Trudy Petrozavodskogo gosudarstvennogo universiteta
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THE SHARP UPPER BOUND FOR $\Re(A_3 - \lambda A_2)$ IN U'_α

I. Naraniecka

In this note we determine the exact value of max $\mathrm{Re}(A_3 - \lambda A_2)$, $\lambda \in \mathbb{R}$, within the linearly invariant family U'_α introduced by V. V. Starkov in [4]. For $\lambda = 0$ the sharp estimate for $|A_3|$ follows. If $\alpha = 1$ the corresponding result is valid for convex univalent functions in the unit disk.

1. For given $\alpha \geq 1$, we consider the class of holomorphic functions in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ of the form

$$f(z) = z + A_2 z^2 + A_3 z^3 + \dots \tag{1}$$

which are defined by the formula

$$f'(z) = \exp\left\{-2\int_0^{2\pi} \log(1 - ze^{-it} d\mu(t))\right\}, \tag{2}$$

where $\mu(t)$ is a complex function of bounded variation on $[0, 2\pi]$ satisfying the conditions

$$\int_0^{2\pi} d\mu(t) = 1, \qquad \int_0^{2\pi} |d\mu(t)| \leq \alpha. \tag{3}$$

The class U'_α has been introduced by Starkov in [4]. The idea of studing such a class is justified by at least two facts:

1) The class U'_α appears to be a linearly invariant family in the sense of Pommerenke of order α, and can be used for studing the universal invariant family U_α [3].

2) The class U'_α generalizes essentially the class $V_{k=2\alpha}$ of functions with bounded boundary variation (Paatero class) and in the sequel convex univalent functions $K \equiv U'_1$.

In [5] V. V. Starkov has found sharp bound for $|A_3|$ within the class U'_α which disproved the Campbell–Cima–Pfaltzgraff conjecture about $\max|A_3|$ in U_α.
In this note we determine

$$\max_{f\in U'_\alpha} \text{ Re } (A_3 - \lambda A_2), \tag{4}$$

for real λ, which as a corollary ($\lambda = 0$) gives the above result of Starkov.

Justification of studing such a functional is highly motivated by corresponding result for the class of univalent functions S [2] (Bombieri Conjecture). As a method we are going to use is the variational method of Starkov for U'_α [5].

2. Problem of finding (4) is equivalent to

$$\max_{f\in U'_\alpha} \text{ Re } (C_2 - \lambda C_1), \qquad \lambda \in \mathbb{R}, \tag{5}$$

where $f'(z) = 1 + C_1 z + C_2 z^2 + \ldots$, $f \in U'_\alpha$.

Because $\mathcal{J}(f) = \text{Re } (C_2 - \lambda C_1)$ is a linear functional, then according to a result of Starkov [4] the extremal function $f_0(z)$ is of the form

$$f_0'(z) = (1 - ze^{-it_1})^{-2a_1}(1 - ze^{-it_2})^{-2a_2}, \tag{6}$$

where

$$t_1,\ t_2 \in [0, 2\pi] \tag{7}$$

and

$$a_1 + a_2 = 1 \quad \text{and} \quad |a_1| + |a_2| = \alpha. \tag{8}$$

One cand find that the coefficients of f_0 are given by

$$c_1 = 2(a_1 e^{-it_1} + a_2 e^{-it_2}), \quad c_2 = \frac{c_1^2}{2} + a_1 e^{-2it_1} + a_2 e^{-2it_2}. \tag{9}$$

Therefore the problem is reduced in finding the maximal value of

$$\begin{aligned}\psi(a_1, a_2; t_1, t_2) = \text{Re}\Big\{ & 2\left[a_1 e^{-it_1} + a_2 e^{-it_2}\right]^2 + \left[a_1 e^{-2it_1} + a_2 e^{-2it_2}\right] \\ & -2\lambda\left[a_1 e^{-it_1} + a_2 e^{-it_2}\right]\Big\}\end{aligned} \tag{10}$$

where t_1, t_2, a_1, a_2 are satysfying the conditions (7) and (8) Moreover same extra conditions follow from the extremality of f_0 [4] (see below).

