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OPTIMAL BOUNDS FOR CERTAIN BIVARIATE MEANS

Abstract. New bivariate means, introduced and investigated in [1], play a central role in this work. The lower and upper bounds for those means are obtained. Bounding quantities are the one-parameter means derived from the harmonic and contraharmonic means by forming convex combinations of the variables of these two means.

Key words: *Bivariate means, one-parameter means, optimal bounds, inequalities.*

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§ 1. Introduction

Recently several researchers have obtained optimal bounds for some bivariate means such as logarithmic mean, two Seiffert means, Neuman-Sándor mean, Neuman means, to mention few. The optimal bounds are the other bivariate means which are simpler than those for which the optimal bounds are sought. For more details the interested reader is referred to [1 − 7] and the references therein.

This paper deals with optimal lower and upper bounds for the family of bivariate means introduced in [1] and is organized as follows. In Section 2 we give definition and some basic properties of the generic one-parameter family of means. Definitions of bivariate means used in this paper are given in Section 3. In particular, we include there formulas for the four new means investigated in [1]. Optimal bounds for these means are established in Section 4. The bounding quantities are the one-parameter generalizations of either the harmonic or the contraharmonic means.

§ 2. Definition and basic properties of the one-parameter family of means

For the reader's convenience we recall definition and basic properties of the one-parameter family of bivariate means. Following [3] we define two nonnegative numbers w_1 and w_2:

$$w_1 = \frac{1+p}{2}, \quad w_2 = \frac{1-p}{2}, \tag{2.1}$$

where the parameter p satisfies $0 \le |p| \le 1$. Clearly $w_1 + w_2 = 1$. In what follows (a, b) will stand for the pair of unequal positive numbers. We associate with (a, b) another pair of positive numbers (x, y), where

$$x = w_1 a + w_2 b, \quad y = w_1 b + w_2 a. \tag{2.2}$$

Thus x and y are the convex combinations of a and b. One can easily verify that (for $0 < p \le 1$) $a < x < y < b$ if $a < b$ or $b < y < x < a$ if $b < a$.

For the sake of presentation let $\mathcal{N}$ stand for a generic bivariate symmetric mean. We define a one-parameter mean $\mathcal{N}_p(a, b) \equiv \mathcal{N}_p$ as follows

$$\mathcal{N}_p(a, b) = \mathcal{N}(x, y). \tag{2.3}$$

In the sequel we will call mean $\mathcal{N}_p$ the p-mean or the p-mean generated by $\mathcal{N}$.

We recall now some elementary properties of the p-means. Using (2.3), (2.1), and (2.2) we see that

$$\mathcal{N}_{-p}(a, b) = \mathcal{N}(y, x) = \mathcal{N}(x, y) = \mathcal{N}_p(a, b).$$

Thus the function $p \to \mathcal{N}_p$ is an even function. Without a loss of generality we may assume that $0 \le p \le 1$. It follows from (2.1) and (2.2) that

$$\mathcal{N}_0 = A, \quad \mathcal{N}_1 = \mathcal{N}. \tag{2.4}$$

Moreover, the function $p \to \mathcal{N}_p$ is strictly decreasing if $\mathcal{N} < A$, i. e.,

$$\mathcal{N}_1 \le \mathcal{N}_p \le N_0 \tag{2.5}$$

or is strictly increasing if $\mathcal{N} > A$, i. e.,

$$\mathcal{N}_0 \le \mathcal{N}_p \le \mathcal{N}_1. \tag{2.6}$$

§ 3. Bivariate means used in this paper

In this section we give definitions and formulas of certain bivariate means used in the next section of this paper.

First we define a number v as follows

$$v = \frac{a-b}{a+b}. \tag{3.1}$$

Clearly $0 < |v| < 1$.

The arithmetic, harmonic and contraharmonic means of a and b, denoted respectively by, A, H and C, are defined in usual way

$$A = \frac{a+b}{2}, \qquad H = \frac{2ab}{a+b} \qquad \text{and} \qquad C = \frac{a^2+b^2}{a+b}.$$

