Probl. Anal. Issues Anal. Vol. 3 (21), No. 2, 2014, pp. 3–15

DOI: 10.15393/j3.art.2014.2689

UDC 510.225

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ABOUT PLANAR (α, β)–ACCESSIBLE DOMAINS

Abstract. The article is devoted to the class $A_\rho^{\alpha,\beta}$ of all (α,β)–accessible with respect to the origin domains D, $\alpha,\beta \in [0,1)$, possessing the property $\rho = \min\limits_{p\in\partial D} |p|$, where $\rho \in (0,+\infty)$ is a fixed number. We find the maximal set of points a such that all domains $D \in A_\rho^{\alpha,\beta}$ are (γ,δ)–accessible with respect to a, $\gamma \in [0;\alpha]$, $\delta \in [0;\beta]$. This set is proved to be the closed disc of center 0 and radius $\rho \sin \dfrac{\varphi\pi}{2}$, where $\varphi = \min\{\alpha-\gamma, \beta-\delta\}$.

Key words: *α–accessible domain, (α,β)–accessible domain, cone condition*

2010 Mathematical Subject Classification: *52A30, 03E15*

In [1] the notion of α–accessible domain was introduced. Let $\alpha \in [0,1)$ be a fixed number. A domain $D \subset \mathbb{R}^n$, $0 \in D$, is called *α–accessible* if for every point $p \in \partial D$ there exists a number $r = r(p) > 0$ such that the cone

$$K_+(p,\alpha,r) = \left\{ x \in \mathbb{R}^n : \|x\| \le r, \left(x-p, \frac{p}{\|p\|} \right) \ge \|x-p\| \cos \frac{\alpha\pi}{2} \right\}$$

is included in $\mathbb{R}^n \backslash D$.

In the case $n = 2$ these domains have been studied earlier by J. Stankiewicz [2—4], D. A. Brannan and W. E. Kirwan [5], W. Ma and D. Minda [6], T. Sugawa [7] and others as a generalization of starlike domains. It was noted (see, for example, [7–10]) that in the planar case it is possible to consider domains that possess a more general property. In [9, 10] such domains were called (α,β)–accessible.

Definition 1. [9], [10] *Let $\alpha,\beta \in [0,1)$, $D \subset \mathbb{C}$, $a \in D$. A domain D is called (α,β)–accessible with respect to a if for every point $p \in \partial D$ there* *exists a number* $r = r(p) > 0$ *such that the cone*

$$K_+(p, a, \alpha, \beta, r) = \\ = \left\{ z \in \mathbb{C} : -\frac{\beta\pi}{2} \le \operatorname{Arg}(z-p) - \operatorname{Arg}(p-a) \le \frac{\alpha\pi}{2}, |z-p| \le r \right\}$$

is contained in $\mathbb{C} \backslash D$.

The notion of (α, β)–accessible domain is a generalization of the notion of α–accessible domain. They are equal if $\alpha = \beta$.

In the article we choose values of arguments so that their difference belongs to $(-\pi; \pi]$.

It was shown in [1] and [9, 10] that α– and (α, β)–accessible domains satisfy the so-called "cone condition", i. e. such domains are also conically accessible from the interior.

The problem of characterization all domains with the "cone condition" is very hard. α– and (α, β)–accessible domains are only special but important cases of such domains. For α–accessible domains the axis of symmetry of the cone is radial, for (α, β)–accessible domains D the angle between this axis and the vector $p - a$, $p \in \partial D$, is fixed.

The following criterion of (α, β)–accessibility is needed for the sequel.

Theorem A. [9, 10] *Let* $D \subset \mathbb{C}$, ∂D *be smooth,* $n(p)$ *be an outward normal to the domain* D *at a point* $p \in \partial D$, $\alpha, \beta \in (0; 1)$. *Then the domain* D *is* (α, β)*–accessible with respect to the origin if and only if*

$$-\frac{(1-\beta)\pi}{2} \le \operatorname{Arg}(p) - \operatorname{Arg}(n(p)) \le \frac{(1-\alpha)\pi}{2} \tag{1}$$

for every $p \in \partial D$.

Let us note that Theorem A can be applied locally. In particular, if Γ is an open smooth curve, $\Gamma \subset \partial D$, a point $p \in \Gamma$, then from the proof of Theorem A it follows that the unbounded cone

$$K_+(p, 0, \alpha, \beta) = \left\{ z \in \mathbb{C} : -\frac{\beta\pi}{2} \le \operatorname{Arg}(z-p) - \operatorname{Arg} p \le \frac{\alpha\pi}{2} \right\}$$

is included in $\mathbb{C} \backslash D$.

