Trudy Petrozavodskogo gosudarstvennogo universiteta

Seria "Matematika" Vypusk 14, 2008

UDK 517

ON PROBLEMS OF UNIVALENCE FOR THE CLASS TR(1/2)

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In this paper we discuss the class $TR(\frac{1}{2})$ consisted of typically real functions given by the integral formula

$$f(z) = \int_{-1}^{1} \frac{z}{\sqrt{1-2zt+z^2}}\, d\mu(t),$$

where μ is the probability measure on $[-1,1]$. The problems of local univalence, univalence, convexity in the direction of real and imaginary axes are examined. This paper is the continuation of research on $TR(\frac{1}{2})$, especially concerning problems, which results were published in [5].

Let $\mathcal{A}$ denote the set of all functions which are analytic in the unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Let TR denote the well known class which consists of typically real functions. Recall that the function $f \in \mathcal{A}$ belongs to TR if and only if the condition

$$\Im z \cdot \Im f(z) \geq 0 \qquad z \in \Delta.$$

is satisfied.

Rogosinski [4] proved that $f \in TR \iff f(z) = \int_{-1}^{1} k_t(z)\, d\mu(t)$, where $k_t(z) = \frac{z}{1-2zt+z^2}$, and μ belongs to $P_{[-1,1]}$, i.e. the collection of all probability measures on $[-1,1]$. Similarly Szynal [6] defined the class $TR(\frac{1}{2}) = \left\{ f \in \mathcal{A} : \quad f(z) = \int_{-1}^{1} f_t(z)\, d\mu(t), \quad \mu \in P_{[-1,1]} \right\}$, where

$$f_t(z) = z\left(\frac{k_t(z)}{z}\right)^{\frac{1}{2}} = \frac{z}{\sqrt{1-2tz+z^2}}. \tag{1}$$

In this paper Szynal considered the coefficients problems. He proved that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $TR(\frac{1}{2})$ then $|a_n| \leq 1$. This fact means that the coefficients of the function $f \in TR(\frac{1}{2})$ are bounded by the same number as the coefficients of functions in the classes CV, $CV(i)$, $ST(\frac{1}{2})$ consisting of convex functions, convex in the direction of the imaginary axis functions, and starlike of order $\frac{1}{2}$ functions, respectively. Moreover, he proved that the functions of the class $TR(\frac{1}{2})$ are typically real, so $TR(\frac{1}{2}) \subset TR$.

We shall point out the property, which in essential manner differs the class $TR(\frac{1}{2})$ from the class TR. However the functions k_t of the class TR are starlike, the functions of the form

$$\alpha k_1 + (1-\alpha)k_{-1}\ ,\ \alpha \in (0,1)$$

are not univalent. These functions are extremal in many univalence problems. One of the most important functions is the function

$$z \mapsto [k_1(z) + k_{-1}(z)]/2 = \frac{z(1+z^2)}{(1-z^2)^2}\ ,$$

which is used, for example, to determining the domain of univalence or the domain of local univalence for TR.

By analogy to the class TR, the kernel functions f_t of the class $TR(\frac{1}{2})$ are starlike of order $1/2$. On the other hand, it is easy to check, that the functions given by the formula

$$\alpha f_1 + (1-\alpha)f_{-1}\ ,\ \alpha \in [0,1]$$

are univalent and convex in the direction of the imaginary axis. Hence, these functions are not extremal in the problems concerning univalence.

The classes CVR, $CVR(i)$, $STR(\frac{1}{2})$ (where AR denotes the subclass of a class A consisting of functions having real coefficients) and the class $TR(\frac{1}{2})$ are connected by the following inclusions, namely

$$CVR \subset STR\left(\frac{1}{2}\right) \subset TR\left(\frac{1}{2}\right) \tag{2}$$

and

$$CVR \subset CVR(i) \subset TR(\frac{1}{2}). \tag{3}$$

The relations (2) result from the equality $\overline{co}\, STR\left(\frac{1}{2}\right) = TR\left(\frac{1}{2}\right)$ given by Hallenbeck [1], where $\overline{co}\, A$ denotes the closed convex hull of A, and the well known theorem of Marx and Strohhäcker .