One can easily verify that

$$H = A(1-v^2) \qquad \text{and} \qquad C = A(1+v^2).$$

The p-means generated by H and C are denoted, respectively, by H_p and C_p. It is an elementary task to show that

$$H_p = A(1-(pv)^2) \qquad \text{and} \qquad C_p = A(1+(pv)^2). \tag{3.2}$$

We recall now formulas for the four means which play a central role in this paper. They have been introduced and studied in [1]:

$$N_{AG} = \frac{1}{2}A\left(1+(1-v^2)\frac{\tanh^{-1} v}{v}\right), \tag{3.3}$$

$$N_{GA} = \frac{1}{2}A\left(\sqrt{1-v^2}+\frac{\sin^{-1} v}{v}\right), \tag{3.4}$$

$$N_{QA} = \frac{1}{2}A\left(\sqrt{1+v^2}+\frac{\sinh^{-1} v}{v}\right), \tag{3.5}$$

$$N_{AQ} = \frac{1}{2}A\left(1+(1+v^2)\frac{\tan^{-1} v}{v}\right), \tag{3.6}$$

where the symbols G and Q stand for the geometric and the root-square means of a and b, respectively. Recall that

$$G = \sqrt{ab} \qquad \text{and} \qquad Q = \sqrt{\frac{a^2+b^2}{2}}.$$

The four means defined in (3.3)–(3.6) are special cases of the Schwab-Borchardt mean SB which is defined as follows

$$SB(a,b) \equiv SB = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)} & \text{if } a<b, \\ \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)} & \text{if } b<a \end{cases}$$

(see, e.g., [9], [10]). This mean has been studied extensively in [11], [12], and in [13]. It is well known that the mean SB is strict, nonsymmetric and homogeneous of degree one in its variables. The Schwab-Borchardt mean is the iterative mean, i. e.,

$$SB = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n\,,$$

where

$$a_0 = a, \quad b_0 = b, \quad a_{n+1} = \frac{a_n+b_n}{2}\,, \quad b_{n+1} = \sqrt{a_{n+1}b_n}$$

$(n = 0, 1, \ldots)$.

The four means under discussion are defined in terms of SB, A, G and Q as follows: $N_{AG} = SB(A,G)$, $N_{GA} = SB(G,A)$, $N_{QA} = SB(Q,A)$ and $N_{AQ} = SB(A,Q)$.

All the means mentioned above are comparable. It is known that (see [5, Theorem 4.2])

$$H < G < N_{AG} < N_{GA} < A < N_{QA} < N_{AQ} < Q < C. \tag{3.7}$$

As in [1] we call means given in (3.3) – (3.6) the four new means.

§ 4. Optimal bounds for the four new means

The goal of this section is to obtain optimal bounds for the four new means defined in the previous section with the bounding quantities being either the p-mean generated the harmonic mean or the p-mean generated by the contraharmonic mean.

In the proofs presented in this section we will utilize the following result (see, e.g., [14]).

Theorem A. *Let the functions f and g be continuous on $[c,d]$, differentiable on (c,d) and such that $g'(t)\neq 0$ on (c,d). If $f'(t)/g'(t)$ is (strictly) increasing (decreasing) on (c,d), then the functions*

$$\frac{f(t)-f(d)}{g(t)-g(d)} \quad \textit{and} \quad \frac{f(t)-f(c)}{g(t)-g(c)}$$

are also (strictly) increasing (decreasing) on (c,d).

In what follows we will assume that the numbers p and q are belong to the unit interval $[0,1]$. We are in a position to prove the following result.

Theorem 1. *The following two-sided inequality*

$$H_p < N_{AG} < H_q \tag{4.1}$$

is valid provided

$$p \geq \sqrt{\frac{1}{2}} = 0.707106\ldots \qquad \textit{and} \qquad q \leq \sqrt{\frac{1}{3}} = 0.577350\ldots. \tag{4.2}$$

Proof. Making use of (3.3) and (3.2) we see that the two-sided inequality (4.1) inequality can be written as follows

$$q^2 < \frac{1}{2v^2}\left[1-(1-v^2)\frac{\tanh^{-1}v}{v}\right] < p^2.$$

Letting $v=\tanh t$ $(t>0)$ we obtain

$$q^2 < \phi_1(t) < p^2,$$

where

$$\phi_1(t) = \frac{f(t)}{g(t)} := \frac{\sinh t\cosh^2 t - t\cosh t}{2\sinh^3 t}.$$

Differentiation yields

$$\frac{f'(t)}{g'(t)} = \frac{3\sinh 2t - 2t}{6\sinh 2t} = \frac{1}{6}\left(3-\frac{2t}{\sinh 2t}\right) := h_1(t).$$

Since the function $2t/\sinh 2t$ is strictly decreasing on the interval $(0,\infty)$ we conclude that the function $h_1(t)$ is strictly increasing on the same domain. Application of Theorem A leads to the conclusion that the function

$\phi_1(t)$ is also strictly increasing on the interval $(0, \infty)$. One can easily verify that $\phi_1(0^+) = 1/3$ and $\phi_1(\infty^-) = 1/2$. This completes the proof of (4.1) with the domains of validity for p and q as stated in (4.2). □

In the next theorem we give optimal bounds for the mean N_{GA} in terms of the p-mean generated by the harmonic mean.

