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LINEAR PROBLEMS IN THE SPACE OF POLINOMIALS OF DEGREE AT MOST 3

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Denote by P_n, $n \in \mathrm{N}$ the linear space of real polynomials p of degree at most n. There are various ways in which we can introduce norm in P_n, here the problem is investigated when $\|p\| = \max\{|p(x)|:\ x \in [-1;1]\}$. Let $B_n = \{p \in P_n :\ \|p\| \le 1\}$ be the unit ball and let EB_n be the set of the extreme points of B_n, i.e. such points $p \in B_n$ that $B_n \setminus \{p\}$ is convex. The sets EB_0, EB_1 and EB_2 are known and it turns out that also EB_3 has a particularly simple form. In this paper we determine EB_3 and give some conclusions and applications of the main results. Moreover, several examples are included. The coefficient region for the polynomials of degree exceeding 3 seems very complicated.

Let $p \in P_n$ and let $N(p)$ be the number of all zeros of the polynomial $1-p^2$ in the interval $[-1;1]$, counted according to multiplicity. In the case of complex polynomials, D.A. Brannan and J.G. Clunie [1] were able to characterize the extreme points partially. In the case of real polynomials, A.G. Konheim and T.J. Rivlin [2] proved for $p \in B_n$ that $p \in EB_n$ if and only if $N(p) > n$. We know that $EB_0 = \{-1, 1\}$ and $EB_1 = \{-1, 1, -x, x\}$. Moreover W. Szumny [3] determined a precise form of EB_2, i.e.

$$\begin{aligned} EB_2 = \quad & \{-1, 1, (c + 2\sqrt{2(1-c)} - 3)x^2 + 2(c + \sqrt{2(1-c)} - 1)x + c, \\ & (c + 2\sqrt{2(1-c)} - 3)x^2 - 2(c + \sqrt{2(1-c)} - 1)x + c, \\ & (3 - c - 2\sqrt{2(1-c)})x^2 + 2(1 - c - \sqrt{2(1-c)})x - c, \\ & (3 - c - 2\sqrt{2(1-c)})x^2 + 2(c + \sqrt{2(1-c)} - 1)x - c; \\ & c \in [\tfrac{1}{2}; 1].\}. \end{aligned}$$

For $p \in EB_3$ we have $N(p) \in \{4,5,6\}$ and

THEOREM 1. *The set EB_3 consists of the functions $p_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1$, $p_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2$, $p_3(x) = a_3x^3 + b_3x^2 + c_3x + d_3$, $p_4(x) \equiv 1$ and $-p_i(x)$, where $i \in \{1,2,3,4\}$ and*

$$\begin{gathered} a_1 = \frac{4}{(\beta-\alpha)^3}, \quad b_1 = -\frac{6(\alpha+\beta)}{(\beta-\alpha)^3}, \\ c_1 = \frac{12\alpha\beta}{(\beta-\alpha)^3}, \quad d_1 = \frac{(\alpha+\beta)(\alpha^2-4\alpha\beta+\beta^2)}{(\beta-\alpha)^3}, \end{gathered} \tag{1}$$

$$\begin{aligned} a_2 &= \frac{1}{(1+\gamma)^2}, \quad b_2 = -\frac{1+2\gamma}{(1+\gamma)^2}, \\ c_2 &= \frac{\gamma(\gamma+2)}{(1+\gamma)^2}, \quad d_2 = \frac{1+2\gamma}{(1+\gamma)^2}, \end{aligned} \tag{2}$$

$$\begin{aligned} a_3 &= \frac{-4\delta}{(1-\delta^2)^2}, \quad b_3 = \frac{2(3\delta^2-1)}{(1-\delta^2)^2}, \\ c_3 &= \frac{4\delta}{(1-\delta^2)^2}, \quad d_3 = \frac{1-4\delta^2-\delta^4}{(1-\delta^2)^2}, \end{aligned} \tag{3}$$

the coefficient region for α, β is given on the fig. 1 and $\gamma \in [-\frac{1}{2}; 1]$, $\delta \in [-\frac{1}{3}; 0]$.