Consider the class $A_\rho^{\alpha,\beta}$, containing all (α, β)–accessible with respect to the origin domains D, possessing the property $\min\limits_{p \in \partial D} |p| = \rho$, where

$\rho \in (0, \infty)$ is a fixed number. Let $\gamma \in [0; \alpha]$, $\delta \in [0; \beta]$, $D \in A_\rho^{\alpha,\beta}$. By $\Omega_D^{\gamma,\delta}$ denote the set of all points $a \in D$ such that the domain D is (γ, δ)–accessible with respect to a.

In the present paper we find the maximal set that is contained in $\Omega_D^{\gamma,\delta}$ for all domains D from $A_\rho^{\alpha,\beta}$.

By $\mathbb{B}\,[0, R]$ denote the disc $\{z \in \mathbb{C} : |z| \leq R\}$.

Theorem 1. *If $\alpha, \beta \in [0; 1)$, $\gamma \in [0; \alpha]$, $\delta \in [0; \beta]$ then*

$$\bigcap_{D \in A_\rho^{\alpha,\beta}} \Omega_D^{\gamma,\delta} = \mathbb{B}\left[0, \rho \sin \frac{\varphi\pi}{2}\right],$$

where $\varphi = \min\{\alpha - \gamma, \beta - \delta\}$.

Proof. Let us show that the set $\bigcap\limits_{D \in A_\rho^{\alpha,\beta}} \Omega_D^{\gamma,\delta}$ is a disc. If $D \in A_\rho^{\alpha,\beta}$, then a domain $U(D)$, obtained by rotation of D with respect to the origin, also belongs to $A_\rho^{\alpha,\beta}$. Therefore the set $\bigcap\limits_{U} \Omega_{U(D)}^{\gamma,\delta}$, where the intersection extends over all rotation transformations of D, is a disc with center at the origin. Hence,

$$\bigcap_{D \in A_\rho^{\alpha,\beta}} \Omega_D^{\gamma,\delta} = \bigcap_{D \in A_\rho^{\alpha,\beta}} \bigcap_{U} \Omega_{U(D)}^{\gamma,\delta}$$

is a disc too.

Let $\alpha, \beta \in [0; 1)$, $D \in A_\rho^{\alpha,\beta}$. Fix $p \in \partial D$. Since the domain D is (α, β)–accessible with respect to the origin, then $K_+(p, 0, \alpha, \beta, r) \subset \mathbb{C} \backslash D$ for some $r > 0$. Let us show that for all points a from the intersection of the domain D and the cone

$$K_-(p, 0, \alpha - \gamma, \beta - \delta) = \\ = \left\{ z \in \mathbb{C} : -\frac{(\beta - \delta)\pi}{2} \leq \operatorname{Arg}(z - p) - \operatorname{Arg}(-p) \leq \frac{(\alpha - \gamma)\pi}{2} \right\}$$

the following inclusion holds

$$K_+(p, a, \gamma, \delta, r) \subset K_+(p, 0, \alpha, \beta, r).$$

Take $a \in K_-(p, 0, \alpha - \gamma, \beta - \delta) \cap D$, then

$$-\frac{(\beta - \delta)\pi}{2} \leq \operatorname{Arg}(a - p) - \operatorname{Arg}(-p) \leq \frac{(\alpha - \gamma)\pi}{2}. \tag{2}$$

Let $z \in K_+(p, a, \gamma, \delta, r)$, $z \neq p$, this means that $|z - p| < r$ and

$$-\frac{\delta\pi}{2} \leq \operatorname{Arg}(z-p) - \operatorname{Arg}(p-a) \leq \frac{\gamma\pi}{2}. \tag{3}$$

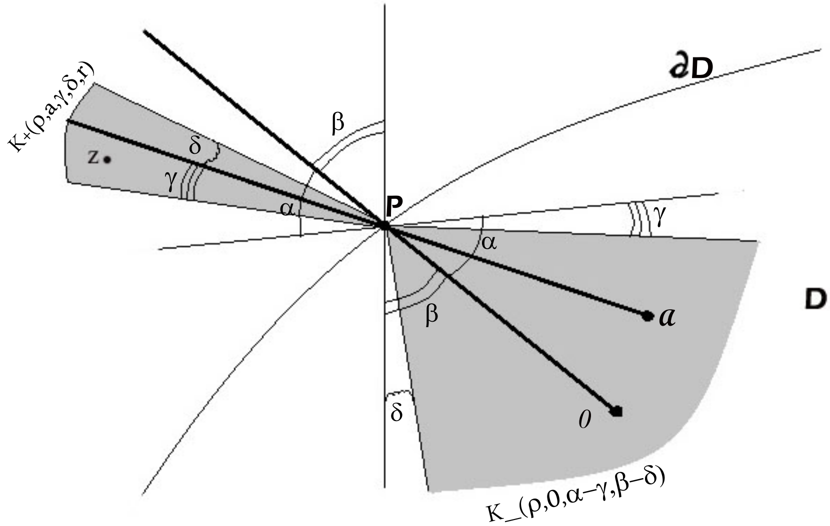

Figure 1: The domain D, case $|\operatorname{Arg}(z-p) - \operatorname{Arg} p| \geq |\operatorname{Arg}(p-a) - \operatorname{Arg} p|$

