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# ON METRIC SPACE VALUED FUNCTIONS OF BOUNDED ESSENTIAL VARIATION

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Let $\emptyset \neq T \subset \mathbb{R}$ and let $X$ be a metric space. For an ideal $\mathcal{J} \subset \mathcal{P}(T)$ and a function $f: T \to X$, we define the essential variation $V_{ess}^{\mathcal{J}}(f,T)$ as the infimum of all variations $V(g,T)$ where $g: T \to X$, $g = f$ on $T \setminus E$, and $E \in \mathcal{J}$. We show that if $X$ is complete then the essential variation of $f$ is equal to $\inf\{V(f, T \setminus E) : E \in \mathcal{J}\}$. This extends former theorems of that type. We list some consequences that are analogues to the recent results by Chistyakov. Some examples of different kinds of essential variation are also investigated.

## Introduction

Let $T$ be a nonempty subset of the real line. Let $\mathcal{J}$ be an ideal of subsets of $T$; thus $\mathcal{J}$ is a nonempty hereditary and additive subfamily of $\mathcal{P}(T)$ (the power set of $T$) with $T \notin \mathcal{J}$. An ideal is usually interpreted as a family of small sets that are negligible in the respective sense. Sometimes, one assumes additionally that $\mathcal{J}$ contains all singletons $\{t\}$, $t \in T$, or/and it does not contain nonempty open sets in $T$. If an ideal is $\sigma$-additive, it is called a $\sigma$-ideal. The following families form well-known ideals on the real line: finite sets, nowhere dense sets, countable sets, Lebesgue null sets, sets of the first Baire category. The last three examples are $\sigma$-ideals. (See [1].) These examples may produce ideals on $T \subset \mathbb{R}$ by taking intersections of their members with $T$, provided that $T$ is not too small (to ensure that $T \notin \mathcal{J}$).

Let $\mathbb{N} = \{1, 2, ...\}$. Fix a metric space $X$ with a metric $d$, and let $\emptyset \neq T \subset \mathbb{R}$. If $x \in \mathbb{R}$ is a right (left) limit point of $T$ then by $f(x+)$, $f(x-)$ we denote the one-sided limits of a function $f : T \to X$ at $x$, provided that they exist. By $\mathcal{L}ip(T,X)$ we denote the set of all Lipschitz functions from $T$ into $X$, and by $L_d(f,T)$ we denote the smallest Lipschitz constant for $f$. A finite sequence $\mathcal{T}=\{t_i\}_{i=0}^n$ such that $t_0<\ldots<t_n$ and $t_i\in T$ for $i=0,\ldots,n$ is called a partition of $T$. The Jordan variation of $f:T\to X$ is defined by

$$V(f,T)=\sup_{\mathcal{T}}\Big\{\sum_{i=1}^{n} d(f(t_i),f(t_{i-1}))\colon n\in\mathbb{N}\Big\}.$$

(See [2], [3], [4].) If $V(f,T)<\infty$, we say that $f$ is of bounded variation and we write $f\in BV(T,X)$. Consider an ideal $\mathcal{J}\subset\mathcal{P}(T)$. The essential variation of $f:T\to X$ with respect to $\mathcal{J}$ is defined as the following quantity:

$$V^{\mathcal{J}}_{ess}(f,T)=\inf\{V(g,T):\ \text{there are } E\in\mathcal{J} \text{ and } g\colon T\to X \text{ such that } g=f \text{ on } T\setminus E\}.$$

If $V^{\mathcal{J}}_{ess}(f,T)<\infty$, we say that $f$ is a function of bounded essential variation with respect to $\mathcal{J}$, and we write $f\in BV^{\mathcal{J}}_{ess}(T,X)$.

Essential variation was considered before in [5], [6], [7], [8], [2] but only for the ideal of Lebesgue null sets on the real line. In the present paper we generalize a characterization of essential variation given in [7] and [2]. Namely, we show that $V^{\mathcal{J}}_{ess}(f,T)$ equals $\inf\{V(f,T\setminus E)\colon E\in\mathcal{J}\}$ (Theorem 2). We obtain several properties of functions from $BV^{\mathcal{J}}_{ess}$ analogous to those proved in [2] (Theorem 3). We show how one can extend a function from $BV(T\setminus E,X)$ to a function defined on the whole $T$ with the same variation (Theorem 1). A related result was given in [9] with another proof. Our paper contains general facts and examples witnessing a significant dependence of $V^{\mathcal{J}}_{ess}(f,T)$ on $\mathcal{J}$.