The fact $\overline{co}\, CVR = CVR(i)$ (compare [5]), the relation (2) and convexity of the class $TR\left(\frac{1}{2}\right)$ (see [6]) give us (3).

Now, we are going to prove that the class $TR(\frac{1}{2})$ is the essential superclass of CVR, $CVR(i)$ and $STR(\frac{1}{2})$. In order to do this we shall find functions belonging to $TR(\frac{1}{2})$ which are not univalent.

Let us consider the functions

$$F_t(z) = \left[f_t(z) + f_{-t}(z)\right]/2, \quad t \in [0,1] \ .$$

Theorem 1. *For all $t \in (0,1)$ there exist $r_t \in (0,1)$ such that functions F_t are not locally univalent in Δ_r, $r \geq r_t$.*

Proof. Let $t \in (0,1)$.
We have $F_t'(z) = \frac{1}{2}\left[\frac{1-tz}{(1-2tz+z^2)^{\frac{3}{2}}} + \frac{1+tz}{(1+2tz+z^2)^{\frac{3}{2}}}\right]$. Hence, the equality $F_t'(ir) = 0$ is equivalent to

$$\Re(1-itr)(1-r^2+2tir)^{\frac{3}{2}} = 0. \tag{4}$$

Using

$$\begin{gathered}\sqrt{1-r^2+2tir} = \sqrt{\frac{1}{2}\left(1-r^2+\sqrt{(1-r^2)^2+4t^2r^2}\right)}+ \\ +i\sqrt{\frac{1}{2}\left(-1+r^2+\sqrt{(1-r^2)^2+4t^2r^2}\right)}\end{gathered} \tag{5}$$

the condition (4) could be written as

$$\begin{gathered}\left[1-r^2-tr(1-2tr+r^2)\right]\left[1-r^2+tr(1+2tr+r^2)\right]+ \\ (1-r^2)(1+t^2r^2)\sqrt{(1-r^2)^2+4t^2r^2} = 0.\end{gathered}$$

Let us denote the left hand side of (5) by $G(t,r)$. The function G is continuous with respect to both variables. Moreover, $G(t,0) = 2$ and $G(t,1) = -4t^2(1-t^2) < 0$ for $t \in (0,1)$. We conclude that there exist $r_t \in (0,1)$ such that $G(t,r_t) = 0$.

Now, we determine the smallest number r_t, which was described above. Solving the system of equations

$$\begin{cases} G(t,r) = 0 \\ \frac{\partial G}{\partial t}(t,r) = 0 \end{cases}$$

we obtain

$$\begin{cases} G(t,r) = 0 \\ t^2 = \frac{-5+6r^2+3r^4}{8r^2} \end{cases} .$$

Hence $(1+r^2)^3(7-9r^2) = 0$ and consequently $r = \frac{\sqrt{7}}{3} = 0,88....$ We have proved that

Corolary. *The radius of locally univalence r_{LU} of $TR(\frac{1}{2})$ satisfies the condition $r_{LU} \leq \frac{\sqrt{7}}{3}$.*

This means that there are the functions of the class $TR(\frac{1}{2})$ which are not univalent in each disk Δ_r, $r > \frac{\sqrt{7}}{3}$.

In the proof of the following theorems we will apply the Krein-Milman Theorem. This theorem concerns the extremalization of linear and continuous functionals in a given $A \subset \mathcal{A}$. By this theorem, such real functionals attain the lowest and the greatest values on the extreme points of A.

Theorem 2. *If $f \in TR\left(\frac{1}{2}\right)$ then $\Re\frac{f(z)}{z} > \frac{1}{2}$ for $z \in \Delta$.*

In the proof of Theorem 2 we use the following lemma.