PROOF. By simple consideration we see easy that $p(x) \equiv 1$ and $p(x) \equiv -1$ are the extreme points of B_3. Because the polynomial $q(x) = bx^2 + cx + d$ cannot be an extreme point of B_3 (apart from $q(x) = 2x^2 - 1$ and $q(x) = -2x^2 + 1$ where $N(q) = 4$) then it is also sufficient to consider only the polynomials $p(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$. For these polynomials we will consider three cases of $p \in EB_3$ described on the figures 2, 3 and 4:

The other extreme points of B_3 may be obtained as their symmetrical.

Suppose that $p(x) = ax^3 + bx^2 + cx + d$ belongs to EB_3 and $p(\alpha) = 1$, $p(\beta) = -1$, $p'(\alpha) = p'(\beta) = 0$ (fig. 2). Thus we have

$$\begin{gathered} p(x) - 1 = a(x-\alpha)^2(x-u), \\ p(x) + 1 = a(x-\beta)^2(x-t) \end{gathered}$$

and

$$\begin{gathered} p'(x) = a\left[2(x-\beta)(x-t) + (x-\beta)^2\right] = a(x-\beta)(3x-2t-\beta), \\ p'(x) = a\left[2(x-\alpha)(x-u) + (x-\alpha)^2\right] = a(x-\alpha)(3x-2u-\alpha). \end{gathered}$$

Hence

$$0 = a(\alpha-\beta)(3\alpha-2t-\beta) \quad \text{and} \quad 0 = a(\beta-\alpha)(3\beta-2u-\alpha).$$

Therefore

$$3\alpha-2t-\beta = 0 \quad \text{and} \quad 3\beta-2u-\alpha = 0,$$

thus

$$t = \frac{1}{2}(3\alpha-\beta) \qquad u = \frac{1}{2}(3\beta-\alpha).$$

Because $p(\alpha) = 1$ then

$$1 = a(\alpha-\beta)^2\left[\alpha-\frac{1}{2}(3\alpha-\beta)\right]-1,$$

$$\frac{1}{2}a(\alpha-\beta)^2(2\alpha-3\alpha+\beta) = 2,$$

$$a(\alpha-\beta)^2(\beta-\alpha) = 4, \qquad a = \frac{4}{(\beta-\alpha)^3}.$$

If $p(x) = ax^3+bx^2+cx+d$ and $p(x) = a(x-\alpha)^2(x-u)+1 = ax^3-a(u+2\alpha)x^2+a\alpha(\alpha+2u)x+1-a\alpha^2u$, then by comparing the coefficients of x^2 and x we see that

$$b = -a(u+2\alpha),\ c = a\alpha(\alpha+2u),\ d = 1-a\alpha^2u.$$

Hence we obtain (1).

Because $t \le -1$ and $u \ge 1$ then $\frac{1}{2}(3\alpha-\beta) \le -1$ and $\frac{1}{2}(3\beta-\alpha) \ge 1$ so we obtain the region given on the fig. 1. This completes the proof for the polynomial p_1.

Now let $p(x) = ax^3+bx^2+cx+d$ belongs to EB_3 and $p(-1) = -1$, $p(1) = 1$, $p'(\gamma) = p'(\varepsilon) = 0$, $p(\gamma) = 1$, $-1 \le p(\varepsilon) \le 1$ (fig. 3). Then

$$p(x)-1 = a(x-\gamma)^2(x-1)$$

and from $p(-1) = -1$ we obtain

$$a = \frac{1}{(1+\gamma)^2}.$$

Because $b = -(1+2\gamma)a$, $c = \gamma(\gamma+2)a$, $d = (1+2\gamma)a$ we have (2) finally. In that

$$p'(x) = a(x-\gamma)(3x-2-\gamma) \text{ and } p'(\varepsilon) = 0 \text{ and } \varepsilon = \gamma \text{ iff } \varepsilon = \gamma = 1$$

so

$$\varepsilon = \frac{1}{3}(\gamma + 2).$$

Since

$$p(\varepsilon) = \frac{1}{(1+\gamma)^2}(\varepsilon - \gamma)^2(\varepsilon - 1) + 1 \text{ and } p(\varepsilon) \geq -1$$

then

$$2(\gamma - 1)^3 + 27(\gamma + 1)^2 \geq 0.$$

Hence $\gamma \in [-\frac{1}{2}; 1]$ and this completes the proof for the polynomial p_2.