## Characterization of essential variation

We are going to prove that a metric space valued function of bounded variation on a subset of a fixed nonempty set $T\subset\mathbb{R}$ can be extended to a function on $T$ with the same variation. A related result was obtained earlier by Chistyakov and Rychlewicz [9, Theorem 1(a)]. In the proof given in [9], the authors apply a structural theorem for functions of bounded variation [4, Theorem 3.1], [2, Lemma 2.1]. Our proof is different and seems more elementary.

We start with the following lemma.

LEMMA 1. *Let $\emptyset \neq D \subset \mathbb{R}$, $X$ be a complete metric space, $f \in BV(D, X)$ and assume that $t$ is a right-sided (left-sided) limit point of a set $D$. Then there exists a right-sided (left-sided) limit of $f$ at $t$.*

A proof is quite similar to that in [3, Theorem I.24(c)].

THEOREM 1. *Let $\emptyset \neq T \subset \mathbb{R}$, $E \subset T$, $\emptyset \neq E \neq T$. Assume that $f\colon T \setminus E \to X$ and $f \in BV(T \setminus E, X)$ where $X$ is a complete metric space. Then there is a function $h\colon T \to X$ such that $V(h, T) = V(f, T \setminus E)$ and $h|_{T\setminus E} = f$.*

PROOF. Define $h\colon T \to X$ as follows. Let $t \in T$. If $t \notin E$ we put $h(t) = f(t)$. Now let $t \in E$. In the case $t \in E$ and $(T \setminus E) \cap (-\infty, t) \neq \emptyset$ we denote $t^* = \sup((T \setminus E) \cap (-\infty, t))$. If this supremum is attained, we put $h(t) = f(t^*)$, and otherwise let $h(t) = f(t^*-)$ (this limit exists by Lemma 1). In the case $(T \setminus E) \cap (-\infty, t) = \emptyset$, from $E \neq T$ it follows that $(T \setminus E) \cap (t, \infty) \neq \emptyset$. Thus we denote $t_* = \inf((T \setminus E) \cap (t, \infty))$. If this infimum is attained, we put $h(t) = f(t_*)$, and otherwise let $h(t) = f(t_*+)$ (this limit exists by Lemma 1).

Since $h|_{T\setminus E} = f$, we have $V(f, T \setminus E) \leq V(h, T)$. Now, we will show the converse. For a partition $\mathcal{T} = \{t_i\}_{i=0}^m$ of $T$ we denote $S(h, \mathcal{T}) = \sum_{t=1}^m d(h(t_i), h(t_{i-1}))$. We will prove that for each partition $\mathcal{T}$ of $T$ there is a partition $\mathcal{T}^*$ of $T$ such that

$$S(h, \mathcal{T}) \leq S(h, \mathcal{T}^*) \leq V(f, T \setminus E)$$

which yields the assertion.

So, let $\mathcal{T} = \{t_i\}_{i=0}^m$ be a given partition of $T$. We modify $\mathcal{T}$ in four steps:

1) If $t_0 \in E$ and $(T \setminus E) \cap (-\infty, t_0) \neq \emptyset$, we insert a point from $(T \setminus E) \cap (-\infty, t_0)$ to the partition.
2) If $t_m \in E$ and $(T \setminus E) \cap (t_m, \infty) \neq \emptyset$, we insert a point from $(T \setminus E) \cap (t_m, \infty)$ to the partition.
3) For every pair $t_i, t_{i+1} \in E$, if $(t_i, t_{i+1}) \cap (T \setminus E) \neq \emptyset$, we insert a point from $(t_i, t_{i+1}) \cap (T \setminus E)$ to the partition.
4) We look for all maximal strings $t_i, t_{i+1}, ..., t_{i+k}$ with $[t_i, t_{i+k}] \cap T \subset E$. (Since $E \neq T$, by our modifications 1), 2), 3) it is impossible that the whole partition is such a string.) Thus $h$ is constant on $[t_i, t_{i+k}] \cap T$. We delete points $t_{i+1}, ..., t_{i+k}$ from the partition. This does not change $S(h, \mathcal{T})$.