If $|\operatorname{Arg}(z-p) - \operatorname{Arg} p| \geq |\operatorname{Arg}(p-a) - \operatorname{Arg} p|$ (see fig. 1), then, by (**??**) and (**??**),

$$-\frac{\beta\pi}{2} = -\frac{\delta\pi}{2} - \frac{(\beta-\delta)\pi}{2} \leq \operatorname{Arg}(z-p) - \operatorname{Arg} p =$$

$$= (\operatorname{Arg}(z-p) - \operatorname{Arg}(p-a)) + (\operatorname{Arg}(p-a) - \operatorname{Arg} p) \leq$$

$$\leq \frac{\gamma\pi}{2} + \frac{(\alpha-\gamma)\pi}{2} = \frac{\alpha\pi}{2},$$

and therefore $z \in K_+(p, 0, \alpha, \beta, r)$. In the case

$$|\operatorname{Arg}(z-p) - \operatorname{Arg} p| < |\operatorname{Arg}(p-a) - \operatorname{Arg} p| \text{ (see fig. 2)},$$

we have $z \in K_+(p, 0, \alpha - \gamma, \beta - \delta, r)$ and therefore $z \in K_+(p, 0, \alpha, \beta, r)$. Consequently for both cases

$$K_+(p, a, \gamma, \delta, r) \subset K_+(p, 0, \alpha, \beta, r) \subset \mathbb{C} \backslash D.$$

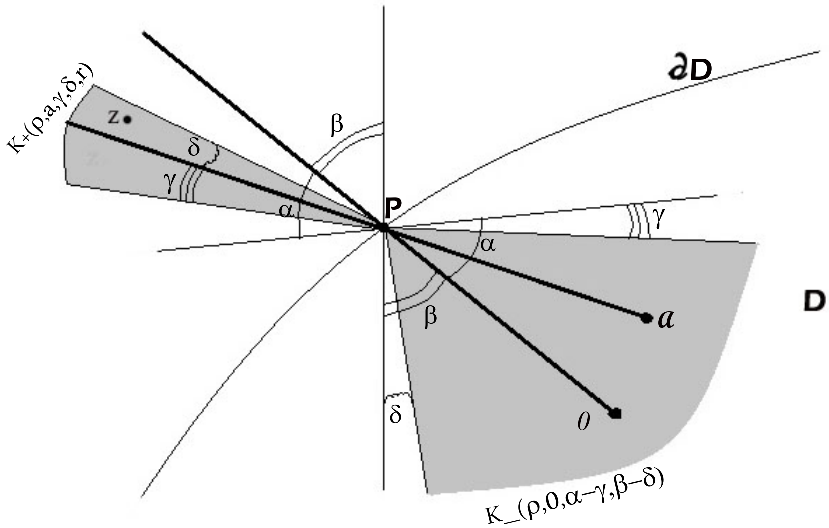

Figure 2: The domain D, case $|\operatorname{Arg}(z-p) - \operatorname{Arg} p| < |\operatorname{Arg}(p-a) - \operatorname{Arg} p|$

Inscribe a disc with center at the origin into the set

$$K_-(p, 0, \alpha-\gamma, \beta-\delta) \cap D$$

and find its radius $R(p)$. Denote by y a point from $\partial K_-(p, 0, \alpha-\gamma, \beta-\delta)$ such that

$$|y| = \operatorname{dist}(0, \partial K_-(p, 0, \alpha-\gamma, \beta-\delta)).$$

Then, from the right triangle $0, y, p$, we obtain $R(p) = |y| = |p| \sin \dfrac{\varphi\pi}{2}$, where $\varphi = \min\{\alpha-\gamma, \beta-\delta\}$. Put

$$R = \min_{p \in \partial D} R(p) = \rho \sin \frac{\varphi\pi}{2}.$$

Then the disc $\mathbb{B}\,[0, R]$ is contained in $K_-(p, 0, \alpha-\gamma, \beta-\delta) \cap D$ for all $p \in \partial D$. Thus $\mathbb{B}\,[0, R] \subset \Omega_D^{\gamma,\delta}$ for all $D \in A_\rho^{\alpha,\beta}$.

