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THE GENERALIZED KOEBE FUNCTION

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We observe that the extremal function for $|a_3|$ within the class $U_\alpha^{'}$ (see Starkov [1]) has as well the property that $\max|A_4| > 4.15$, if $\alpha = 2$. The problem is equivalent to the global estimate for Meixner-Pollaczek polynomials $P_3^1(x;\theta)$.

In [1] Starkov has found $\max|a_3|$ within the class $U_\alpha^{'}$, which for $\alpha = 2$ disproved the Campbell-Cima-Pfaltzgraff conjecture, that $|a_3| \leq 3$ for U_2.

The extremal function $f_0(z) = \sum_{n=1}^{\infty} A_n z^n, \quad z \in \mathbb{D} = \{z : |z| < 1\}$ has the form

$$f_0^{'}(z) = \frac{1}{(1 - ze^{i\theta})^{1-i\sqrt{\alpha^2-1}}(1 - ze^{-i\theta})^{1+i\sqrt{\alpha^2-1}}},$$

with appropriate θ, $\theta \in (-\pi, \pi], \alpha > 1, z \in \mathbb{D}$ which appears to be very closely connected with Meixner-Pollaczek (M-P) polynomials [2].

For $\lambda > 0,\ x \in \mathbb{R},\ \theta \in (0, \pi)$ the Meixner-Pollaczek polynomials of the variable x are defined by the generating function

$$G^\lambda(x;\theta;z) = \frac{1}{(1 - ze^{i\theta})^{\lambda-ix}(1 - ze^{-i\theta})^{\lambda+ix}} = \sum_{n=0}^{\infty} P_n^\lambda(x;\theta)z^n, \quad z \in \mathbb{D}.$$

Therefore, we see that $nA_n = P_{n-1}^1(\sqrt{\alpha^2 - 1};\theta)$ and the estimate of $P_n^1(x;\theta)$ as the function of $\theta \in (0, \pi)$ is of independent interest and will lead to the bound for $|A_n|$. In this note we find sharp bound for $|P_n^1(x;\theta)|$, $n = 1, 2, 3$, which implies that $\max|a_4| > 4.15$ for U_2, supporting the result of Starkov [1].

Theorem A [2]. *(i) The M-P polynomials $P_n^\lambda(x;\theta)$ satisfy the three-term recurrence relation:*

$$nP_n^\lambda(x;\theta) = 2[x\sin\theta + (n-1+\lambda)\cos\theta]P_{n-1}^\lambda(x;\theta) - \\ -(2\lambda+n-2)P_{n-2}^\lambda(x;\theta), \quad n \geq 2.$$

(ii) The polynomials $P_n^\lambda(x;\theta)$ are given by the formula:

$$P_n^\lambda(x;\theta) = e^{in\theta}\sum_{j=0}^{n}\frac{(\lambda+ix)_j(\lambda-ix)_{n-j}}{j!(n-j)!}e^{-2ij\theta}, \quad n \in \mathbb{N}\cup\{0\}.$$

(iii) The polynomials $P_n^\lambda(x;\theta)$ have the hypergeometric representation

$$P_n^\lambda(x;\theta) = e^{in\theta}\frac{(2\lambda)_n}{n!}F(-n,\lambda+ix,2\lambda;1-e^{-2i\theta}).$$

Symbol $(a)_n$ denotes the Pochhammer symbol:

$$(a)_n = a(a+1)...(a+n-1),\ n\in\mathbb{N},\ (a)_0 = 1,$$

and $F(a,b,c;z)$ denotes the Gauss Hypergeometric Function.
(iiii) The polynomials $y(x) = P_n^\lambda(x;\theta)$ satisfy the following difference equation

$$e^{i\theta}(\lambda-ix)y(x+i)+2i[x\cos\theta-(n+\lambda)\sin\theta]y(x)-e^{-i\theta}(\lambda+ix)y(x-i) = 0.$$

From Theorem A we have the form of $P_n^1(x;\theta)$, $n = 1,2,3$, convenient for further calculations:

$$\begin{aligned} P_0^1(x;\theta) &= 1, \\ P_1^1(x;\theta) &= 2(x\sin\theta+\cos\theta), \\ P_2^1(x;\theta) &= 3x\sin 2\theta + (2-x^2)\cos 2\theta + (x^2+1), \\ P_3^1(x;\theta) &= (x^2+1)(x\sin\theta + \\ &+2\cos\theta) + \frac{1}{3}(x(11-x^2)\sin 3\theta + 6(1-x^2)\cos 3\theta). \end{aligned} \tag{1}$$

Remark. *In our calculations we will use the obvious convenient formula*

$$A\sin\alpha + B\cos\alpha = \sqrt{A^2+B^2}\sin(\alpha+\varphi),$$

where $\cos\varphi = \dfrac{A}{\sqrt{A^2+B^2}}, \quad \sin\varphi = \dfrac{B}{\sqrt{A^2+B^2}}.$