Denote the modified partition by $\mathcal{T}^*$. The operation described in 1), 2), 3) can only enlarge $S(h,\mathcal{T})$, so we have $S(h,\mathcal{T}) \le S(h,\mathcal{T}^*)$. In partition $\mathcal{T}^*$, at least one of any two consecutive points belongs to $T \setminus E$. For simplicity assume that $\mathcal{T}^* = \{t_i\}_{i=0}^m$. By 1), 2) we have ensured that $t_0, t_m \in T \setminus E$, if it is possible.

Now fix $i \in \{0, ..., m\}$. Consider three cases:

**Case 1.** $2 \le i \le m$ and $t_{i-2} \in T \setminus E$, $t_{i-1} \in E$, $t_i \in T \setminus E$. If $(T \setminus E) \cap (-\infty, t_{i-1})$ has a maximal element $t_{i-1}^*$, then $t_{i-2} \le t_{i-1}^* < t_{i-1}$ and $d(h(t_{i-2}), h(t_{i-1})) + d(h(t_{i-1}), h(t_i)) = d(f(t_{i-2}), f(t_{i-1}^*)) + d(f(t_{i-1}^*), f(t_i))$.

If $(T \setminus E) \cap (-\infty, t_{i-1})$ has no maximal element, we set $t_{i-1}^* = \sup(T \setminus E) \cap (-\infty, t_{i-1})$. Of course $t_{i-2} < t_{i-1}^* \le t_{i-1}$. Pick a sequence $\{t_{i-1,n}^*\}_{n=1}^\infty$ of numbers from $(T \setminus E) \cap (-\infty, t_{i-1})$ such that $\lim_{n\to\infty} t_{i-1,n}^* = t_{i-1}^*$. We have

$$\begin{aligned} &d(h(t_{i-2}), h(t_{i-1})) + d(h(t_{i-1}), h(t_i)) = \\ &= d(f(t_{i-2}), h(t_{i-1})) + d(h(t_{i-1}), f(t_i)) = \\ &= \lim_{n\to\infty} [d(f(t_{i-2}), f(t_{i-1,n}^*)) + d(f(t_{i-1,n}^*), f(t_i))] \le \\ &\le V(f, [t_{i-2}, t_i] \cap (T \setminus E)). \end{aligned}$$

The last equality results from Lemma 1.

**Case 2.** Let $t_0 \in E$, $t_1 \in T \setminus E$. If $(T \setminus E) \cap (t_0, \infty)$ has a minimal element $t_{0*}$, then $t_0 < t_{0*} \le t_1$ and $d(h(t_0), h(t_1)) = d(f(t_{0*}), f(t_1))$. If $(T \setminus E) \cap (t_0, \infty)$ has no minimal element, we set $t_{0*} = \inf(T \setminus E) \cap (t_0, \infty)$ and of course $t_0 \le t_{0*} < t_1$. Pick a sequence $\{t_{0*,n}\}_{n=1}^\infty$ of numbers from $(T \setminus E) \cap (t_0, \infty)$ such that $\lim_{n\to\infty} t_{0*,n} = t_{0*}$. We have

$$\begin{aligned} &d(h(t_0), h(t_1)) = d(h(t_0), f(t_1)) = \\ &= \lim_{n\to\infty} d(f(t_{0*,n}), f(t_1)) \le \\ &\le V(f, [t_0, t_1] \cap (T \setminus E)). \end{aligned}$$

The last equality results from Lemma 1.

**Case 3.** Let $t_{m-1} \in T \setminus E$, $t_m \in E$. If $(T \setminus E) \cap (-\infty, t_m)$ has a maximal element $t_m^*$, then $t_{m-1} \le t_m^* < t_m$ and $d(h(t_{m-1}), h(t_m)) = d(f(t_{m-1}), f(t_m^*))$. If $(T \setminus E) \cap (-\infty, t_m)$ has no maximal element, we set $t_m^* = \sup(T \setminus E) \cap (-\infty, t_m)$. Of course $t_{m-1} < t_m^* \le t_m$. Pick a sequence $\{t_{m,n}^*\}_{n=1}^\infty$ of numbers from $(T \setminus E) \cap (-\infty, t_m)$ with $\lim_{n\to\infty} t_{m,n}^* = t_m^*$. We have

$$d(h(t_{m-1}), h(t_m)) = d(f(t_{m-1}), h(t_m)) =$$

$$= \lim_{n\to\infty} d(f(t_{m-1}), f(t^*_{m,n})) \leq$$
$$\leq V(f, [t_{m-1}, t_m] \cap (T \setminus E)).$$

The last equality results from Lemma 1.

