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ON SOME HARMONIC FUNCTIONS RELATED TO HOLOMORPHIC FUNCTIONS WITH A POSITIVE REAL PART

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In the paper we examine some holomorphic functions and complex harmonic functions, which satisfy certain conditions of a Mocanu kind. We also consider their relations with appropriate coefficient conditions. The paper is a natural supplement to the results published in [1] and [2].

A. Let us first consider functions f holomorphic in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and such that $f(0) = f'(0) - 1 = 0$, i. e. functions of the form

$$f(z) = z + \sum_{n=2}^{+\infty} a_n z^n, \qquad a_n \in \mathbb{C}, \quad n = 2, 3, \ldots. \tag{1}$$

For a fixed number $\alpha \in \langle 0, 1\rangle$ by $J(\alpha)$ we denote the class of all functions f of the form (1) satisfying the condition

$$\operatorname{Re}\left[\alpha \frac{f(z)}{z} + (1-\alpha) f'(z)\right] > 0, \qquad z \in \Delta. \tag{2}$$

Remark 1. *Some properties of the class $J(\alpha)$, $\alpha \in \langle 0, 1\rangle$, were examined in 1977 by P. N. Chichra* [2]. *In 1915 J. W. Alexander* [3] *proved that $J(0)$ is a class of univalent functions (see also the Noshiro — Warszawski lemma, 1935). The class $J(0)$ was examined by T. H. MacGregor* [4] *and others as well.*

Observe that the identity function belongs to every class $J(\alpha)$, $\alpha \in \langle 0,1\rangle$.

Directly from the definition of the class $J(\alpha)$, $\alpha \in \langle 0,1\rangle$, we get

PROPOSITION 1. *Let $\alpha \in \langle 0,1\rangle$. If $f \in J(\alpha)$ then functions*

$$z \mapsto r^{-1} f(rz), \qquad z \mapsto e^{-it} f(e^{it} z), \qquad z \in \Delta,\ r \in (0,1),\ t \in \mathbb{R},$$

also belong to $J(\alpha)$.

Let $\wp$ denote the known class of Carathéodory functions with a positive real part, i. e. the functions p holomorphic in Δ such that $p(0) = 1$ and $\operatorname{Re} p(z) > 0$, $z \in \Delta$.

From the definitions of the classes $\wp$, $J(\alpha)$, $\alpha \in \langle 0,1\rangle$, we immediately obtain

PROPOSITION 2. *Let $\alpha \in \langle 0,1\rangle$. If $f \in J(\alpha)$, then the function p of the form*

$$p(z) = \alpha \frac{f(z)}{z} + (1-\alpha) f'(z), \qquad z \in \Delta, \tag{3}$$

belongs to the class $\wp$. Conversely, if $p \in \wp$, then the function f of the form (1), which is a solution of the equation (3), belongs to $J(\alpha)$.

EXAMPLE 1. Let $k \in \mathbb{N}$, $k \geq 2$, $\alpha \in \langle 0,1\rangle$. Consider the functions $f(z,k)$ of the form

$$f(z,k) = z + a_k z^k, \qquad 0 < a_k \leq \frac{1}{\alpha + (1-\alpha)k}, \quad z \in \Delta. \tag{4}$$

It is easy to check that every function $f(z,k)$ of the form (4) belongs to $J(\alpha)$.

We shall prove the next theorem.

THEOREM 1. *Let $\alpha \in \langle 0,1\rangle$. If a holomorphic function f of the form (1) satisfies the condition*

$$\sum_{n=2}^{+\infty} (\alpha + (1-\alpha)n)\,|a_n| \leq 1, \tag{5}$$

then $f \in J(\alpha)$.

PROOF. Assume that for $\alpha \in \langle 0,1\rangle$ a function f of the form (1) satisfies (5). It suffices to show that $\left|\alpha\frac{f(z)}{z} + (1-\alpha)f'(z) - 1\right| < 1$, $z \in \Delta$. By (1) and (5) we obtain

$$\left|\alpha\frac{f(z)}{z} + (1-\alpha)f'(z) - 1\right| = \left|\sum_{n=2}^{+\infty}(\alpha+(1-\alpha)n)\,a_n z^{n-1}\right| \leq$$

$$\leq \sum_{n=2}^{+\infty}(\alpha+(1-\alpha)n)\,|a_n||z^{n-1}| < 1, \quad z \in \Delta,$$

hence $f \in J(\alpha)$. □

REMARK 2. *Every function of the form (4) satisfies the condition (5), of course.*

Let $\mathcal{F}$ denote the class of functions of the form (1) such that

$$|a_n| \leq 1, \qquad n = 2, 3, \ldots .$$

It is clear that the known class S^c of convex functions is a subclass of $\mathcal{F}$.

