proves the theorem completely. □

Acknowledgement. The author is highly thankful to the referees for their valuable comments and precious suggestions.

References

[1] Aziz A., Dawood Q. M. *Inequalities for a polynomial and its derivative,* J. Approx. Theory, 1988, vol. 54, pp. 306 – 313.
DOI: https://doi.org/10.1016/0021-9045(88)90006-8

[2] Dubinin V. N. *Distortion theorems for polynomials on the circle*, Sb. Math., 2000, vol. 191, No. 12, pp. 1797 – 1807.

[3] Lax P. D. *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc. (N.S), 1944 vol. 50, pp. 509 – 513.

[4] Xin Li. *A comparizon inequality for Rational Functions*, Proc. Amer. Math. Soc., 2011, vol. 139, No. 5, pp. 1659 – 1665.
DOI: https://doi.org/S0002-9939(2012)11349-8

[5] Xin Li, Mohapatra R. N., Rodgriguez R. S. *Bernstein inequalities for rational functions with prescribed poles*, J. London Math. Soc., 1995, vol. 51, pp. 523 – 531. DOI: https://doi.org/10.1112/jlms/51.3.523

[6] Rahman Q. I., Schmeisser G. *Analytic Theory of Polynomials*, Oxford University Press, New York, 2002.

[7] Turán P. *Uber die ableitung von polynomen*, Compositio Math., 1939, vol. 7, pp. 89 – 95.

[8] Wali S. L., Shah W. M. *Some applications of Dubinin's Lemma to rational functions with prescribed poles*, J. Math. Anal. Appl, 2017, vol. 450, pp. 769 – 779. DOI: https://doi.org/10.1016/j.jmaa.2017.01.069

Received August 14, 2021.
In revised form, December 28, 2021.
Accepted December 30, 2021.
Published online January 25, 2022.

Department of Mathematics,
National Institute of Technology, Srinagar,
India, 190006
E-mail: idreesf3@gmail.com