Lemma 1. *Let $\frac{f(z)}{z} = \left(\frac{g(z)}{z}\right)^2$. Then $f \in ST \iff g \in ST(\frac{1}{2})$.*

Proof of Theorem 2.: The functional $\Re\frac{f(z)}{z}$ is linear and continuous so

$$\min\left\{\Re\frac{f(z)}{z}, f \in TR\left(\frac{1}{2}\right)\right\} = \min\left\{\Re\frac{f_t(z)}{z}, t \in [-1,1]\right\}.$$

Let f_t be given by (1). From Lemma 1 it follows that there exists the function $g_t \in ST(\frac{1}{2})$ which satisfies $\frac{f_t(z)}{z} = \sqrt{\frac{g_t(z)}{z}}$. Using the known inequality $\Re\sqrt{\frac{h(z)}{z}} > \frac{1}{2}$ for $h \in ST$ we obtain the conclusion of this theorem.

Theorem 3. *The radius of bounded rotation $r_{P'}$ of $TR(\frac{1}{2})$ is equal to $r_{P'} = \frac{\sqrt{2}}{2} = 0,707...$*

Proof. From the Krein-Milman Theorem we have

$$\min\left\{\Re f'(z), f \in TR\left(\frac{1}{2}\right), |z| = r\right\} > 0 \iff$$
$$\min\left\{\Re f_t'(z), t \in [-1,1], |z| = r\right\} > 0.$$

Let f_t be given by (1). Since $f_t \in ST(\frac{1}{2})$, there is

$$\frac{zf_t'(z)}{f_t(z)} \prec \frac{1}{1-z} \ .$$

It means that there exists a function ω_1 of the class $B = \{\omega \in \mathcal{A} : \omega(0) = 0, |\omega(z)| < 1, z \in \Delta\}$ such that

$$\frac{zf_t'(z)}{f_t(z)} = \frac{1}{1-\omega_1(z)}.$$

Hence, we have

$$f_t'(z) = \frac{f_t(z)}{z} \cdot \frac{1}{1-\omega_1(z)}. \tag{6}$$

From Theorem 1 it follows that

$$\Re\frac{f_t(z)}{z} > \frac{1}{2}, \ z \in \Delta,$$

and consequently

$$\frac{f_t(z)}{z} \prec \frac{1}{1-z}.$$

Therefore, there exists a function $\omega_2 \in B$ such that

$$\frac{f_t(z)}{z} = \frac{1}{1-\omega_2(z)}. \tag{7}$$

Finally, the function f_t' can be written in the form

$$f_t'(z) = \frac{1}{1-\omega_2(z)} \cdot \frac{1}{1-\omega_1(z)}. \tag{8}$$

The condition $\Re f_t'(z) > 0$ is equivalent to the condition $|\mathrm{Arg}\, f_t'(z)| < \frac{\pi}{2}$. Using (8) and simple estimation we have

$$|\mathrm{Arg}\, f_t'(z)| = \left|\mathrm{Arg}\, \frac{1}{1-\omega_2(z)} \cdot \frac{1}{1-\omega_1(z)}\right| \le \max_{\omega \in B} 2\left|\mathrm{Arg}\, \frac{1}{1-\omega(z)}\right| \le 2\arcsin|z|$$

Hence, if $2\arcsin|z| < \frac{\pi}{2}$ or equivalently $|z| < \sin\frac{\pi}{4}$ then $\Re f_t'(z) > 0$.
The equality in (**??**) appears for $\omega_1(z) \equiv z$, $\omega_2(z) \equiv z$. Hence, from (7) we get the function $f_t(z) = \frac{z}{1-z}$ for which $\Re f'(z)$ has negative values while $|z| > \frac{\sqrt{2}}{2}$.

THEOREM 4. *The radius of convexity in the direction of the imaginary axis $r_{CV(i)}$ of $TR(\frac{1}{2})$ is equal to $r_{CV(i)} = \sqrt{2\sqrt{3}-3} = 0,68...$*

PROOF. It is known that, if $f \in \mathcal{A}$ then

$$f \in CVR(i) \iff zf'(z) \in TR.$$

Hence

$$f \in CVR(i) \iff \Im z \Im z f'(z) \geq 0, \quad z \in \Delta.$$

Let $z \in \Delta$, $\Im z > 0$ and $f \in TR\left(\frac{1}{2}\right)$.