Theorem 1 and the definition of $\mathcal{F}$ imply

COROLARY 1. *If $\varphi(z) = z + \sum_{n=2}^{+\infty} c_n z^n$, $z \in \Delta$, is a function of the class $\mathcal{F}$ and f of the form (1) satisfies the condition (5) for $\alpha \in \langle 0,1\rangle$, then the Hadamard product*

$$(f * \varphi)(z) = z + \sum_{n=2}^{+\infty} a_n c_n z^n, \qquad z \in \Delta,$$

belongs to $J(\alpha)$.

EXAMPLE 2. Fix $\alpha \in \langle 0,1\rangle$ and denote (see [2])

$$f_0(z) = z + \sum_{n=2}^{+\infty}\frac{2z^n}{\alpha+(1-\alpha)n}, \qquad z \in \Delta. \tag{6}$$

The function f_0 of the form (6) is holomorphic in Δ and it is suitably normalized. Moreover, for any $z \in \Delta$ we have

$$\mathrm{Re}\left[\alpha\frac{f_0(z)}{z} + (1-\alpha)f_0'(z)\right] = \mathrm{Re}\left(1 + \sum_{n=1}^{+\infty} 2z^n\right) = \mathrm{Re}\frac{1+z}{1-z} > 0,$$

which gives $f_0 \in J(\alpha)$. However, we observe that f_0 of the form (6) does not satisfy (5).

It appears that with some additional assumptions the condition (5) is not only sufficient but also necessary for a function to belong to $J(\alpha)$.

Let $J^-(\alpha)$, $\alpha \in \langle 0,1\rangle$, denote the class of functions $f \in J(\alpha)$ which are of the form

$$f(z) = z - \sum_{n=2}^{+\infty} |a_n| z^n, \qquad z \in \Delta, \tag{7}$$

(see [5], [6]). We have

THEOREM 2. *Let $\alpha \in \langle 0,1\rangle$. A holomorphic function f of the form (7) belongs to the class $J^-(\alpha)$ if and only if it satisfies (5).*

PROOF. If f of the form (7) belongs to $J^-(\alpha)$, $\alpha \in \langle 0,1\rangle$, then for any $x \in (0,1)$ we have

$$0 < \mathrm{Re}\left[\alpha \frac{f(x)}{x} + (1-\alpha) f'(x)\right] = 1 - \sum_{n=2}^{+\infty} (\alpha + (1-\alpha)n)\, |a_n| x^{n-1}.$$

Therefore we obtain (5).

The converse statement follows from Theorem 1. □

COROLARY 2. *Let $\alpha \in \langle 0,1\rangle$. If $f \in J^-(\alpha)$ and*

$$\varphi(z) = z + \sum_{n=2}^{+\infty} |c_n| z^n, \qquad z \in \Delta,$$

is a function of $\mathcal{F}$, then $f * \varphi \in J^-(\alpha)$.

It is known [7] that if a holomorphic function f of the form (1) satisfies the condition

$$\sum_{n=2}^{+\infty} n|a_n| \le 1, \tag{8}$$

then f is univalent and starlike in Δ.

Denote $\Delta_r = \{z \in \mathbb{C}\colon |z| < r\}$ for $r > 0$, with $\Delta_1 = \Delta$. In the paper [2] we can find the theorem on the disc where all functions of the class $J(\alpha)$, $\alpha \in \langle 0,1\rangle$, are univalent. The radius of this disc is a solution of an equation. It appears that in the class of functions satisfying (5) we have the next theorem.

THEOREM 3. *Let $\alpha \in \langle 0,1\rangle$ and $r_* = 1-\frac{\alpha}{2}$. If a function f of the form (1) satisfies the condition (5), then it is univalent and starlike in Δ_{r_*}. The constant $1-\frac{\alpha}{2}$ is the best possible.*

PROOF. The proof is based on Proposition 1, Theorem 1, the known inequality [1]

$$nr^{n-1} \le \alpha + (1-\alpha)n, \qquad \alpha \in \langle 0,1\rangle, \quad r \in (0, r_*\rangle, \quad n = 2, 3, \dots,$$

and the condition (8).