Let us prove that it is impossible to enlarge the constant R in the last inclusion. For this aim we find a domain $D_0 \in A_\rho^{\alpha,\beta}$ such that $\mathbb{B}\,[0, R] \subset$ $\subset \Omega_{D_0}^{\gamma,\delta}$, but for every $\varepsilon > 0$

$$\mathbb{B}\,[0, R+\varepsilon] \not\subseteq \Omega_{D_0}^{\gamma,\delta}.$$

a) Let us begin with the case $\alpha, \beta \in (0; 1)$, $\gamma \in [0; \alpha]$, $\delta \in [0; \beta]$. Consider the simply connected domain D_0, $0 \in D_0$, bounded by the logarithmic spirals

$l_\alpha(\varphi) = \rho\, e^{i\varphi}\, e^{\varphi\, \mathrm{tg}\, \frac{(1-\alpha)\pi}{2}}, 0 \le \varphi \le \frac{\pi}{2},$

$l_\beta(\varphi) = \rho\, e^{i\varphi}\, e^{-\varphi\, \mathrm{tg}\, \frac{(1-\beta)\pi}{2}}, -\frac{\pi}{2} \le \varphi \le 0,$

and the circle

$l(\varphi) = \rho\, e^{\frac{\pi}{2}\, \mathrm{tg}\, \frac{(1-\alpha)\pi}{2}}\, e^{i\varphi}, -\pi < \varphi \le \pi,$ if $\alpha \ge \beta$

or $l(\varphi) = \rho\, e^{\frac{\pi}{2}\, \mathrm{tg}\, \frac{(1-\beta)\pi}{2}}\, e^{i\varphi}, -\pi < \varphi \le \pi,$ if $\alpha < \beta$ (see fig. 3).

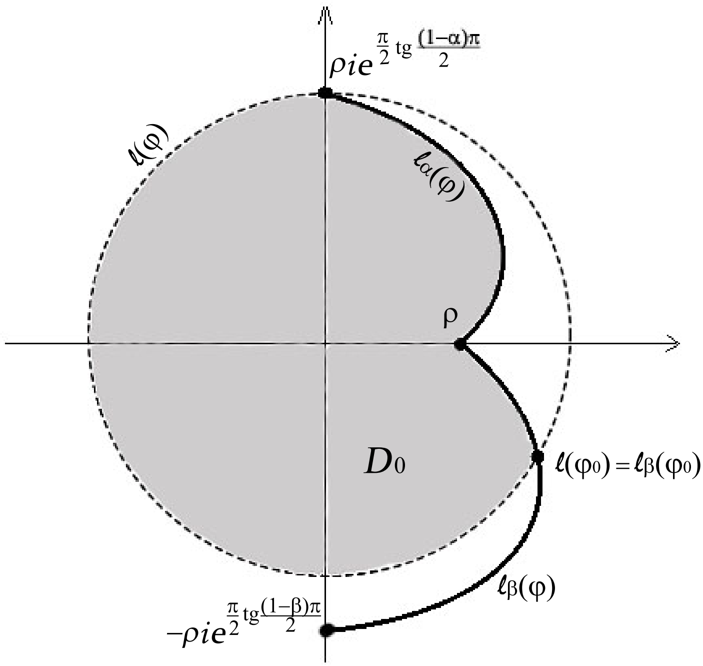

Figure 3: The domain D_0, case $\alpha, \beta \in (0;1)$

We will consider the case $\alpha \ge \beta$ only. The proof for $\alpha < \beta$ is analogous.

Let us check that $D_0 \in A_\rho^{\alpha,\beta}$. Note that $\min\limits_{p \in \partial D} |p| = |l_\alpha(0)| = |l_\beta(0)| = \rho$. Show that domain D_0 is (α, β)–accessible with respect to the origin. Take $p \in \partial D_0$. Prove that the unbounded cone $K_+(p, 0, \alpha, \beta)$ is contained in $\mathbb{C} \backslash D_0$. Denote by $n(p)$ the outward normal to the domain D_0 at the point p if such a normal exists. Divide the proof into six cases.

a 1) Let $p = l_\alpha(\varphi)$, $\varphi \in \left(0; \frac{\pi}{2}\right)$. Then

$$\operatorname{Arg} p - \operatorname{Arg} n(p) = \operatorname{Arg} p - \left(\operatorname{Arg}\, l'_\alpha(\varphi) - \frac{\pi}{2}\right) =$$

$$= \varphi - \arg\left(\rho e^{i\varphi} e^{\varphi \operatorname{tg}\frac{(1-\alpha)\pi}{2}}\left(i + \operatorname{tg}\frac{(1-\alpha)\pi}{2}\right)\right) + \frac{\pi}{2} =$$

$$= \varphi - \varphi - \operatorname{arctg}\frac{1}{\operatorname{tg}\frac{(1-\alpha)\pi}{2}} + \frac{\pi}{2} = -\operatorname{arctg}\left(\operatorname{tg}\frac{\alpha\pi}{2}\right) + \frac{\pi}{2} = \frac{(1-\alpha)\pi}{2}.$$

By Theorem A, for such p we have $K_+(p,0,\alpha,\beta) \subset \mathbb{C}\backslash D_0$.