Denote

$$\sin\beta_0 = \frac{2}{\sqrt{x^2+4}}, \quad \cos\beta_0 = \frac{x}{\sqrt{x^2+4}}, \tag{2}$$
$$\sin\beta_1 = \frac{6(1-x^2)}{\sqrt{x^2+4}\sqrt{x^2+9}\sqrt{x^2+1}},$$
$$\cos\beta_1 = \frac{x(11-x^2)}{\sqrt{x^2+4}\sqrt{x^2+9}\sqrt{x^2+1}},$$

x is fixed, and

$$\Psi(\theta) = 3\sqrt{x^2+1}\sin(\theta+\beta_0) + \sqrt{x^2+9}\sin(3\theta+\beta_1),\ \theta\in[-\pi,\pi].$$

THEOREM 1. *For the Meixner-Pollaczek polynomials $P_n^1(x;\theta)$, $x \geq 0$, $\theta\in(0,\pi)$ we have the sharp estimates:*

$$|P_1^1(x;\theta)| \leq 2\sqrt{x^2+1},$$
$$|P_2^1(x;\theta)| \leq \sqrt{x^2+1}(\sqrt{x^2+1}+\sqrt{x^2+4}),$$
$$|P_3^1(x;\theta)| \leq \frac{1}{3}\sqrt{x^2+1}\sqrt{x^2+4}\max_{\theta\in[0,\pi]}|\Psi(\theta)| =$$
$$= \frac{1}{3}\sqrt{x^2+1}\sqrt{x^2+4}\Big(3\sqrt{x^2+1}\sin(\hat\theta+\beta_0)+\sqrt{(x^2+1)\sin^2(\hat\theta+\beta_0)+8}\Big) <$$
$$< \sqrt{x^2+1}\sqrt{x^2+4}(\sqrt{x^2+1}+\frac{1}{3}\sqrt{x^2+9}),$$

where $\hat\theta\in(0,\pi)$ is the root of the equation

$$H(\theta) = \frac{\cos(3\theta+\beta_1)}{\cos(\theta+\beta_0)} = -\sqrt{\frac{x^2+1}{x^2+9}}.$$

REMARK. *Due to the property: $\Psi(\pi+\theta) = -\Psi(\theta)$ and $H(\pi+\theta) = H(\theta)$, the estimates for $|P_n^1(x;\theta)|$, $n = 1,2,3$ are valid for $\theta\in[-\pi;\pi]$.*

PROOF. Using Remark 1, we have for $x > 0$:

$$P_1^1(x;\theta) = 2\sqrt{x^2+1}\sin(\theta+\varphi_1) \leq 2\sqrt{x^2+1}$$

with equality for θ_1, such that $\sin(\theta_1+\varphi_1) = 1$, where

$$\cos\varphi_1 = \frac{x}{\sqrt{x^2+1}}, \qquad \sin\varphi_1 = \frac{1}{\sqrt{x^2+1}}.$$

For $P_2^1(x;\theta)$ we have

$$P_2^1(x;\theta) = 3x\sin 2\theta + (2-x^2)\cos 2\theta + (x^2+1) =$$
$$= \sqrt{x^2+1}\sqrt{x^2+4}\sin(2\theta+\varphi_2)+(x^2+1) \le \sqrt{x^2+1}(\sqrt{x^2+4}+\sqrt{x^2+1}),$$

with equality for θ_2, such that $\sin(2\theta_2+\varphi_2)=1$, where

$$\cos\varphi_2 = \frac{3x}{\sqrt{x^2+1}\sqrt{x^2+4}}, \qquad \sin\varphi_2 = \frac{2-x^2}{\sqrt{x^2+1}\sqrt{x^2+4}}.$$

Finally, for $P_3^1(x;\theta)$ we have

$$P_3^1(x;\theta) = (x^2+1)(x\sin\theta+2\cos\theta)+\frac{1}{3}(x(11-x^2)\sin 3\theta+6(1-x^2)\cos 3\theta) =$$
$$= (x^2+1)\sqrt{x^2+4}\sin(\theta+\beta_0)+\frac{1}{3}\sqrt{x^2+1}\sqrt{x^2+4}\sqrt{x^2+9}\sin(3\theta+\beta_1) =$$
$$= \frac{1}{3}\sqrt{x^2+1}\sqrt{x^2+4}(3\sqrt{x^2+1}\sin(\theta+\beta_0)+\sqrt{x^2+9}\sin(3\theta+\beta_1)) =$$
$$= \frac{1}{3}\sqrt{x^2+1}\sqrt{x^2+4}\cdot\Psi(\theta),$$

where β_0 and β_1 are given by (2).