We will start with simple technical lemma.

LEMMA 1. *If* $a_1 = |a_1|e^{i\beta_1}$ *and* $a_2 = |a_2|e^{i\beta_2}$ *and*

$$\begin{cases} a_1 + a_2 = 1 \\ |a_1| + |a_2| = \alpha > 1 \end{cases} \tag{11}$$

then

$$|a_1| = \frac{\sin\beta_2}{\sin(\beta_2 - \beta_1)}, \qquad |a_2| = \frac{-\sin\beta_1}{\sin(\beta_2 - \beta_1)}. \tag{12}$$

Moreover, β_1 *and* β_2 *satisfy the condition*

$$\cos\frac{\beta_2 + \beta_1}{2} = \alpha\cos\frac{\beta_2 - \beta_1}{2} \Longleftrightarrow \tan\frac{\beta_2}{2}\tan\frac{\beta_1}{2} = -\frac{\alpha - 1}{\alpha + 1} := A. \tag{13}$$

PROOF. *The system* (11) *can be written in the real form:*

$$\begin{cases} |a_1|\cos\beta_1 + |a_2|\cos\beta_2 = 1 \\ |a_1|\sin\beta_1 + |a_2|\sin\beta_2 = 0 \\ |a_1| + |a_2| = \alpha. \end{cases}$$

Solution of the first two equations by Cramer's rule is unique and given by (12)*. (If* $\sin(\beta_2 - \beta_1) = 0$ *then the above system has no solution).*

Substitution of (12) into the equation $|a_1| + |a_2| = \alpha$ gives (13) after slight calculations.

The following lemma plays important rule.

LEMMA 2. *The extremal function* f_0 *for functional* (5) *has real coefficient* c_1*.*

PROOF. If $f_0 \in U'_\alpha$ is an extremal function, then for any $\varepsilon \in (0,1)$, the following variation f_ε of f belongs to U'_α:

$$\begin{aligned} f_\varepsilon(z) &= \int_0^z \left(f_0'(s)\right)^{1-\varepsilon} \overline{\left(f_0'(\overline{s})\right)^{\varepsilon}} ds \\ &= 1 + c_1(\varepsilon)z + c_2(\varepsilon)z^2 + \ldots \in U'_\alpha. \end{aligned} \tag{14}$$

But

$$\begin{aligned} c_2(\varepsilon) - \lambda c_1(\varepsilon) &= c_2 - 2i\varepsilon \mathrm{Im}\ c_2 + \varepsilon(1-\varepsilon)(|c_1|^2 - \mathrm{Re}\ c_1^2) - \lambda(c_1 - 2i\varepsilon \mathrm{Im}\ c_1) \\ &= (c_2 - \lambda c_1) - 2i\varepsilon \mathrm{Im}\ c_2 + 2i\lambda\varepsilon \mathrm{Im}\ c_1 + \varepsilon(|c_1|^2 - \mathrm{Re}\ c_1^2) + o(\varepsilon) \end{aligned}$$

which implies

$$\mathcal{J}(f_\varepsilon) = \mathcal{J}(f_0) + \varepsilon(|c_1|^2 - \text{Re } c_1^2) + \text{Re } o(\varepsilon).$$

The extermality of f_0: $\mathcal{J}(f_\varepsilon) \le \mathcal{J}(f_0)$, when $\varepsilon \to 0$, gives the condition

$$|c_1|^2 - \text{Re } c_1^2 \le 0 \text{ which implies Im } c_1 = 0,$$

due to the form of our functional (5).

. If Im $\left[a_1 e^{-it_1} + a_2 e^{-it_2}\right] = 0$ then either: both e^{-it_1} and e^{-it_2} are real, or $e^{-it_2} = e^{it_1} = \overline{e^{-it_1}}$.