a 2) If $p = l_\beta(\varphi)$, $\varphi \in (\varphi_0; 0)$, where φ_0 is the solution of the equation $l(\varphi_0) = l_\beta(\varphi_0)$. Then

$$\operatorname{Arg} p - \operatorname{Arg} n(p) = \operatorname{Arg} p - \left(\operatorname{Arg}\, l'_\beta(\varphi) - \frac{\pi}{2}\right) =$$

$$= \varphi - \arg\left(\rho e^{i\varphi} e^{-\varphi \operatorname{tg}\frac{(1-\beta)\pi}{2}}\left(i - \operatorname{tg}\frac{(1-\beta)\pi}{2}\right)\right) + \frac{\pi}{2} =$$

$$= \varphi - \varphi - \pi + \operatorname{arctg}\frac{1}{\operatorname{tg}\frac{(1-\beta)\pi}{2}} + \frac{\pi}{2} = -\frac{(1-\beta)\pi}{2}. \tag{4}$$

Consequently, by Theorem A, $K_+(p,0,\alpha,\beta) \subset \mathbb{C}\backslash D_0$.

a 3) Let $p = l(\varphi)$, $\varphi \in (-\pi;\varphi_0) \cup \left(\frac{\pi}{2};\pi\right]$. In this case

$$K_+(p,0,\alpha,\beta) \subset K_+(p,0,1,1) \subset \mathbb{C}\backslash D_0.$$

a 4) Consider

$$p = l\left(\frac{\pi}{2}\right) = l_\alpha\left(\frac{\pi}{2}\right) = \rho i e^{\frac{\pi}{2}\operatorname{tg}\frac{(1-\alpha)\pi}{2}}.$$

Since $\arg l'_\alpha\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \frac{\alpha\pi}{2}$, $\arg l'\left(\frac{\pi}{2}\right) = \pi$ (here and below in *a 5), a 6), b 3), b 5)* we consider one–sided derivatives), and $\arg_\alpha l(\varphi)$, $\arg l(\varphi)$ are monotone we get

$$K_+(p,0,\alpha,\beta) \subset K_+(p,0,1,1) \subset \mathbb{C}\backslash D_0.$$

a 5) Let $p = l_\alpha(0) = l_\beta(0) = \rho$. Since $\arg l'_\alpha(0) = \frac{\alpha\pi}{2}$ and $\arg l'_\beta(0) = = \pi - \frac{\beta\pi}{2}$, then $K_+(p,0,\alpha,\beta) \subset \mathbb{C}\backslash D_0$ and for all $\varepsilon_1,\varepsilon_2 \geq 0$, ${\varepsilon_1}^2 + {\varepsilon_2}^2 \neq 0$,

$$K_+(p,0,\alpha+\varepsilon_1,\beta+\varepsilon_2) \cap D_0 \neq \emptyset.$$

Moreover, both rays of $\partial K_+(p,0,\alpha+\varepsilon_1,\beta+\varepsilon_2)$ intersect the domain D_0.

a 6) The last case is $p = l(\varphi_0) = l_\beta(\varphi_0)$. Here $\arg l'(\varphi_0) = \varphi_0 + \dfrac{\pi}{2}$ and $\arg l'_\beta(\varphi_0) = \varphi_0 + \pi - \dfrac{\pi\beta}{2}$. Thus, $K_+(p, 0, \alpha, \beta) \subset K_+(p, 0, 1, 1) \subset \mathbb{C}\backslash D_0$.

Consequently $K_+(p, 0, \alpha, \beta) \subset \mathbb{C}\backslash D_0$ for all $p \in \partial D_0$. Therefore $D_0 \in \in A_\rho^{\alpha,\beta}$.

Now we will show that $\mathbb{B}\left[0, \rho\sin\dfrac{\varphi\pi}{2}\right]$ is the maximal disc, contained in $\Omega_{D_0}^{\gamma,\delta}$.

Let $t \in (0, 1-\varphi)$. Fix

$$a^* \in \partial\mathbb{B}\left[0, \rho\sin\frac{(\varphi+t)\pi}{2}\right] \cap K_-(\rho, 0, \varphi+t, \varphi+t),$$

such that $\operatorname{Im} a^* > 0$ if $\varphi = \beta - \delta$ and $\operatorname{Im} a^* < 0$ if $\varphi = \alpha - \gamma$ (see fig. 4).