In order to find sharp estimate for $P_3^1(x;\theta)$ we have to find $\max\limits_{0\le\theta\le\pi}|\Psi(\theta)|$ for fixed $x>0$.

The equation $\Psi'(\theta)=0$ is equivalent to

$$H(\theta) = \frac{\cos(3\theta+\beta_1)}{\cos(\theta+\beta_0)} = -\sqrt{\frac{x^2+1}{x^2+9}}, \tag{3}$$

which is pretty difficult for discussion. However we can restrict ourselves to the case $\theta\in[0,\pi]$, because $\Psi(\pi+\theta)=-\Psi(\theta)$ and $H(\pi+\theta)=H(\theta)$.

Corolary. *In the case $\alpha=2\Leftrightarrow x^2=3$, the equation* (3) *is equivalent to*

$$\cos(\theta+\beta_0)+\sqrt{3}\sin(3\theta+\beta_0)=0,\ \sin\beta_0=\frac{2}{\sqrt{7}},\ \cos\beta_0=\frac{\sqrt{3}}{\sqrt{7}}$$

or

$$5t^3+5\sqrt{3}t-7t-3\sqrt{3}=0, \quad \textit{where} \quad t=\operatorname{tg}\theta. \tag{4}$$

The approximate calculations shows that, the maximal value of $\Psi(\theta)$ is given by $\hat{t} = tg\hat{\theta} \simeq 0.938$. For $t = \operatorname{tg}\theta \simeq 0.938$ we obtain $\max|A_4| = \max\frac{1}{4}|P_3^1(x;\theta)| > 4.17$, which show that for U_2', $|A_4|$ can be greater than 4.

Our result follows simply by taking $\theta = \frac{\pi}{4}$ in $\Psi(\theta)$. We get

$$A_4 = \sqrt{7}(1 + \frac{\sqrt{3}}{3})\sin(\frac{\pi}{4} + \beta_0) = \frac{1}{6}\sqrt{2}(5\sqrt{3} + 9) > 4.15.$$

REMARK. *Another important extremal problem solved by Starkov [3], namely* $\max|argf^{'}(z)|,\ f \in U_\alpha^{'}$, *has the extremal function:*

$$f_0(z) = \frac{1}{(e^{it_2} - e^{it_1})i\sqrt{\alpha^2 - 1}}\left[(\frac{1 - ze^{it_1}}{1 - ze^{it_2}})^{i\sqrt{\alpha^2-1}} - 1\right], \quad t_1 \neq t_2 + 2k\pi,$$

with $t_1 = \pi - \operatorname{arctg}\frac{1}{\alpha} - \operatorname{arctg}\frac{r}{\alpha}$, $t_2 = -\pi + \arcsin\frac{1}{\alpha} - \arcsin\frac{r}{\alpha}$, $r = |z| < 1,\ t_1 \neq -t_2$.

The coefficients of this function are not M-P polynomials. Inspired by that we are going to study the properties of the generalized Koebe function defined by the formula:

$$k_c(\theta, \psi; z) = \frac{1}{(e^{i\psi} - e^{i\varphi})c}\Big[\Big(\frac{1 - ze^{i\theta}}{1 - ze^{i\psi}}\Big)^c - 1\Big], c \in \mathbb{C}\setminus\{0\},\ e^{i\psi} \neq e^{i\theta},\ z \in \mathbb{D},$$

and

$$k_0(\theta, \psi; z) = \frac{1}{(e^{i\psi} - e^{i\varphi})} log\frac{1 - ze^{i\theta}}{1 - ze^{i\psi}}, \quad e^{i\psi} \neq e^{i\theta}, \quad z \in \mathbb{D},$$

for which

$$k_c^{'}(\theta, \psi; z) = \frac{1}{(1 - ze^{i\theta})^{1-c}(1 - ze^{i\psi})^{1+c}}, \quad c \in \mathbb{C}.$$

This is evidently connected with the polynomials which we call the generalized M-P polynomials (GMP) given by generating function $(\theta, \psi \in \mathbb{R},\ x \in \mathbb{R},\ \lambda > 0):$

$$G^\lambda(x; \theta, \psi; z) = \frac{1}{(1 - ze^{i\theta})^{\lambda - ix}(1 - ze^{i\psi})^{\lambda + ix}} = \sum_{n=0}^{\infty} P_n^\lambda(x; \theta, \psi)z^n,\ z \in \mathbb{D}.$$

This set of polynomials will be studied somewhere else.

Bibliography

[1] Starkov V. V. *The estimates of coefficients in locally-univalent family* U'_α // Vestnik Lenin. Gosud. Univ. 13(1984). P. 48–54. (in Russian).

[2] Koekoek R., Swarttouw R. F. *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue* // Report 98–17. Delft University of Technology. 1998.

[3] Starkov V. V. *Linear-invariant families of functions* // Dissertation. Ekatirenburg, 1999. 1-287. (in Russian).

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