From the Krein-Milman Theorem

$$\min\left\{\Im z f'(z), f \in TR\left(\frac{1}{2}\right), z \in \Delta\right\} = \min\left\{\Im z f_t'(z), t \in [-1,1], z \in \Delta\right\},$$

where f_t is given by (1).

Now we use the theorem established by MacGregor in [3]

Theorem A. *If $f \in ST(\frac{1}{2})$ then $f(\Delta_r)$ is convex for $r \leq \sqrt{2\sqrt{3}-3}$.*

Since $f_t \in ST(\frac{1}{2})$, from Theorem A in particular it follows that the set $f_t(\Delta_r)$ is convex in the direction of the imaginary axis for $r \leq \sqrt{2\sqrt{3}-3}$.

We are going to prove that for $r > \sqrt{2\sqrt{3}-3}$ there exists a function f_{t_0} of the form (1) such that $\Im z \Im z f_{t_0}'(z) < 0$ for some $z \in \Delta_r$.
Let $G_t(z) \equiv zf_t'(z)$. We have $G_t(re^{i\varphi}) = re^{i\varphi}\frac{1-tre^{i\varphi}}{(1-2tre^{i\varphi}+r^2e^{2i\varphi})^{\frac{3}{2}}}$. The argument of the tangent vector to the curve $\Gamma = \partial G_t(\Delta_r)$ in the point $G_t(r)$ is equal to

$$\arg\left(\frac{\partial G_t}{\partial \varphi}(r)\right) = \arg\left(i \cdot w_t(r)\right) = \frac{\pi}{2} + \arg w_t(r),$$

where $w_t(r) = \frac{r(1-tr-2r^2+t^2r^2+tr^3)}{(1-2tr+r^2)^{\frac{5}{2}}}$.

The inequality $w_t(r) \geq 0$ is true for all $t \in [-1,1]$ if $r \leq \sqrt{2\sqrt{3}-3}$. For $r > \sqrt{2\sqrt{3}-3}$ and $t_0 = \frac{1-r^2}{2r}$ the inequality $w_{t_0}(r) < 0$ holds.

It means that for $r > \sqrt{2\sqrt{3}-3}$ the argument of the tangent vector to Γ in $G_{t_0}(r)$ is equal to $\frac{-\pi}{2}$. Hence, there exists φ_0 such that

$$\Im G_{t_0} < 0 \quad \text{for} \quad \varphi \in [0, \varphi_0) \ .$$

Furthermore, $f(\Delta_r)$ is convex in the direction of the imaginary axis in the disk $|z| < \sqrt{2\sqrt{3}-3}$ and this number is best possible. The extremal function is

$$f_{t_0}(z) = \frac{z}{\sqrt{1 - \frac{1-r^2}{r} z + z^2}} \ .$$

Using the similar method to that from the proof of Theorem 2, we estimate the radius of convexity in the direction of the real axis in $TR(\frac{1}{2})$. Koczan in [2] determined the representation formula for the class $CVR(1)$. Namely

Theorem B. *The function f belongs to $CVR(1)$ if and only if $f \in \mathcal{A}$, f is real on $(-1, 1)$, and there exists $\beta \in [0, \pi]$ such that*

$$\Re\left[(1 - 2z\cos\beta + z^2) f'(z)\right] > 0, \quad z \in \Delta.$$

We make use of the following fact

$$\max\left\{\operatorname{Arg}\frac{1-z}{1-\zeta} : |\zeta| \le |z| < 1\right\} = 2\arcsin|r|. \tag{9}$$

Indeed, from the maximum principle for analytic functions we have

$$\max\left\{\operatorname{Arg}\frac{1-z}{1-\zeta} : |\zeta| \le |z| < 1\right\} = \max\left\{\operatorname{Arg}\frac{1-z}{1-\zeta} : |z| = |\zeta| < 1\right\}.$$

Using twice the inequality $\operatorname{Arg}(1-w) \le \arcsin|w|$ for $w \in \Delta$ we obtain (9) .