The result is sharp because the function f_* of the form

$$f_*(z) = z - \frac{z^2}{2-\alpha}, \qquad z \in \Delta,$$

satisfies the condition (5) and $f_*'(1-\frac{\alpha}{2}) = 0$. □

B. Let us next consider complex functions harmonic in the disc Δ of the form

$$f = h + \overline{g}, \quad h(z) = z + \sum_{n=2}^{+\infty} a_n z^n, \quad g(z) = \sum_{n=2}^{+\infty} b_n z^n, \quad z \in \Delta. \tag{9}$$

If f is a function of the form (9), then $F = h+g$ is a function holomorphic in Δ and $F(0) = F'(0) - 1 = 0$.

For an arbitrarily fixed $\alpha \in \langle 0,1\rangle$ by $J_H(\alpha)$ we denote the class of functions f of the form (9) and such that the function $F = h+g$ belongs to $J(\alpha)$. It means that the function f of the form (9) belongs to the class $J_H(\alpha)$, $\alpha \in \langle 0,1\rangle$, if and only if

$$\operatorname{Re}\left[\alpha\frac{h(z)+g(z)}{z} + (1-\alpha)\left(h'(z)+g'(z)\right)\right] > 0, \quad z \in \Delta. \tag{10}$$

Obviously, $J(\alpha) \subset J_H(\alpha)$, $\alpha \in \langle 0,1\rangle$.

According to the definition of the class $J_H(\alpha)$, $\alpha \in \langle 0,1\rangle$, and Theorem 1 we obtain the theorem.

THEOREM 4. *Let $\alpha \in \langle 0,1\rangle$. If a harmonic function f of the form (9) satisfies the condition*

$$\sum_{n=2}^{+\infty} (\alpha + (1-\alpha)n)\left(|a_n| + |b_n|\right) \le 1, \tag{11}$$

then $f \in J_H(\alpha)$.

Let $\mathcal{F}_{\mathcal{H}}$ stand for the class of functions $\chi = \varphi + \overline{\psi}$, where

$$\varphi(z) = z + \sum_{n=2}^{+\infty} c_n z^n, \quad \psi(z) = \sum_{n=2}^{+\infty} d_n z^n, \qquad z \in \Delta,$$

and $|c_n| \le 1$, $|d_n| \le 1$, $n = 2, 3, \dots$

From the definition of the class $\mathcal{F}_{\mathcal{H}}$ and by Theorem 4 we obtain:

COROLLARY 3. *If* $\chi = \varphi + \overline{\psi} \in \mathcal{F}_{\mathcal{H}}$ *and* f *of the form (9) satisfies the condition (11), then the Hadamard product of harmonic functions*

$$(f * \chi)(z) = h * \varphi + \overline{g * \psi}$$

satisfies (10) and consequently belongs to the class $J_H(\alpha)$.

Denote by $J_H^-(\alpha)$, $\alpha \in \langle 0, 1\rangle$, the class of functions $f \in J_H(\alpha)$ which are of the form

$$f = h + \overline{g}, \quad h(z) = z - \sum_{n=2}^{+\infty} |a_n| z^n, \quad g(z) = -\sum_{n=2}^{+\infty} |b_n| z^n, \quad z \in \Delta. \quad (12)$$

By the definition of $J_H^-(\alpha)$, $\alpha \in \langle 0, 1\rangle$, and in view of Theorems 1, 2, 4 and the condition (11) we get the theorem.

THEOREM 5. *Let* $\alpha \in \langle 0, 1\rangle$. *A harmonic function* f *of the form (12) belongs to the class* $J_H^-(\alpha)$ *if and only if it satisfies (11).*

REMARK 3. *We can consider harmonic functions* $\chi = \varphi + \overline{\psi}$, *where*

$$\varphi(z) = z + \sum_{n=2}^{+\infty} |c_n| z^n, \quad \psi(z) = \sum_{n=2}^{+\infty} |d_n| z^n, \qquad z \in \Delta,$$

$|c_n| \le 1$, $|d_n| \le 1$, $n = 2, 3, \dots$ *and their Hadamard products with functions* f *of the form (12). Then we obtain a corollary analogous to Corollary 2.*

In the paper [8] we can find the sufficient condition for harmonic functions to be starlike. Applying this theorem, which generalizes the mantioned result of A. W. Goodman ([7]), we can prove the theorem.