Now let $p \in EB_3$ and $p(1) = p(-1) = p(r) = -1$, $p(\delta) = p(s) = 1$, $p'(\delta) = p'(\zeta) = 0$, $\delta \in (-1; 1)$, $1 \leq \zeta \leq r < s$ (fig. 4). Thus we have

$$p(x) + 1 = a(x-1)(x+1)(x-r) \text{ and } p(\delta) = 1,$$

hence

$$2 = a(\delta^2 - 1)(\delta - r)$$

$$r = \delta - \frac{2}{a(\delta^2 - 1)}.$$

Because

$$\begin{aligned} p'(x) &= a[2x(x-r) + x^2 - 1] = a[3x^2 - 2rx - 1] = \\ &= a\left[3x^2 - 2x\left(\delta - \frac{2}{a(\delta^2-1)}\right) - 1\right] \end{aligned}$$

and $p'(\delta) = 0$ then

$$3\delta^2 - 2\delta\left(\delta - \frac{2}{a(\delta^2 - 1)}\right) - 1 = 0$$

$$a = \frac{-4\delta}{(\delta^2 - 1)^2}.$$

Thus

$$p(x) = \frac{-4\delta}{(\delta^2-1)^2}(x^2 - 1)\left(x - \frac{3\delta^2 - 1}{2\delta}\right) - 1 =$$

$$= \frac{-4\delta}{(\delta^2-1)^2}x^3 + \frac{2(3\delta^2-1)}{(\delta^2-1)^2}x^2 + \frac{4\delta}{(\delta^2-1)^2}x - \frac{2(3\delta^2-1)}{(\delta^2-1)^2} - 1.$$

Hence the polynomial p has coefficients the form (3). Because $r = \frac{3\delta^2-1}{2\delta}$ and $r \geq 1$ then $\delta(3\delta+1)(\delta-1) \geq 1$, where $\delta < 1$. Therefore $\delta \in [-\frac{1}{3}; 0]$ and this completes the proof for p_3 and endes the proof of theorem.

Corolary. *1. By using Theorem we can construct the extreme points of B_3, for example:*
a) *extreme points of B_3 described on fig. 2.:*

$$\begin{array}{llll}
\text{if} & \alpha = -\dfrac{1}{2}, \quad \beta = \dfrac{1}{2} & \text{then} & p_1(x) = 4x^3 - 3x, \\
\text{if} & \alpha = -1, \quad \beta = \dfrac{1}{3} & \text{then} & p_1(x) = \dfrac{27}{16}x^3 - \dfrac{27}{16}x^2 - \dfrac{27}{16}x + \dfrac{11}{16}, \\
\text{if} & \alpha = -1, \quad \beta = 1 & \text{then} & p_1(x) = \dfrac{1}{2}x^3 - \dfrac{3}{2}x, \\
\text{if} & \alpha = -\dfrac{3}{4}, \quad \beta = \dfrac{3}{4} & \text{then} & p_1(x) = \dfrac{32}{27}x^3 - 2x;
\end{array}$$

b) *extreme points of B_3 described on fig. 3.:*

$$\begin{array}{llll}
\text{if} & \gamma = -\dfrac{1}{2} & \text{then} & p_2(x) = 4x^3 - 3x, \\
\text{if} & \gamma = -\dfrac{1}{3} & \text{then} & p_2(x) = \dfrac{9}{4}x^3 - \dfrac{3}{4}x^2 - \dfrac{5}{4}x + \dfrac{3}{4}, \\
\text{if} & \gamma = 0 & \text{then} & p_2(x) = x^3 - x^2 + 1, \ p_2(-x) = -x^3 - x^2 + 1, \\
 & & & -p_2(x) = -x^3 + x^2 - 1, \ -p_2(-x) = x^3 + x^2 - 1, \\
\text{if} & \gamma = \dfrac{1}{2} & \text{then} & p_2(x) = \dfrac{4}{9}x^3 - \dfrac{8}{9}x^2 + \dfrac{5}{8}x + \dfrac{8}{9}, \\
\text{if} & \gamma = 1 & \text{then} & p_2(x) = \dfrac{1}{4}x^3 - \dfrac{3}{4}x^2 + \dfrac{3}{4}x + \dfrac{3}{4};
\end{array}$$

c) *extreme points of B_3 described on fig. 4.:*

$$\begin{array}{llll}
\text{if} & \delta = -\dfrac{1}{3} & \text{then} & p_3(x) = \dfrac{27}{16}x^3 - \dfrac{27}{16}x^2 - \dfrac{27}{16}x + \dfrac{11}{16}, \\
\text{if} & \delta = -\dfrac{1}{4} & \text{then} & p_3(x) = \dfrac{256}{225}x^3 - \dfrac{416}{225}x^2 - \dfrac{256}{225}x + \dfrac{191}{225}, \\
\text{if} & \delta = 0 & \text{then} & p_3(x) = -2x^2 + 1.
\end{array}$$