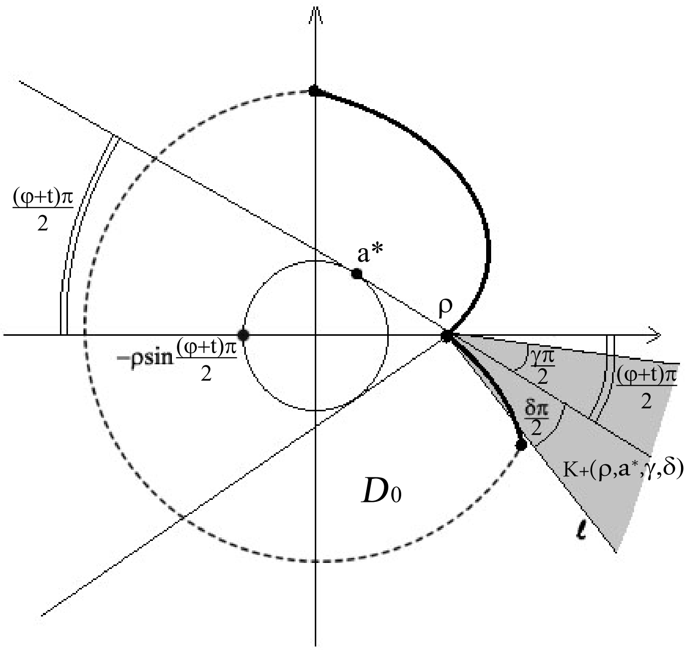

Figure 4: Intersection D_0 and $K_+(\rho, a^*, \gamma, \delta)$, case $\alpha, \beta \in (0;1)$

Further we consider the case $\varphi = \beta - \delta$ only, because the proof for the case $\varphi = \alpha - \gamma$ is analogous. Let us show that $a^* \notin \Omega_{D_0}^{\gamma,\delta}$. By definition of a^*,

$$\operatorname{Arg}\rho - \operatorname{Arg}(\rho - a^*) = (\varphi+t)\frac{\pi}{2}. \tag{5}$$

Denote by l the ray, consisting of points $w \in \partial K_+(\rho, a^*, \gamma, \delta)$, $w \neq \rho$, such that

$$\operatorname{Arg}(w-\rho) - \operatorname{Arg}(\rho - a^*) = -\frac{\delta\pi}{2}. \tag{6}$$

By (**??**) and (**??**), for every $w \in l$

$$\operatorname{Arg}(w-\rho) - \operatorname{Arg}(\rho) = (\operatorname{Arg}(w-\rho) - \operatorname{Arg}(\rho - a^*)) + (\operatorname{Arg}(\rho - a^*) - \operatorname{Arg}\rho) =$$

$$= -\frac{\delta\pi}{2} - \frac{(\varphi+t)\pi}{2} = -\frac{(\beta+t)\pi}{2}.$$

Consequently, l is one of the rays of $\partial K_+(\rho, 0, \alpha, \beta+t)$. As it was proved above (see *a5*)) for every $t > 0$

$$l \cap D_0 \neq \emptyset.$$

Therefore $a^* \notin \Omega_{D_0}^{\gamma,\delta}$. Since positive t is arbitrary, we obtain that

$$\mathbb{B}\left[0, \rho \sin\frac{\varphi\pi}{2}\right]$$

is the maximal disc, contained in $\Omega_{D_0}^{\gamma,\delta}$.

b) Let $\alpha = 0$, $\beta \in (0;1)$, $\delta \in [0;\beta]$. Consider the domain $D_0 \subset \mathbb{C}$, $0 \in D_0$, bounded by the logarithmic spiral

$$l_\beta(\varphi) = \rho\, e^{i\varphi}\, e^{-\varphi \operatorname{tg}\frac{(1-\beta)\pi}{2}}, -\frac{\pi}{2} \leq \varphi \leq 0,$$

the circle

$$l(\varphi) = \rho\, e^{\frac{\pi}{2}\operatorname{tg}\frac{(1-\beta)\pi}{2}}\, e^{i\varphi}, \quad \varphi \in \left(-\pi, -\frac{\pi}{2}\right] \cup [0, \pi],$$

and the segment $\left[\rho; \rho e^{\frac{\pi}{2}\operatorname{tg}\frac{(1-\beta)\pi}{2}}\right]$ (see fig. 5). Show that $D_0 \in A_\rho^{0,\beta}$.

Let us prove that domain D_0 is $(0, \beta)$–accessible with respect to the origin. Fix $p \in \partial D_0$. Show that $K_+(p, 0, 0, \beta) \subset \mathbb{C} \backslash D_0$.

b 1) If $p = l_\beta(\varphi)$, $\varphi \in \left(-\frac{\pi}{2}; 0\right)$, then (**??**) is true (see *a1*)) and

$$K_+(p, 0, 0, \beta) \subset K_+(p, 0, \alpha, \beta) \subset \mathbb{C} \backslash D_0.$$

b 2) If $p = l(\varphi)$, $\varphi \in \left(-\pi; -\frac{\pi}{2}\right) \cup (0; \pi)$, then

$$K_+(p, 0, 0, \beta) \subset K_+(p, 0, 1, 1) \subset \mathbb{C} \backslash D_0.$$

