DOI: 10.15393/j3.art.2022.11851

UDC 517.98

M. El Hamma, A. Mahfoud

GENERALIZATION OF TITCHMARSH' S THEOREM FOR THE FIRST HANKEL-CLIFFORD TRANSFORM IN THE SPACE $L^p_\mu((0, +\infty))$

Abstract. Using a generalized translation operator, we intend to establish generalizations of the Titchmarsh theorem ([14], theorem 84) for the first Hankel-Clifford transform for certain classes of functions in the space $L^p_\mu((0, +\infty))$, where $1 < p \leqslant 2$.

Key words: *first Hankel-Clifford transform, generalized translation operator, Clifford-Lipschitz class, Dini-Clifford-Lipschitz class.*

2020 Mathematical Subject Classification: *47G30*

1. Introduction. Titchmarsh ([14], Theorem 84) characterized the set of functions in $L^p(\mathbb{R})$, $1 < p \leqslant 2$, satisfying the Lipschitz condition, by means of an asymptotic estimate growth of the norm of their Fourier transform; namely, we have:

Theorem 1. *Let f belong to $L^p(\mathbb{R})$, $1 < p \leqslant 2$, such that*

$$\int_{-\infty}^{+\infty} |f(x+h) - f(x-h)|^p dx = O(h^{\alpha p}), \quad 0 < \alpha \leqslant 1, \quad as\ h \longrightarrow 0.$$

Then its Fourier transform $\mathcal{F}(f)$ belongs to $L^\beta(\mathbb{R})$ for

$$\frac{p}{p + \alpha p - 1} < \beta \leqslant \frac{p}{p-1}.$$

On the other hand, Younis in ([15], Theorem 3.3) studied the same phenomena for the wider Dini-Lipschitz class, as well as for some other allied classes of functions. More precisely,

Theorem 2. *Let $f \in L^p(\mathbb{R})$ with $1 < p \leqslant 2$, such that*

$$\Big(\int\limits_{-\infty}^{+\infty} |f(x+h) - f(x)|^p dx\Big)^{\frac{1}{p}} = O\Big(\frac{h^\alpha}{(\log \frac{1}{h})^\gamma}\Big), \ h \longrightarrow 0, \ 0 < \alpha \leqslant 1, \gamma > 0.$$

Then $\mathcal{F}(f) \in L^\beta(\mathbb{R})$ for

$$\frac{p}{p + \alpha p - 1} \leqslant \beta < p' = \frac{p}{p-1}$$

and $\dfrac{1}{\beta} < \gamma$, where $\mathcal{F}(f)$ stands for the Fourier transform of f.

There are many analogues of these theorems: for the Bessel transform on $\mathbb{R}^+$, for the Dunkl transform on $\mathbb{R}^d$, for the q-Dunkl transform on $\mathbb{R}_q$, etc (for example, see [2], [3], [4], [5], [10]).

The aim of this paper is to provide generalizations of Theorems 1 and 2 for the first Hankel-Clifford transform. For this purpose, we use the generalized translation operator.

2. Preliminaries. Let us we briefly collect the pertinent definitions and facts relevant for first Hankel-Clifford analysis, which can be founded in [11], [12], [13], [16].

Assume that $L^p_\mu = L^p_\mu((0, +\infty))$, $1 \leqslant p < \infty$ and $\mu \geqslant 0$, is the space of all real-valued measurable functions f on $(0, +\infty)$, such that

$$\|f\|_{p,\mu} = \Big(\int\limits_0^{+\infty} |f(x)|^p x^\mu dx\Big)^{\frac{1}{p}} < \infty.$$

Let c_μ be the Bessel-Clifford function of the first kind defined by (see [6])

$$c_\mu(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k x^k}{k!\Gamma(\mu + k + 1)}, \tag{1}$$

which satisfies the differential equation

$$xy'' + (\mu + 1)y' + y = 0.$$

For $\mu \geqslant -\frac{1}{2}$, we introduce the normalized spherical Bessel function j_μ of index μ, defined by

$$j_\mu(x) = \Gamma(\mu + 1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(k + \mu + 1)} \Big(\frac{x}{2}\Big)^{2k}, \ x \in \mathbb{C}, \tag{2}$$

where $\Gamma(x)$ is the gamma-function.