THEOREM 6. *Let $\alpha \in \langle 0, 1\rangle$. If a harmonic function f of the form (9) satisfies the condition (11), then f is univalent, sens-preserving and starlike in every disc Δ_r, where $r \in (0, r_*\rangle$, $r_* = 1 - \frac{\alpha}{2}$. The constant $1 - \frac{\alpha}{2}$ is the best possible.*

Theorems 2, 3, 5, 6 imply

COROLARY 4. *The univalence and starlikeness radius for the classes $J^-(\alpha)$, $J_H^-(\alpha)$, $\alpha \in \langle 0, 1\rangle$, is equal to $r_* = 1 - \frac{\alpha}{2}$.*

We can show

LEMMA 1. *Let $\alpha \in \langle 0, 1\rangle$, $l \geq 1$. Then for any $r \in (0, \frac{2-\alpha}{2^l}\rangle$ and $n = 2, 3, \ldots$ we have*

$$n^l r^{n-1} \leq \alpha + (1 - \alpha)n.$$

The above inequality is a kind of generalization of the inequality (see [1]) applied in the proof of Theorem 3 and follows from it.

Applying the known from [8] condition concerning convexity of harmonic functions we get the theorem.

THEOREM 7. *Let $\alpha \in \langle 0, 1\rangle$. If a harmonic function f of the form (9) satisfies the condition (11), then for any $r \in (0, r_c\rangle$, $r_c = \frac{2-\alpha}{4}$, the set $f(\Delta_r)$ is a convex domain. The constant $\frac{2-\alpha}{4}$ is the best possible.*

PROOF. Fix $\alpha \in \langle 0, 1\rangle$, $r \in (0, r_c\rangle$ and assume that f of the form (9) satisfies the condition (11). By Theorem 4 the function f belongs to $J_H(\alpha)$. Consider the function $f_r(z) = r^{-1} f(rz)$, $z \in \Delta$. We have

$$f_r(z) = z + \sum_{n=2}^{+\infty} a_n r^{n-1} z^n + \overline{\sum_{n=2}^{+\infty} b_n r^{n-1} z^n}, \quad z \in \Delta.$$

Using Lemma 1 (for $l = 2$) and (11) we obtain

$$\sum_{n=2}^{+\infty} n^2 \left(|a_n r^{n-1}| + |b_n r^{n-1}|\right) \leq \sum_{n=2}^{+\infty} (\alpha + (1 - \alpha)n) \left(|a_n| + |b_n|\right) - \leq 1.$$

Hence $f(\Delta_r)$, $r \in (0, r_c\rangle$, is a convex domain (see [8]).

The constant $\frac{2-\alpha}{4}$ cannot be improved because of e. g. the mentioned function $f_* \in J^-(\alpha) \subset J_H^-(\alpha) \subset J_H(\alpha)$. It follows from the known fact that a holomorphic function of tht form $z \mapsto z + az^n$ is convex in Δ if and only if $|a| \leq \frac{1}{n^2}$, $n \in \mathbb{N}$, $n \geq 2$. $\square$

Theorems 5 and 7 give:

COROLARY 5. *The convexity radius for the classes* $J^{-}(\alpha)$, $J^{-}_{H}(\alpha)$, $\alpha \in \langle 0,1\rangle$, *is equal to* $r_c = \frac{2-\alpha}{4}$.

From Proposition 2, the inequality (10) and the known coefficient estimates for functions of the class $\wp$ we obtain

PROPOSITION 3. *Let* $\alpha \in \langle 0,1\rangle$. *If a function* f *of the form (9) belongs to the class* $J_H(\alpha)$, *then*

$$||a_n| - |b_n|| \leq \frac{2}{\alpha + (1-\alpha)n}, \qquad n = 2, 3, \ldots \tag{13}$$

The estimates (13) are sharp.

The sharpness of the estimates (13) we can observe, among others, for the function f_0 of the form (6), where $a_n = \frac{2}{\alpha+(1-\alpha)n}$, $b_n = 0$, $n = 2, 3, \ldots$

C. For a fixed $\alpha \in \langle 0,1\rangle$ denote by $K(\alpha)$ the class of holomorphic functions f of the form (1) such that

$$\operatorname{Re}\left[f'(z) + (1-\alpha)f''(z)\right] > 0, \qquad z \in \Delta. \tag{14}$$

The class $K(\alpha)$, $\alpha \in \langle 0,1\rangle$, was partially examined in the paper [2]. The classes $J(\alpha)$, $K(\alpha)$, $\alpha \in \langle 0,1\rangle$, are closely related by the following theorem.