Taking into account cases 1), 2), 3), by adding the respective sides of inequalities we obtain

$$S(h, \mathcal{T}^*) \leq V(f, [t_0, t_m] \cap (T \setminus E)) \leq V(f, T \setminus E).$$

The theorem has been proved. □

Now we are ready to prove our main result, a characterization of essential variation. In the case when $T = [a, b]$, $X = \mathbb{R}$, and $\mathcal{J}$ is the family of all Lebesgue null sets in $T$, this theorem was proved by Banaś and El-Sayed in [7]. If $T$ is a density-open subset of $\mathbb{R}$ and $\mathcal{J}$ is a family of all Lebesgue null sets in $T$, and $X$ is a complete metric space and $f \in BV^{\mathcal{J}}_{ess}(T, X)$, the result was obtained by Chistyakov in [2]. In our paper we consider a more general situation where essential variation is associated with an arbitrary ideal of subsets of a given nonempty set $T$ on the real line.

THEOREM 2. *Let $X$ be a complete metric space and $\emptyset \neq T \subset \mathbb{R}$ and let $\mathcal{J}$ be a proper ideal of subsets of $T$. Then for every function $f: T \to X$ we have*

$$V^{\mathcal{J}}_{ess}(f, T) = \inf\{V(f, T \setminus E) : E \in \mathcal{J}\}.$$

PROOF. (Compare with [2, Theorem 2.1].). Denote $v = \inf\{V(f, T \setminus E) : E \in \mathcal{J}\}$. If $v = \infty$, then $V(f, T \setminus E) = \infty$ for all $E \in \mathcal{J}$. Hence for each function $g: T \to X$ such that $f|_{T\setminus E} = g|_{T\setminus E}$ with $E \in \mathcal{J}$, we have

$$V(g, T) \geq V(g, T \setminus E) = V(f, T \setminus E) = \infty.$$

Thus $V^{\mathcal{J}}_{ess}(f, T) = \infty$. Now assume that $v < \infty$. For a fixed $\varepsilon > 0$ there exists a set $E_0 \in \mathcal{J}$ such that $V(f, T \setminus E_0) < v + \varepsilon$. By Theorem 1 we find a function $g_0 \in BV(T, X)$ such that $f|_{T\setminus E_0} = g_0|_{T\setminus E_0}$ and $V(g_0, T) = V(f, T \setminus E_0)$. Then

$$V^{\mathcal{J}}_{ess}(f, T) \leq V(g_0, T) = V(f, T \setminus E_0) < v + \varepsilon.$$

Consequently, $V^{\mathcal{J}}_{ess}(f, T) \leq v$. It suffices to show the reverse inequality. By definition of $V^{\mathcal{J}}_{ess}(f, T)$, for any number $\alpha > V^{\mathcal{J}}_{ess}(f, T)$ we find a function

$g_1\colon T \to X$ and a set $E_1 \in \mathcal{J}$ such that $V(g_1, T) \leq \alpha$ and $g_1|_{T\setminus E_1} = f|_{T\setminus E_1}$. We have

$$V(f, T \setminus E_1) = V(g_1, T \setminus E_1) \leq V(g_1, T) \leq \alpha.$$

It follows that $v \leq V(f, T \setminus E_1) \leq \alpha$ and since $\alpha > V^{\mathcal{J}}_{ess}(f, T)$ is arbitrary, we obtain $v \leq V^{\mathcal{J}}_{ess}(f, T)$. □

## Some examples and properties

Let us start with a simple observation when one considers two ideals on $T$.

PROPOSITION 1. *Let $\emptyset \neq T \subset \mathbb{R}$ and let $X$ be a metric space. Assume that $\mathcal{I}, \mathcal{J} \subset \mathcal{P}(T)$ are ideals and $f\colon T \to X$. Then we have:*

(a) *if $\mathcal{I} \subset \mathcal{J}$ then $V^{\mathcal{I}}_{ess}(f, T) \geq V^{\mathcal{J}}_{ess}(f, T)$,*

(b) *$V^{\mathcal{I}\cap\mathcal{J}}_{ess}(f, T) \geq \max\{V^{\mathcal{I}}_{ess}(f, T), V^{\mathcal{J}}_{ess}(f, T)\}$.*

PROOF. Assertion (a) is an immediate corollary from the definition of essential variation. Assertion (b) follows from (a). □

Now, we will to show that the essential variation in one sense can be small and in another sense – can be large.