Denote:

$$\begin{aligned} &\cos\beta = \frac{1}{\alpha}, \quad \sin\beta = \frac{\sqrt{\alpha^2-1}}{\alpha}, \\ &\cos\varphi = \frac{3-\alpha^2}{\alpha\sqrt{\alpha^2+3}}, \quad \sin\varphi = \frac{3\sqrt{\alpha^2-1}}{\alpha\sqrt{\alpha^2+3}} \\ &\tau = \beta + \frac{\varphi}{2}, \quad x = t + \beta \end{aligned} \tag{15}$$

We have:

THEOREM 1. *If $f \in U'_\alpha$ and $f'(z) = 1 + c_1 z + c_2 z^2 + \dots$ then*

$$\max_{f\in U'_\alpha} Re(c_2 - \lambda c_1) = \Phi(t_0)$$

where

$$\Phi(t) = \alpha^2 + (3-\alpha^2)\cos 2t + 3\sqrt{\alpha^2-1}\sin 2t - 2\lambda\left(\cos t - \sqrt{\alpha^2-1}\sin t\right) \tag{16}$$

and

$$t_0 = t_0(\alpha,\lambda) \in (0, 2\pi) \tag{17}$$

is the root of the equation: $\lambda \sin x - \sqrt{\alpha^2+3}\cdot\sin(2x - 2\tau) = 0$, *for which* $\varphi''(t_0) < 0$.

PROOF. Let $f \in U'_\alpha$ and

$$\begin{aligned} f'(z) &= \exp\left\{-2\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\mu(t)\right\} = \exp\{\varphi(z)\} \\ &= 1 + C_1 z + C_2 z^2 + \dots . \end{aligned} \tag{18}$$

The functional $\mathcal{J}(f) = \mathrm{Re}(C_2 - \lambda C_1)$ is a linear and continuous on the compact family U'_α and therefore it attains its sharp bounds on it. V. V. Starkov [3] has proved, that if $F(f) = F(\varphi)$is Fréchet differentiable and its differential functional on U'_α with differential $L_\varphi(h)$, and max Re $F(\varphi)$ is attained for a jump function $\mu(t)$ with n jumps at points t_j, $j = 1, \dots, n$ and jumps $\theta_j = \arg\, d\mu_n(t_j)$ (we assume that at least two jumps θ_j are different), then the following system of equations holds

$$\begin{cases} \mathrm{Re}\ \left\{ e^{i\theta_j} L_{\varphi_n} \left[\frac{\partial g}{\partial t}(z, t_j) \right] \right\} = 0 \\ \mathrm{Im}\ \left[\left(e^{i\theta_j} - e^{i\theta_m} \right) \left(L_{\varphi_n}[g(z,t_j)] - L_{\varphi_n}[g(z,t_m)] \right) \right] = 0. \end{cases} \tag{19}$$

In our case $\mathcal{J}(f) = \mathrm{Re}\ (C_2 - \lambda C_1)$ s Fréchet differentiable and its differential is given by the formula:

$$L_\varphi(h) = \{h \exp \varphi\}_2 - \lambda \{h \exp \varphi\}_1 ,$$

where $h = -2\log(1 - ze^{-it}) = g(z,t)$ and $\{F(z)\}_p$ denotes the p–th coefficient of F.

In our problem the extremal function has the form :

$$f_0'(z) = 1 + c_1 z + c_2 z^2 + \dots = (1 - ze^{-it_1})^{-2a_1} (1 - ze^{-it_2})^{-2a_2}$$

where $t_1, t_2 \in [0, 2\pi]$, $a_1 + a_2 = 1$, $|a_1| + |a_2| = \alpha$.
Because the Fréchet differential is equal to

$$L_{(\log f_0')} \left[-2\log(1 - ze^{-it}) \right] = e^{-2it} + 2e^{-it}(c_1 - \lambda),$$

the conditions (19) take the form

$$\begin{cases} \mathrm{Im} \left[e^{i\beta_1} \left(e^{-2it_1} + e^{-it_1}(c_1 - \lambda) \right) \right] = 0 \\ \mathrm{Im} \left[e^{i\beta_2} \left(e^{-2it_2} + e^{-it_2}(c_1 - \lambda) \right) \right] = 0 \\ \mathrm{Im} \left[\left(e^{i\beta_1} - e^{i\beta_2} \right) \left(e^{-2it_1} - e^{-2it_2} + 2(c_1 - \lambda) \left(e^{-it_1} - e^{-it_2} \right) \right) \right] = 0. \end{cases} \tag{20}$$