Moreover, from (2) we see that

$$\lim_{x\to 0} \frac{j_\mu(x)-1}{x^2} \neq 0;$$

by consequence, there exist $C>0$ and $\eta>0$ satisfying

$$|x| \leqslant \eta \Longrightarrow |j_\mu(x)-1| \geqslant C|x|^2. \tag{3}$$

The function $j_\mu(x)$ is infinitely differentiable, even, and, moreover, entire analytic.

From [1], we have the following lemma:

Lemma 1. *Let $\mu \geqslant -\frac{1}{2}$. The following inequalities are fulfilled:*

1) $|j_\mu(x)| \leqslant 1$;
2) $1-j_\mu(x) = O(x^2), \quad 0 \leqslant x \leqslant 1$;
3) $1-j_\mu(x) = O(1), \quad x \geqslant 1$.

By formulas (1) and (2), we have the following relation, which connect the Bessel-Clifford function and the normalized spherical Bessel function:

$$c_\mu(x) = \frac{1}{\Gamma(\mu+1)} j_\mu(2\sqrt{x}) \tag{4}$$

Definition 1. *[8], [9] For $\mu \geqslant 0$, the first Hankel-Clifford transform for a function $f \in L^1_\mu$ is defined by*

$$h_{1,\mu}(f)(\lambda) = \lambda^\mu \int_0^{+\infty} c_\mu(\lambda x) f(x)\,dx.$$

Proposition 1. *If $f \in L^1_\mu$ and $h_{1,\mu}(f) \in L^1_\mu$, then*

$$f(x) = x^\mu \int_0^{+\infty} c_\mu(\lambda x) h_{1,\mu}(f)(\lambda)\,d\lambda, \ \forall x \in (0,+\infty).$$

For $\mu \geqslant 0$, let $F(\lambda) = h_{1,\mu}(f)(\lambda)$ and $G(\lambda) = h_{1,\mu}(g)(\lambda)$ denote the first Hankel-Clifford transform of order μ of $f(x)$ and $g(x)$, respectively. Méndez et al. [9] established the following Parseval relation:

$$\int_0^{+\infty} F(\lambda)G(\lambda)\lambda^\mu\,d\lambda = \int_0^{+\infty} f(x)g(x)x^\mu\,dx.$$

Then the first Hankel-Clifford transform $h_{1,\mu}\colon f(x) \longrightarrow h_{1,\mu}(f)(\lambda)$ is a linear isomorphism of the space L^2_μ into itself, and for any function $f \in L^2_\mu$ we have the Parseval identity

$$\|\lambda^{-\mu} h_{1,\mu}(f)(\lambda)\|_{2,\mu} = \|x^{-\mu} f(x)\|_{2,\mu}.$$

Parseval's identity and the Marcinkiewicz interpolation theorem (see [14]) are true for $f \in L^p_\mu$ with $1 < p \leqslant 2$ and p', such that $\frac{1}{p} + \frac{1}{p'} = 1$

$$\|\lambda^{-\mu} h_{1,\mu}(f)(\lambda)\|_{p',\mu} \leqslant C_0 \|x^{-\mu} f(x)\|_{p,\mu}. \tag{5}$$

Let $\Delta = \Delta(x, y, z)$ be area of the triangle with sides x, y, z (see [7], [16]). For $\mu \geqslant 0$, set

$$D_\mu(x, y, z) = \frac{\Delta^{2\mu+1}}{2^{2\mu}(xyz)^\mu \Gamma(\mu + \frac{1}{2})\sqrt{\pi}}$$

if Δ exists, and zero otherwise. Note that $D_\mu(x, y, z) \geqslant 0$ and it is symmetric in x, y, z.