Moreover we have

Corolary. *2. Let $p(x) = ax^3 + bx^2 + cx + d$ belongs to EB_3 and $p'(x_1) = p'(x_2) = 1 - |p(x_1)| = 0$ for some different $x_1, x_2 \in [-1; 1]$.*

a) If $|p(x_2)| = 1$ then $|a| \in [\frac{1}{2}; 4]$, $|b| \le \frac{27}{16}$, $|c| \in [\frac{3}{2}; 3]$, $|d| \le \frac{11}{16}$. The estimations of coefficients are sharp and being attained by polynomials given in Corollary 1.

b) If $|p(x_2)| < 1$ then $|p(1)| = |p(-1)| = 1$ and $|a| \in (\frac{1}{4}; 4)$, $|b| \le 1$, $|c| < 3$, $|d| \le 1$, moreover $a + c = 1$, $b + d = 0$. The estimations of coefficients are sharp, it is easy to see looking at Corollary 1.

Corolary. *3. Let $p(x) = ax^3+bx^2+cx+d$ belongs to EB_3 and $p'(x_1) = p'(x_2) = 0$, $|p(x_1)| = |p(-1)| = |p(1)| = 1$. If $|x_1| < 1$ and $|x_2| > 1$ then $|a| < \frac{27}{16}$, $|b| \in (\frac{27}{16}; 2)$, $|c| < \frac{27}{16}$, $|d| \in (\frac{11}{16}; 1)$, moreover $a+c = 0$, $b+d = -1$. The estimations are sharp, extremal polynomials are given in Corollary 1.*

Also we have

Remark. *1. Let $p \in EB_3$ be described as on fig. 2.*

a) If $t < -1$ and $n > 1$ then $N(p) = 4$.
b) If $t = -1$ and $n > 1$ then $N(p) = 5$ and $\beta = 3\alpha+2$, where $\alpha \in [-\frac{1}{2}; -\frac{1}{3}]$.
c) If $t < -1$, $n = 1$ then $N(p) = 5$ and $\beta = \frac{1}{3}\alpha + \frac{2}{3}$, where $\alpha \in [-1; -\frac{1}{2}]$.
d) If $t = -1$, $n = 1$ then $N(p) = 6$ and $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$, $p(x) = 4x^3 - 3x$, i.e. p is the Chebyshev polynomial of order 3.

Remark. *2. We have some particular cases:*
a) $\gamma = 1$ (fig. 3) implies $p_2(x) = \frac{1}{4}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x + \frac{3}{4}$, i.e. $p_2(x) - 1 = \frac{1}{4}(x-1)^3$ and $p_2'(x) \ge 0$ for all $x \in \mathrm{R}$;
b) $\delta = 0$ (fig. 4) implies $p_3 = -2x^2+1$, i.e. p_3 is the Chebyshev polynomial of order 2;
c) $\gamma = -\frac{1}{2}$ implies $p_2 = p_1$ for $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$;
d) $\delta = -\frac{1}{3}$ implies $p_3 = p_1$ for $\alpha = -\frac{1}{3}$, $\beta = 1$.

Remark. *3. If $p \in EB_4$ then $N(p) > n$ and only these polynomials of degree 3 belong to EB_4 which have $N(p) = 6$ or $N(p) = 5$, i.e. $p(x) = 4x^3 - 3x$ or $p(x) = \frac{1}{2(\alpha+1)^3}(x - 3\alpha - 2)(x+1) - 1$ where $\alpha \in [-\frac{1}{2}; -\frac{1}{3}]$ and their symmetrical.*

Remark. *4. Only two polynomials of degree 3 belong to EB_5, i.e. $p(x) = 4x^3 - 3x$ and $p(x) = -4x^3 + 3x$.*

Bibliography

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