Theorem 2. *The inequalities*

$$H_p < N_{GA} < H_q \tag{4.3}$$

are satisfied if

$$p \geq \sqrt{1 - \frac{\pi}{4}} = 0.463252\ldots \qquad \textit{and} \qquad q \leq \sqrt{\frac{1}{6}} = 0.408248\ldots. \tag{4.4}$$

Proof. First we shall write the double inequality (4.3) in the equivalent form. Making use of (3.4) together with the application of the first part of (3.2), followed by a substitution $v = \sin t$ $(0 < t < \pi/2)$, yields after a little algebra,

$$q^2 < \phi_2(t) < p^2, \tag{4.5}$$

where

$$\phi_2(t) = \frac{f(t)}{g(t)} := \frac{2\sin t - t - \sin t \cos t}{2\sin^3 t}.$$

Differentiation gives

$$\frac{f'(t)}{g'(t)} = \frac{1}{6}\sec^2\left(\frac{t}{2}\right) := h_2(t).$$

Clearly the function $h_2(t)$ is strictly increasing on the interval $(0, \pi/2)$. We utilize Theorem A again to conclude that the function ϕ_2 is also strictly increasing on the same interval. It is easy to verify that

$$\phi_2(0^+) = \frac{1}{6} \qquad \text{and} \qquad \phi_2\left(\frac{\pi}{2}\right) = 1 - \frac{\pi}{4}.$$

This in conjunction with (4.5) yields the bounds for p and q as stated in (4.4). □

In the following theorem we shall establish optimal bounds C_p and C_q for the mean N_{QA}.

Theorem 3. *If* $p = 0$ *and* $q \geq \sqrt{\frac{1}{6}} = 0.408248\ldots,$ (4.6)

then

$$C_p < N_{QA} < C_q. \tag{4.7}$$

Proof. We follow the lines of the proofs of two theorems already established in this section. Making use of the second formula of (3.2) and (3.5) we can write inequality (4.7) in the equivalent form as

$$p^2 < \frac{1}{v^2}\left[\frac{1}{2}\left(\sqrt{1+v^2} + \frac{\sinh^{-1} v}{v}\right) - 1\right] < q^2.$$

With the substitution $v = \sinh t$ $(t > 0)$ the last double inequality becomes

$$p^2 < \phi_3(t) < q^2, \tag{4.8}$$

where

$$\phi_3(t) = \frac{f(t)}{g(t)} := \frac{\sinh t \cosh t + t - 2\sinh t}{2\sinh^3 t}.$$

Differentiating $f(t)$ and $g(t)$ we obtain, upon simplifications,

$$\frac{f'(t)}{g'(t)} = \frac{1}{6\cosh^2(t/2)} := h_3(t).$$

Clearly function $h_3(t)$ is strictly decreasing on the interval $(0, \infty)$. This in conjunction with Theorem A leads to the conclusion that the function $\phi_3(t)$ is strictly decreasing on the the positive semi-axis. Utilizing L'Hôpital rule we obtain $\phi_3(0^+) = 1/6$ and $\phi_3(\infty^-) = 0$. Combining this with (4.8) yields (4.6). The proof is complete. □

We close this section with the following.

Theorem 4. *The two-sided inequality*

$$C_p < N_{AQ} < C_q \tag{4.9}$$

holds true if

$$p \leq \frac{\sqrt{\pi - 2}}{2} = 0.534226\ldots \quad \text{and} \quad q \geq \sqrt{\frac{1}{3}} = 0.577350\ldots. \tag{4.10}$$

Proof. Making use of the second formula of (3.2) and (3.6) one can rewrite inequality (4.9) as follows

$$p^2 < \frac{1}{2v^2}\left[(1+v^2)\frac{\tan^{-1} v}{v} - 1\right] < q^2.$$

With $v = \tan t$ $(0 < t < \pi/4)$ the last two-sided inequality becomes

$$p^2 < \phi_4(t) < q^2, \tag{4.11}$$

where

$$\phi_4(t) = \frac{f(t)}{g(t)} := \frac{t\cos t - \sin t \cos^2 t}{2\sin^3 t}.$$

This yields

$$\frac{f'(t)}{g'(t)} = \frac{1}{6}\big(3 - \frac{2t}{\sin 2t}\big) := h_4(t).$$

Taking into account that the function $2t/\sin 2t$ is strictly increasing on the interval $(0, \pi/4)$ we conclude that the function $h_4(t)$ is strictly decreasing on the same interval. This in turn, in view of Theorem A, implies that the function $\phi_4(t)$ is strictly decreasing. Simple calculations yield

$$\phi_4(0^+) = \frac{1}{3} \qquad \text{and} \qquad \phi_4(\pi/4) = \frac{\pi - 2}{4}.$$

This in conjunction with (4.11) yield the bounds (4.10). □

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