From [12], we define the generalized translation operator by the relation

$$\tau_h(f)(x) = \int_0^{+\infty} f(z) D_\mu(h, x, z) z^\mu dz\ , \quad 0 < x\ , \ \ h < \infty.$$

Assume that $\mu \geqslant 0$. Let M be the map of L^2_μ defined by

$$Mf(x) = x^\mu f(x) \tag{6}$$

Prasad et al proved the following well-known proposition:

Proposition 2. *[12] Let $f \in L^2_\mu$ and fix $h > 0$. Then $\tau_h(f)(x) \in L^2_\mu$ and*

$$h_{1,\mu}(M\tau_h f(\cdot))(\lambda) = c_\mu(\lambda h) h_{1,\mu}(Mf(\cdot))(\lambda), \ \ \lambda \in (0, +\infty).$$

3. Main results. Before giving our first main result, we define the Clifford-Lipschitz class.

Definition 2. *Let $0 < \delta \leqslant 1$. A function $f \in L^p_\mu$, $1 < p \leqslant 2$, is said to be in the Clifford-Lipschitz class, denoted $Lip_c(\delta, p, \mu)$, if*

$$\|\Gamma(\mu + 1)\tau_h f(x) - f(x)\|_{p,\mu} = O(h^\delta) \ \ ash \longrightarrow 0$$

Theorem 3. *Let f belong to the Clifford-Lipschitz class $Lip_c(\delta, p, \mu)$, $0 < \delta \leqslant 1$ and $1 < p \leqslant 2$. Then $h_{1,\mu}(Mf) \in L^{\beta}_{\mu}((0, +\infty))$ for all β satisfying*

$$\frac{\mu p + p}{p - \mu + \delta p - 1} < \beta \leqslant p' = \frac{p}{p-1}.$$

Proof. Assume that $f \in Lip_c(\delta, p, \mu)$; then we have

$$\|\Gamma(\mu+1)\tau_h f(x) - f(x)\|_{p,\mu} = O(h^{\delta}) \text{ as } h \longrightarrow 0.$$

Using the formula (6), we have

$$\|\Gamma(\mu+1)\tau_h f(x) - f(x)\|_{p,\mu} = \|x^{-\mu}\left(\Gamma(\mu+1)x^{\mu}\tau_h f(x) - x^{\mu} f(x)\right) = \\ = \|x^{-\mu}\left(\Gamma(\mu+1)M\left(\tau_h f(x)\right) - M\left(f(x)\right)\right)\|_{p,\mu}.$$

From proposition 2 and formula (4), we get

$$h_{1,\mu}\left(\Gamma(\mu+1)M\left(\tau_h f(x)\right) - M\left(f(x)\right)\right)(\lambda) = \\ = \left(j_{\mu}(2\sqrt{\lambda h}) - 1\right) h_{1,\mu}(Mf(x))(\lambda).$$

By the Hausdorff-Young formula (5), we have

$$\int_0^{+\infty} \lambda^{-\mu p'} |1 - j_{\mu}(2\sqrt{\lambda h})|^{p'} |h_{1,\mu}(Mf)(\lambda)|^{p'} \lambda^{\mu} d\lambda \leqslant \\ \leqslant C_0^{p'} \left\|x^{-\mu}\left(\Gamma(\mu+1)M\left(\tau_h f(x)\right) - M\left(f(x)\right)\right)\right\|^{p'}_{p,\mu} \leqslant \\ \leqslant C_0^{p'} \|\Gamma(\mu+1)\tau_h f(x) - f(x)\|^{p'}_{p,\mu} \leqslant C_1 h^{\delta p'}.$$

Hence,

$$\int_0^{+\infty} |1 - j_{\mu}(2\sqrt{\lambda h})|^{p'} |h_{1,\mu}(Mf)(\lambda)|^{p'} \lambda^{(1-p')\mu} d\lambda \leqslant C_1 h^{\delta p'}.$$