PROPOSITION 2. *Let $\emptyset \neq T \subset \mathbb{R}$ and let $\mathcal{I}, \mathcal{J} \subset \mathcal{P}(T)$ be two ideals such that there exist a set $A \in \mathcal{I} \setminus \mathcal{J}$ and a strictly monotonic sequence $(x_n)_{n=0}^{\infty}$ of numbers from $T$ such that $A_n \notin \mathcal{J}$ for all $n \in \mathbb{N}$ where $A_n = A \cap [\min\{x_{n-1}, x_n\}, \max\{x_{n-1}, x_n\})$, $n \in \mathbb{N}$. Then for every complete metric space $X$ of cardinality $\geq 2$ there exists a function $f\colon T \to X$ with $V^{\mathcal{I}}_{ess}(f, T) = 0$ and $V^{\mathcal{J}}_{ess}(f, T) = \infty$.*

PROOF. Let $A$ be as in the assumption. Pick distinct points $x, y \in X$. Define $f\colon T \to X$ by putting $f(t) = x$ if $t \in \bigcup_{n=1}^{\infty} A_{2n}$ and $f(t) = y$ for $t \in T \setminus \bigcup_{n=1}^{\infty} A_{2n}$. Since $A \in \mathcal{I}$ and $f|_{T\setminus A}$ is constant, by Theorem 2 we obtain $V^{\mathcal{I}}_{ess}(T, X) = 0$. To show that $V^{\mathcal{J}}_{ess}(T, X) = \infty$, fix $E \in \mathcal{J}$. We have $A_n \setminus E \notin \mathcal{J}$ for each $n \in \mathbb{N}$, thus we can pick $t_{n-1} \in A_n \setminus E$, $n \in \mathbb{N}$. The sequence $(t_n)_{n=0}^{\infty}$ is strictly monotonic. Its beginning part $(t_k)_{k=0}^{n}$, $n \in \mathbb{N}$, can be treated as a partition of $T \setminus E$ (if it is decreasing, we reverse the numeration). Consequently,

$$V(f, T \setminus E) \geq \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} d(f(t_k), f(t_{k-1})) = \sup_{n\in\mathbb{N}} nd(x, y) = \infty.$$

Hence by Theorem 2 we obtain $V_{ess}^{\mathcal{J}}(f,T)=\infty$. $\square$

We say that two ideals $\mathcal{I},\mathcal{J}\subset\mathcal{P}(T)$ are orthogonal if there are disjoint sets $A,B\subset T$ such that $A\in\mathcal{I}$, $B\in\mathcal{J}$ and $A\cup B=T$. The $\sigma$-ideals of Lebesgue null sets and of sets of the first category in $\mathbb{R}$ make a well-known example of a pair of orthogonal ideals for $T=\mathbb{R}$ (See [1]). For other pairs of orthogonal ideals, see for instance [10].

COROLLARY. *Let $T$ be an infinite subset of $\mathbb{R}$ and let $\mathcal{I},\mathcal{J}\subset\mathcal{P}(T)$ be two orthogonal ideals consisting of sets with empty interior in $T$. Then for every complete metric space $X$ with cardinality $\geq 2$ there exists a function $f\colon T\to X$ with $V_{ess}^{\mathcal{I}}(f,T)=0$ and $V_{ess}^{\mathcal{J}}=\infty$.*