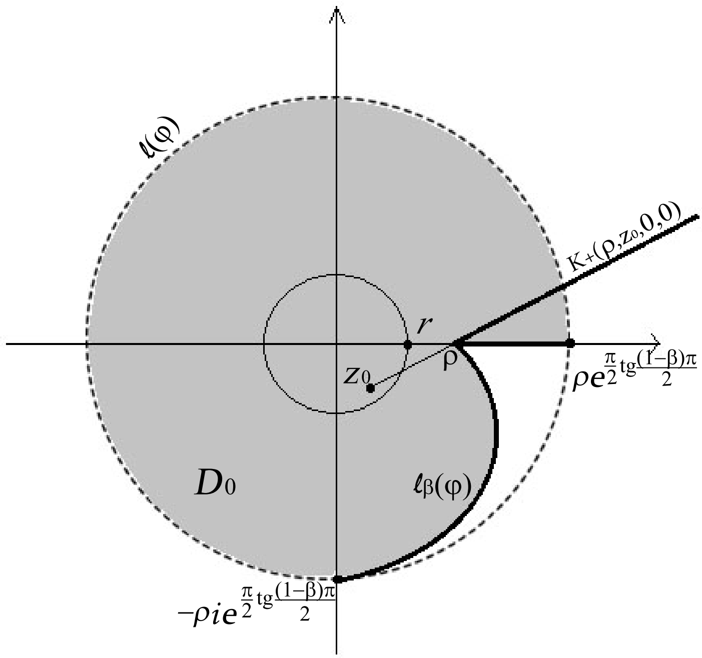

Figure 5: The domain D_0, case $\alpha = 0$, $\beta \in (0;1)$

b 3) Let $p = l_\beta(0) = \rho$. In this case $\arg l'_\beta(0) = \pi - \dfrac{\beta\pi}{2}$. In addition $\arg l_\beta(\varphi)$ is monotone. Hence, $K_+(p,0,0,\beta) \subset \mathbb{C}\backslash D_0$.

b 4) Consider the case $p \in \left(\rho; \rho e^{\frac{\pi}{2}\,\mathrm{tg}\,\frac{(1-\beta)\pi}{2}}\right)$. By *b3*), we have

$$K_+(p,0,0,\beta) \subset K_+(\rho,0,0,\beta) \subset \mathbb{C}\backslash D_0.$$

b 5) Let $p = l\left(-\dfrac{\pi}{2}\right) = l_\beta\left(-\dfrac{\pi}{2}\right) = -\rho i e^{\frac{\pi}{2}\,\mathrm{tg}\,\frac{(1-\beta)\pi}{2}}$. Since

$$\arg l'\left(-\frac{\pi}{2}\right) = 0 \text{ and } \arg l'_\beta\left(-\frac{\pi}{2}\right) = \frac{(2-\beta)\,\pi}{2} - \frac{\pi}{2},$$

and $\arg l_\beta(\varphi)$ is monotone, we get

$$K_+(p,0,0,\beta) \subset K_+(p,0,1,1) \subset \mathbb{C}\backslash D_0.$$

Summarizing everything proved above we obtain that $K_+(p,0,0,\beta) \subset \subset \mathbb{C}\backslash D_0$ for all $p \in \partial D_0$. Since, in addition, $\min\limits_{p\in\partial D_0} |p| = |l_\beta(0)| = \rho$, we conclude that $D_0 \in A_\rho^{0,\beta}$.

Let us check that $\mathbb{B}\,[0,0] = \{0\}$ is the maximal disc, contained in $\Omega_{D_0}^{0,\delta}$. Suppose that for some $r > 0$

$$\mathbb{B}\,[0,r] \subset \Omega_{D_0}^{0,\delta}.$$

Let $z_0 \in \mathbb{B}\,[0,r]$ and $\operatorname{Im} z_0 < 0$. Then, by construction of D_0,

$$K_+(\rho, z_0, 0, 0) \cap D_0 \neq \emptyset,$$

see fig. 3. Since $K_+(\rho, z_0, 0, 0) \subset K_+(\rho, z_0, 0, \delta)$, then

$$K_+(\rho, z_0, 0, \delta) \cap D_0 \neq \emptyset.$$

Therefore, $z_0 \notin \Omega_{D_0}^{0,\delta}$. This contradiction shows that

$$\mathbb{B}\,[0,r] \nsubseteq \Omega_{D_0}^{0,\delta}$$

for every $r > 0$.

c) In the case $\beta = 0$, $\alpha \in (0;1)$, $\gamma \in [0;\alpha]$, we consider the domain $D_0 \subset \mathbb{C}$, $0 \in D_0$, bounded by the logarithmic spiral

$$l_\alpha(\varphi) = \rho\, e^{i\varphi}\, e^{\varphi \operatorname{tg} \frac{(1-\alpha)\pi}{2}}, 0 \le \varphi \le \frac{\pi}{2},$$

the circle

$$l(\varphi) = \rho\, e^{\frac{\pi}{2} \operatorname{tg} \frac{(1-\alpha)\pi}{2}}\, e^{i\varphi}, \quad \varphi \in (-\pi, 0] \cup \left[\frac{\pi}{2}, \pi\right],$$

and the segment $\left[\rho; \rho e^{\frac{\pi}{2} \operatorname{tg} \frac{(1-\alpha)\pi}{2}}\right]$ (see fig. 6).