The information that for the extremal function f_0 the coefficient c_1 is real i.e.

$$\mathrm{Im}\ c_1 = 0 \Longleftrightarrow \sin\beta_2 \sin(\beta_1 - t_1) - \sin\beta_1 \sin(\beta_2 - t_2) = 0 \tag{21}$$

implies $e^{it_2} = e^{-it_1}$ which gives $t_2 = -t_1$, or that e^{-it_1} and e^{-it_2} are real. In the case when e^{-it_1} and e^{-it_2} are real we obtain either contradiction or the result for $U_1 = K$ which is in the Corollary at the end of the papers.

In the case $e^{it_2} = e^{-it_1}$ i.e. $t_2 = -t_1$ the first two equations of (20) are

$$\begin{cases} \sin(\beta_1 - 2t_1) + (c_1 - \lambda)\sin(\beta_1 - t_1) = 0 \\ \sin(\beta_2 - 2t_2) + (c_1 - \lambda)\sin(\beta_2 - t_2) = 0 \end{cases} \tag{22}$$

which together with (21) for $t_2 = -t_1$ implies that $\beta_2 = -\beta_1$. Substitution $\beta_2 = -\beta_1$ into (12) and (13) give

$$\begin{aligned} &|a_1| = |a_2| = \frac{\alpha}{2}; \qquad \cos\beta_1 = \frac{1}{\alpha}; \qquad \cos\beta_2 = \frac{1}{\alpha}; \\ &\sin\beta_1 = \frac{\sqrt{\alpha^2 - 1}}{\alpha}; \qquad \sin\beta_2 = \frac{-\sqrt{\alpha^2 - 1}}{\alpha}. \end{aligned} \tag{23}$$

Putting now $t_1 = t \in [0, 2\pi]$, $t_2 = -t_1 = -t$, $a_1 = \frac{\alpha}{2}e^{i\beta}$, $a_2 = \frac{\alpha}{2}e^{-i\beta}$ we obtain

$$\begin{aligned} \mathrm{Re}(c_2 - \lambda c_1) := \Phi(t) = \alpha^2 + (3 - \alpha^2)\cos 2t + 3\sqrt{\alpha^2 - 1}\sin 2t \\ - 2\lambda(\cos t - \sqrt{\alpha^2 - 1}\sin t). \end{aligned} \tag{24}$$

Using notations (15) we obtain:

$$\Phi(t) = \Phi(x) = \alpha^2 + \alpha\sqrt{\alpha^2 + 3}\cos(2x - 2\tau) - 2\lambda\alpha\cos x. \tag{25}$$

The equation $\Phi'(x) = 0$ is equivalent to

$$-\lambda\sin x + \sqrt{\alpha^2 + 3}\sin(2x - 2\tau) = 0 \tag{26}$$

or

$$\begin{aligned} 4(\alpha^2 + 3)\sin^4 x - 4\lambda\sqrt{\alpha^2 + 3}\sin 2\tau\sin^3 x + [\lambda^2 - 4(\alpha^2 + 3)]\sin^2 x \\ + 2\lambda\sqrt{\alpha^2 + 3}\sin 2\tau\sin x + (\alpha^2 + 3)\sin^2 2\tau = 0, \end{aligned} \tag{27}$$

which ends the proof.

Corolary. *If $f \in U_1 = K$ then*

$$max(A_3 - \lambda A_2) = 1 + |\lambda|, \quad \lambda \in \mathbb{R}.$$

The extremal functions have the form $f_0(z) = \dfrac{z}{1 \pm z}$.

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Department of Applied Mathematics
Faculty of Economics, Maria Curie–Skłodowska University,
20-031 Lublin, Poland
E-mail: iwona.naraniecka@hektor.umcs.lublin.pl