If $0 < \lambda < \frac{\eta^2}{4h}$, then $0 < 2\sqrt{\lambda h} < \eta$ and inequality (3) implies

$$|1 - j_{\mu}(2\sqrt{\lambda h})| \geqslant 4C\lambda h.$$

From this, we get

$$\int_0^{\frac{\eta^2}{4h}} |\lambda h|^{p'} |h_{1,\mu}(Mf)(\lambda)|^{p'} \lambda^{(1-p')\mu} d\lambda \leqslant$$

$$\leqslant \frac{1}{(4C)^{p'}} \int_0^{\frac{\eta^2}{4h}} |1 - j_\mu(2\sqrt{\lambda h})|^{p'} |h_{1,\mu}(Mf)(\lambda)|^{p'} \lambda^{(1-p')\mu} d\lambda \leqslant$$

$$\leqslant \frac{1}{(4C)^{p'}} \int_0^{+\infty} |1 - j_\mu(2\sqrt{\lambda h})|^{p'} |h_{1,\mu}(Mf)(\lambda)|^{p'} \lambda^{(1-p')\mu} d\lambda = O(h^{\delta p'}).$$

So that

$$\int_0^{\frac{\eta^2}{4h}} |\lambda h_{1,\mu}(Mf)(\lambda)|^{p'} \lambda^{(1-p')\mu} d\lambda = O(h^{(\delta-1)p'}).$$

Thus,

$$\int_0^{t} |\lambda h_{1,\mu}(Mf)(\lambda)|^{p'} \lambda^{(1-p')\mu} d\lambda = O(t^{(1-\delta)p'}).$$

Let

$$\psi(t) = \int_1^{t} |\lambda h_{1,\mu}(Mf)(\lambda)|^{\beta} \lambda^{(1-p')\mu\beta/p'} d\lambda.$$

Now, if $\beta \leqslant p'$, by the Hölder inequality we obtain

$$\psi(t) \leqslant \left(\int_1^{t} |\lambda h_{1,\mu}(Mf)(\lambda)|^{p'} \lambda^{(1-p')\mu} d\lambda \right) \left(\int_1^{t} d\lambda \right)^{1-\beta/p'} =$$

$$= O(t^{(1-\delta)p' \times \beta/p'} t^{1-\beta/p'}) = O(t^{(1-\delta)\beta} t^{1-\beta/p'}) = O(t^{1-\delta\beta+\beta/p}).$$

Therefore,

$$\int_1^{t} |h_{1,\mu}(Mf)(\lambda)|^{\beta} \lambda^{\mu} d\lambda = \int_1^{t} \lambda^{-\beta-(1-p')\mu\beta/p'} \psi'(\lambda) \lambda^{\mu} d\lambda =$$

$$= t^{-\beta-(1-p')\mu\beta/p'+\mu}\psi(t)+$$

$$+ (\beta + (1-p')\mu\beta/p' - \mu)\int_1^t \lambda^{-\beta-(1-p')\mu\beta/p'+\mu-1}\psi(\lambda)d\lambda =$$

$$= O(t^{-\beta-(1-p')\mu\beta/p'+\mu}\, t^{1-\delta\beta+\beta/p}) + O\left(\int_1^t \lambda^{-\beta-(1-p')\mu\beta/p'+\mu-1}\lambda^{1-\delta\beta+\beta/p}d\lambda\right) =$$

$$= O(t^{-\beta-(1-p')\mu\beta/p'+\mu+1-\delta\beta+\beta/p})$$

and the right-hand side of this estimate is bounded as $t \longrightarrow \infty$ if

$$-\beta - (1-p')\mu\beta/p' + \mu + 1 - \delta\beta + \beta/p < 0.$$

That is,

$$\beta > \frac{\mu p + p}{p - \mu + \delta p - 1}.$$

Thus, the proof is finished. □

In the rest of this paper, we give our second main result, which is a generalization of Theorem 2. For this objective, we need to define the Dini-Clifford Lipschitz class.