THEOREM A [2]. *Let* $\alpha \in \langle 0,1\rangle$. *If* $f \in K(\alpha)$, *then the function* φ *of the form* $\varphi(z) = zf'(z)$, $z \in \Delta$, *belongs to the class* $J(\alpha)$. *Conversely, if* $\varphi \in J(\alpha)$, *then the function* f *of the form* $f(z) = \int_0^z \frac{\varphi(\zeta)}{\zeta} d\zeta$, $z \in \Delta$, *belongs to* $K(\alpha)$.

By Theorems 1 and A we get the theorem.

THEOREM 8. *If a holomorphic function* f *of the form (1) satisfies the condition*

$$\sum_{n=2}^{+\infty} \left(\alpha n + (1-\alpha)n^2\right)|a_n| \leq 1 \tag{15}$$

for a fixed $\alpha \in \langle 0,1\rangle$, *then* $f \in K(\alpha)$.

REMARK 4. *(i) We can also consider the Hadamard products of functions* f *of the form (1) satisfying the condition (15) with functions of the class* $\mathcal{F}$.

(ii) The condition (15) is not necessary for a function f to belong to the class $K(\alpha)$, $\alpha \in \langle 0,1\rangle$. It follows from Example 2 and Theorem A.

Next denote by $K^{-}(\alpha)$, $\alpha \in \langle 0,1\rangle$, the class of functions f of the form (7) which belong to $K(\alpha)$. As in the class $J^{-}(\alpha)$, $\alpha \in \in \langle 0,1\rangle$, we obtain the theorem.

THEOREM 9. *A holomorphic function f of the form (7) belongs to the class $K^{-}(\alpha)$, $\alpha \in \langle 0,1\rangle$, if and only if f satisfies the condition (15).*

Let now $K_H(\alpha)$, $\alpha \in \langle 0,1\rangle$, stand for the class of harmonic functions of the form (9) ($f = h+\overline{g}$) such that the holomorphic functions $F = h+g$ belong to $K(\alpha)$. Moreover, $K_H^{-}(\alpha)$ denotes the subclass of $K_H(\alpha)$, $\alpha \in \langle 0,1\rangle$, of functions f of the form (12).

According to the above definition a function f of the form (9) belongs to the class $K_H(\alpha)$, $\alpha \in \langle 0,1\rangle$, if and only if

$$\operatorname{Re}\left[h'(z)+g'(z)+(1-\alpha)\left(h''(z)+g''(z)\right)\right] > 0, \qquad z \in \Delta. \tag{16}$$

We see at ones that $K(\alpha) \subset K_H(\alpha)$, $\alpha \in \langle 0,1\rangle$.

From the definitions of the considered classes and by Theorem A we obtain the theorem.

THEOREM 10. *Let $\alpha \in \langle 0,1\rangle$. If $f \in K_H(\alpha)$, then the function φ of the form $\varphi(z) = zh'(z)+\overline{zg'(z)}$, $z \in \Delta$, belongs to the class $J_H(\alpha)$. Conversely, if a function $\varphi = H+\overline{G}$ is in the class $J_H(\alpha)$, then the function f of the form $f(z) = \int_0^z \frac{H(\zeta)}{\zeta}d\zeta + \overline{\int_0^z \frac{G(\zeta)}{\zeta}d\zeta}$, $z \in \Delta$, belongs to the class $K_H(\alpha)$.*

Theorems 8, 9, 10 and the condition (16) give the following two theorems.

THEOREM 11. *If a harmonic function f of the form (9) satisfies the condition*

$$\sum_{n=2}^{+\infty}\left(\alpha n+(1-\alpha)n^2\right)\left(|a_n|+|b_n|\right) \le 1 \tag{17}$$

for a fixed $\alpha \in \langle 0,1\rangle$, then $f \in K_H(\alpha)$.

THEOREM 12. *A harmonic function f of the form (12) belongs to the class $K_H^{-}(\alpha)$, $\alpha \in \langle 0,1\rangle$, if and only if f satisfies the condition (17).*

REMARK 5. *The class of harmonic functions of the form (9) and satisfying the condition (17) for a fixed $\alpha \in \langle 0,1\rangle$, was considered in the paper*

[1]. *The class* $K_H(\alpha)$, $\alpha \in \langle 0, 1\rangle$, *is a certain generalization of this class. By Remark 4(ii) we observe that these classes are not equal. It seems that the above presented theorems form a natural supplement to the results contained in the article* [1].

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