Theorem 5. *The radius of convexity in the direction of the real axis $r_{CV(1)}$ of $TR(\frac{1}{2})$ satisfies the inequality* $\sin\frac{\pi}{8} = 0.38\ldots < r_{CV(1)} \le \sqrt{2}-1$.

Proof. Let $f \in TR(\frac{1}{2})$.
Then

$$\Re\left[(1 - 2z\cos\beta + z^2) f'(z)\right] = \int_{-1}^{1} \Re\left[(1 - 2z\cos\beta + z^2) f_t'(z)\right] d\mu(t).$$

From (8) we have

$$(1-2z\cos\beta+z^2)f_t'(z)=\frac{1-2z\cos\beta+z^2}{(1-\omega_1(z))(1-\omega_2(z))},$$

where $\omega_1,\omega_2\in B$.
Let us consider the inequality

$$\left|\operatorname{Arg}\frac{1-2z\cos\beta+z^2}{(1-\omega_1(z))(1-\omega_2(z))}\right|<\frac{\pi}{2},$$

or equivalently

$$\left|\operatorname{Arg}\frac{1-ze^{-i\beta}}{1-\omega_1(z)}\cdot\frac{1-ze^{i\beta}}{1-\omega_2(z)}\right|<\frac{\pi}{2}. \tag{10}$$

We have $|\omega_k(z)|\le|z|$, $k=1,\ 2$. From (9) it follows now that if $4\arcsin|z|<\frac{\pi}{2}$ then the inequality (10) is satisfied. Consequently, if $|z|<\sin\frac{\pi}{8}$ then

$$\Re\left[(1-2z\cos\beta+z^2)f_t'(z)\right]>0\ . \tag{11}$$

This and Theorem B leads to $r_{CV(1)}\ge\sin\frac{\pi}{8}$. The extremal function in the inequality (11) does not have real coefficients so

$$r_{CV(1)}>\sin\frac{\pi}{8}.$$

Moreover, for the function

$$f(z)=\frac{z}{1-z^2}=\frac{1}{2}\left(\frac{z}{1+z}+\frac{z}{1-z}\right)\in TR(\frac{1}{2}) \tag{12}$$

the set $f(\Delta_{r_0})$, $r_0=\sqrt{2}-1$ is convex in the direction of the real axis and the number r_0 is best possible. It results from the fact that the function

$$\frac{f(iz)}{i}=\frac{z}{1+z^2}$$

is convex in the direction of the imaginary axis in the set $i\cdot H=\{re^{i\theta}:1-r^2>2r|\cos\theta|\}$. Hence, the function (12) is convex in the direction of the real axis in the set H of the form $\{re^{i\theta}:1-r^2>2r|\sin\theta|\}$. Therefore $r_{CVR(1)}\le\sqrt{2}-1$.

From given above theorems we obtain the corollaries concerning starlikeness and convexity of functions from $TR(\frac{1}{2})$.

Corolary. *The radius of starlikeness r_{ST} of $TR(\frac{1}{2})$ satisfies the inequality $\frac{\sqrt{2}}{2} \leq r_{ST} \leq \frac{\sqrt{7}}{3}$.*

The left hand side inequality results from the fact that the functions of the class $\{f \in \mathcal{A} : \Re\frac{f(z)}{z} > \frac{1}{2}, z \in \Delta\}$ are starlike in the disk $\Delta_{\frac{\sqrt{2}}{2}}$, (see [7]) and from Theorem 2. The upper estimation is the consequence of the inequality proved in Theorem 1.
From Theorem 4 we obtain

Corolary. *The radius of convexity r_{CV} of $TR(\frac{1}{2})$ satisfies the inequality $r_{CV} \leq \sqrt{2} - 1$.*

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