Definition 3. *Let $0 < \delta \leqslant 1$, $\gamma > 0$. A function $f \in L^p_\mu$, $1 < p \leqslant 2$, is said to be in the Dini-Clifford-Lipschitz class, denoted D-Lip$_c(\delta,\gamma,p,\mu)$, if*

$$\|\Gamma(\mu+1)\tau_h f(x) - f(x)\|_{p,\mu} = O\left(\frac{h^\delta}{(\log\frac{1}{h})^\gamma}\right) \quad as\ h \longrightarrow 0.$$

Theorem 4. *Let $f \in L^p_\mu$, $1 < p \leqslant 2$. If f belongs to D-Lip$_c(\delta,\gamma,p,\mu)$, then $h_{1,\mu}(Mf)$ belongs to $L^\beta_\mu((0,+\infty))$, such that*

$$\frac{\mu p + p}{p - \mu + \delta p - 1} < \beta \leqslant p' = \frac{p}{p-1} \quad and \quad \beta > \frac{1}{\gamma}.$$

Proof. Similary to the proof of theorem 3, we can establish the following result:

$$\int_0^{\frac{\eta^2}{4h}} |\lambda h_{1,\mu}(Mf)(\lambda)|^{p'}\lambda^{(1-p')\mu}d\lambda = O\left(\frac{h^{(\delta-1)p'}}{(\log\frac{1}{h})^{\gamma p'}}\right).$$

Thus,

$$\int_0^t |\lambda h_{1,\mu}(Mf)(\lambda)|^{p'} \lambda^{(1-p')\mu} d\lambda = O\left(\frac{t^{(1-\delta)p'}}{(\log t)^{\gamma p'}}\right).$$

Let us consider again the function ψ, defined by

$$\psi(t) = \int_1^t |\lambda h_{1,\mu}(Mf)(\lambda)|^{\beta} \lambda^{(1-p')\mu\beta/p'} d\lambda.$$

Then, if $\beta \leqslant p'$, using the Hölder inequality we obtain

$$\psi(t) = O\left(\frac{t^{1-\delta\beta+\beta/p}}{(\log t)^{\gamma\beta}}\right).$$

Hence,

$$\int_1^t |h_{1,\mu}(Mf)(\lambda)|^{\beta} \lambda^{\mu} d\lambda = \int_1^t \lambda^{-\beta-(1-p')\frac{\mu\beta}{p'}} \psi'(\lambda)\lambda^{\mu} d\lambda =$$

$$= t^{-\beta-(1-p')\mu\beta/p'+\mu} \ \psi(t)+(\beta+(1-p')\frac{\mu\beta}{p'}-\mu)\int_1^t \lambda^{-\beta-(1-p')\mu\beta/p'+\mu-1} \ \psi(\lambda)d\lambda =$$

$$= O\Big(t^{-\beta-(1-p')\mu\beta/p'+\mu} \ \frac{t^{1-\delta\beta+\beta/p}}{(\log t)^{\gamma\beta}}\Big)+O\Big(\int_1^t \lambda^{-\beta-(1-p')\mu\beta/p'+\mu-1} \frac{\lambda^{1-\delta\beta+\beta/p}}{(\log \lambda)^{\gamma\beta}} d\lambda\Big) =$$

$$= O\Big(\frac{t^{-\beta-(1-p')\frac{\mu\beta}{p'}+\mu+1-\delta\beta+\beta/p}}{(\log t)^{\gamma\beta}}\Big).$$

and this is bounded as $t \longrightarrow \infty$ if

$$-\beta-(1-p')\mu\beta/p'+\mu+1-\delta\beta+\beta/p < 0 \ \text{ and } \ -\gamma\beta < -1,$$

which gives

$$\beta > \frac{\mu p + p}{p-\mu+\delta p-1} \ \text{ and } \ \beta > \frac{1}{\gamma}.$$

And this ends the proof. □

Acknowledgment. The authors would like to thank the referee for his valuable comments and suggestions.