PROOF. Since $T$ is infinite, we can find a strictly monotonic sequence $(y_n)_{n=0}^{\infty}$ of numbers from $T$. Put $x_n=y_{2n}$ for $n\in\mathbb{N}\cup\{0\}$. Let $A,B\subset T$ be such that $A\in\mathcal{I}$, $B\in\mathcal{J}$, $A\cap B=\emptyset$, $A\cup B=T$. It suffices to show that the sets $A_n$, $n\in\mathbb{N}$, defined as in Proposition 2, are not in $\mathcal{J}$. Assume for instance that $(y_n)_{n=0}^{\infty}$ is increasing. Suppose that $A_k\in\mathcal{J}$ for some $k\in\mathbb{N}$. Observe that $U=(x_{k-1},x_k)\cap T$ is open in $T$ and $U\neq\emptyset$ since $y_{2k-1}\in U$. Because $A_k\in\mathcal{J}$, we have $A\cap U=A_k\cap U\in\mathcal{J}$. From $B\in\mathcal{J}$ it follows that $B\cap U\in\mathcal{J}$. Consequently $U=(U\cap A)\cup(U\cap B)\in\mathcal{J}$ which yields a contradiction. $\square$

**Example.** Let $\mathcal{I}$ be the ideal of all subsets of $[0,1]$ of Lebesgue measure zero and let $\mathcal{J}$ be the ideal of all subsets of $[0,1]$ of the first category. It is well known that $\mathcal{I}$ and $\mathcal{J}$ are two orthogonal $\sigma$-ideals, so pick two disjoint sets $A$ and $B$ such that $A\in\mathcal{I}$ and $B\in\mathcal{J}$ and $[0,1]=A\cup B$. Since $A$ is residual, its intersection with every nondegenerate interval is of the second category. Let $A_n=[1-\frac{1}{n},1-\frac{1}{n+1})\cap A$ for $n\in\mathbb{N}$. Define $f\colon[0,1]\to\mathbb{R}$ by putting $f(t)=0$ if $t\in B\cup\{1\}$ and $f(t)=\frac{1}{2^n}$ if $t\in A_n$. Since $A\in\mathcal{I}$ and $f|_{[0,1]\setminus A}$ is constant, by Theorem 2 we obtain $V_{ess}^{\mathcal{I}}(f,[0,1])=0$. Since $A_n$ is of the second category, for each $E\in\mathcal{J}$ and for every $n\in\mathbb{N}$ we have $A_n\setminus E\neq\emptyset$. Hence $V_{ess}^{\mathcal{J}}(f,[0,1])=\sum_{n=1}^{\infty}(\frac{1}{2^n}-\frac{1}{2^{n+1}})=\frac{1}{2}$. For each $E\in\mathcal{I}\cap\mathcal{J}$ and for every interval $[a,b]\subset[0,1]$ there exist $t,s\in[a,b]$ such that $t\in B\setminus E$ and $s\in A\setminus E$. Hence $V_{ess}^{\mathcal{I}\cap\mathcal{J}}(f,[0,1])=\infty$. This shows that $V_{ess}^{\mathcal{I}\cap\mathcal{J}}(f,[0,1])>\max\{V_{ess}^{\mathcal{I}}(f,[0,1]),V_{ess}^{\mathcal{J}}(f,[0,1])\}$. Hence in the assertion (b) of Proposition 1, we cannot use equality.

Let $X$ be a metric space. We say that $A\subset X$ is precompact in $X$ if the closure $\overline{A}$ is compact. The following theorem collects some consequences of Theorem 2 analogous to those presented in [2].

THEOREM 3. *Let $\emptyset \neq T \subset \mathbb{R}$, $X$ be a complete metric space and $f : T \to X$. Let $\mathcal{J}$ be a $\sigma$-ideal of subsets of $T$. Then*

(a) *$f \in BV^{\mathcal{J}}_{ess}(T,X)$ if and only if there exists a set $E \in \mathcal{J}$ such that $f|_{T\setminus E} \in BV(T\setminus E, X)$; moreover, $E$ can be chosen such that $V(f, T\setminus E) = V^{\mathcal{J}}_{ess}(f,T)$.*

(b) *If $\{f_n\}_{n=1}^{\infty} \subset BV^{\mathcal{J}}_{ess}(T,X)$ and $d(f_n(t), f(t)) \to 0$ as $n \to \infty$ for $t \in T \setminus E$, where $E \in \mathcal{J}$, then $V^{\mathcal{J}}_{ess}(f,T) \leq \liminf_{n\to\infty} V^{\mathcal{J}}_{ess}(f_n,T)$.*

(c) *(Structural Theorem) $f \in BV^{\mathcal{J}}_{ess}(T,X)$ if and only if there exists a nondecreasing bounded function $\varphi : T \to \mathbb{R}$ and a function $g \in \mathcal{L}ip(D,X)$, where $D = \varphi(T)$ and $L_d(g,D) \leq 1$, such that $f = g \circ \varphi$ on $T \setminus E$, where $E \in \mathcal{J}$.*