References

[1] Abilov V. A., Abilova F. V. *Approximation of Functions by Fourier-Bessel sums.* IZV. Vyssh. Uchebn Zaved. Mat., 2001, no. 8, pp. 3 – 9.

[2] Daher R., Djellab N., El Hamma M. *some theorems of the Jacobi-Lipshitz class for the Jacobi transform.* Bulletin of the Transilvania University of Brasov, Series III: Mathematics and Computer Science., 2021, vol. 1(63), no. 2, pp. 29 – 36.

[3] Daher R., El Hamma M., Akhlidj A. *Dini-Lipschitz functions for the Bessel transform.* Nonlinear studies., 2017, vol. 24, no. 2, pp. 297 – 301.

[4] Daher R., Tyr O. *Growth properties of the q-Dunkl transform in the space* $L^p_{q,\alpha}(\mathbb{R}_q, |x|^{2\alpha+1}d_qx)$. The Ramanujan Journal., 2022, vol. 57(1), pp. 119 – 134.

[5] El Hamma M., Daher R., *On some theorems of the Dunkl-Lipschitz class for the Dunkl transform.* Lobachevskii Journal of Mathematics., 2019, vol. 40, no. 8, pp. 1157 – 1163.

[6] Gray A., Matthecos G. B., MacRobert T. M. *A Treatise on Bessel functions and their applications to physics.* Macmillan, London, 1987.

[7] Haimo D. T. *Integral equations associated with Hankel convolution.* Trans. Amer. Math. Soc., 1965, vol.116.

[8] Malgonde S. P., Bandewar S. R. *On the generalized Hankel-Clifford transformation of arbitrary order.* Proc. Indian Alod Sci .Math Sci., 2000, vol. 110(3), pp. 293 – 304.

[9] Méndez Pérez J. M. R., Socas Robayna M. M. *A pair of generalized Hankel-Clifford transformation and their applications.* J. Math. Anal. Appl., 1991, vol. 154, pp. 543 – 557.

[10] Negzaoui S. *Lipschitz conditions in Lagurre Hypergroup.* Mediterr. J. Math., 2017, 14:191.
DOI: https://doi.org/10.1007/s00009-017-0989-4

[11] Prasad A., Kumar M. *Continuity of pseu-differential operator* $h_{1,\mu}$, *involving Hankel translation and Hankel convolution some Gevrey spaces.* Integral. Transforms Spec. Funct., 2010, vol. 21(6), pp. 465 – 477.

[12] Prasad P., Singh V. K., Dixit M. M. *Pseudo-differential operators involving Hankel-Clifford transformations.* Asian-European. J. Math., 2012, vol. 5, no. 3, 15 pages.

[13] Pathak R. S., Pandey P. K. *A class of pseudo-differential operator associated with Bessel operators.* J. Math. Anal. Appl., 1995, vol. 196, pp. 736 – 747.

[14] Titchmarsh E. C. *Introduction to the theory of Fourier integrals.* Clarendon Press, Oxford, 1948.

[15] Younis M. S. *Fourier transforms of Dini-Lipschitz functions.* J. Math. Math. Sci., 1986, vol. 9, no. 2, pp. 301 – 312.

[16] Zemanian A. H. *Generalized integral transformations.* Interscience Publishers, New York, 1968.

Received May, 21, 2022.
In revised form, August 30, 2022.
Accepted September 2, 2022.
Published online October 3, 2022.

Laboratoire Mathématiques Fondamentales et appliquées
Faculté des Sciences Aïn Chock
Université Hassan II, Casablanca, Maroc

Mohamed El Hamma
E-mail: m_elhamma@yahoo.fr

Ayoub Mahfoud
E-mail: mahfoudayoub00@gmail.com