(d) *(Helly's type Theorem) If $\mathcal{F} = \{f_n\}_{n=1}^{\infty} \subset BV^{\mathcal{J}}_{ess}(T,X)$,*

$$\sup_{n\in\mathbb{N}} V^{\mathcal{J}}_{ess}(f_n,T)$$

*is finite and the set $\{f_n(t)\}_{n=1}^{\infty}$ is precompact in $X$ for $t \in T \setminus E$, where $E \in \mathcal{J}$, then $\mathcal{F}$ contains a subsequence which converges in metric $d$ on $T \setminus E$ to a function from $BV^{\mathcal{J}}_{ess}(T,X)$.*

With application of Theorem 2, the proof of Theorem 3 goes similarly as for [2, Theorem 2.2]. Thus we omit it.

REMARK. *The assumption that $\mathcal{J}$ is a $\sigma$-ideal (closed under countable unions) is essential in Theorem 3(a). Indeed, put $E = \{\frac{1}{2^n}: n \in \mathbb{N}\}$ and define $f : [0,1] \to \mathbb{R}$ by putting $f(t) = t$ if $t \in E$ and $f(t) = 0$ if $t \in [0,1] \setminus E$. We have $V(f,[0,1]) = 2$. Consider as $\mathcal{J}$ the ideal of all finite subsets of $[0,1]$. We put $E_n = \{\frac{1}{2^m}: m \in \mathbb{N},\quad m \leq n\}$, $n \in \mathbb{N}$. Thus $E_n \in \mathcal{J}$. For all $n \in \mathbb{N}$ we define $g_n: [0,1] \to \mathbb{R}$ by putting $g_n(t) = t$ if $t \in E \setminus E_n$ and $g_n(t) = 0$ for the remaining $t$ in $[0,1]$. Hence $g_n(t) = f(t)$ for all $n \in \mathbb{N}$ and $t \in [0,1] \setminus E_n$, and $V(g_n,[0,1]) = 2 \cdot \frac{1}{2^n} = \frac{1}{2^{n-1}}$ for all $n \in \mathbb{N}$. Consequently, $\lim\limits_{n\to\infty} V(g_n,[0,1]) = 0$ and $V^{\mathcal{J}}_{ess}(f,[0,1]) = 0$. However, for all $D \in \mathcal{J}$ we have $V(f,[0,1] \setminus D) > 0$.*

## Refrences

[1] Oxtoby J. C. *Measure and Category* / J. C. Oxtoby. N. Y.: Springer, 1971.

[2] Chistyakov V. V. *Selections of bounded variation* /V. V. Chistyakov // J. Appl. Anal. 10. 1 (2004). P. 1–82.

[3] Chistyakov V. V. *Variation* / V. V. Chistyakov. Nizhnii Novgorod: Nizhnii Novgorod State University, 1992 (Russian).

[4] Chistyakov V. V. *On the theory of set-valued maps of bounded variation of one real variable* / V. V. Chistyakov // Sbornik: Mathematics 189:5 (1998). P. 797–819.

[5] Orlicz W. *On functions of finite variation, depending on a parameter* / W. Orlicz // Studia Math. 13 (1953). P. 218–232.

[6] Das A. G. *On functions of bounded essential variation* / A. G. Das, B. K. Lahiri // Comment. Math. 26 (1986). P. 219–227.

[7] Banaś J. *Functions of generalized bounded variation* / J. Banaś, W. G. El-Sayed // Zeszyty Naukowe Politechniki Rzeszowskiej Mat. Fiz. 85 (1991). P. 91–109.

[8] Goffman C. *Homeomorphisms in Analysis* / C. Gofman, T. Nishiura, D. Waterman // Math. Surveys and Monographs 54. American Mathematical Society, 1997.

[9] Chistyakov V. V. *On the extension and generation of set-valued mappings of bounded variation* / V.V. Chistyakov, A. Rychlewicz // Studia Math. 153. 3 (2002). P. 235–247.

[10] Balcerzak M. *Orthogonal $\sigma$-ideals and almost disjoint families* / M. Balcerzak // Acta Universitatis Lodziensis. Folia Mathematica 5 (1992). P. 3–8.

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