Arguing as in case b), taking z_0, $\operatorname{Im} z_0 > 0$, we prove that $D_0 \in A_\rho^{\alpha,0}$ and $\mathbb{B}\,[0,0]$ is the maximal disc, contained in $\Omega_{D_0}^{\gamma,0}$.

d) Let $\alpha = \beta = 0$. In this case, the class of $(0,0)$–accessible domains coincides with the class of 0–accessible domains and the class of starlike with respect to the origin domains (see [11, 12]).

Consider domain $D_0 = \mathbb{C} \backslash l_\rho \in A_\rho^{0,0}$, where $l_\rho = \{\rho t, t \ge 1\}$. Then the set $\Omega_{D_0}^{0,0}$ consists of all points $a \in D_0$ such that $K_+(p, a, 0, 0) \subset l_\rho$ for every $p = \rho\tau$, $\tau \ge 1$. Consequently, $\Omega_{D_0}^{0,0} = \{\rho k, k < 1\}$. Therefore, for all $\varepsilon > 0$

$$\mathbb{B}\,[0,\varepsilon] \nsubseteq \Omega_{D_0}^{0,0} \text{ and } \bigcap_{D \in A_\rho^{0,0}} \Omega_{D_0}^{0,0} = \{0\}.$$

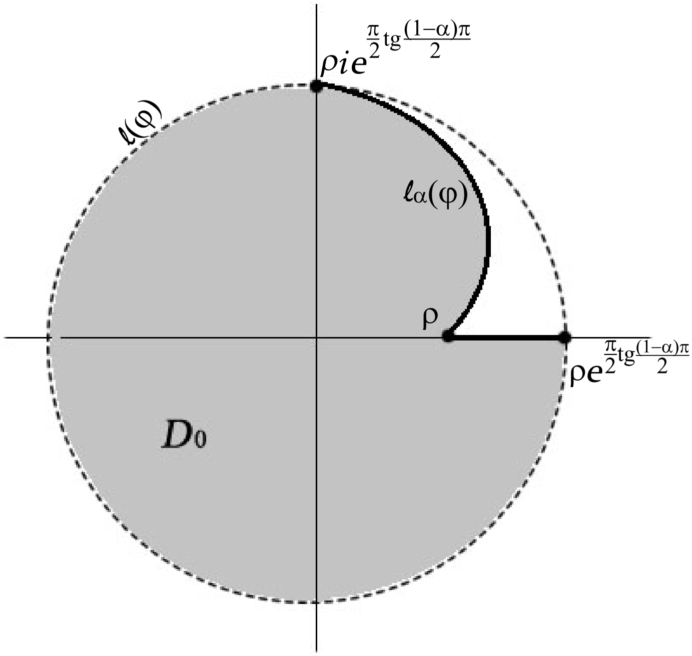

Figure 6: The domain D_0, case $\alpha \in (0;1)$, $\beta = 0$

□

Acknowledgment. This work was supported by Program of Strategic Development of Petrozavodsk State University.

References

[1] Liczberski P., Starkov V. V. *Domains in* $\mathbb{R}^n$ *with conical accessible boundary.* J. Math. Anal. Appl., 2013, vol. 408, no. 2, pp. 547–560.

[2] Stankiewicz J. *Quelques problèmes extrémaux dans les classes des fonctions* α*–angulairement étoilées.* Ann. Univ. Mariae Curie-Sklodowska, Sectio A, 1966, vol. XX, pp. 59–75.

[3] Stankiewicz J. *Some remarks concerning starlike functions.* Bulletin de l'académie Polonaise des sciences. Série des sciences math., astr. et phys., 1970, vol. XVIII, no. 3, pp. 143–146.

[4] Stankiewicz J. *On a family of starlike functions.* Ann. Univ. Mariae Curie-Sklodowska, Sectio A, 1968/1969/1970, vol. XXII/XXIII/XXIV, pp. 175–181.

[5] Brannan D. A. and Kirwan W. E. *On some classes of bounded univalent functions.* J. London Math. Soc., 1969, vol. 2, no. 1, pp. 431–443.

[6] Ma W. and Minda D. *An internal geometric characterization of strongly starlike functions.* Ann. Univ. Mariae Curie-Sklodowska, Sectio A, 1991, vol. XLV, pp. 89–97.

[7] Sugawa T. *A self-duality of strong starlikeness.* Kodai Math. J., 2005, vol. 28, pp. 382–389. DOI: 10.2996%2Fkmj%2F1123767018.

[8] Lecko A. and Stankiewicz J. *An internal geometric characterization of some subclasses of starlike functions.* XVIth Rolf Nevanlinna Colloquium, Joensuu, 1995 (I. Laine and O. Martio, eds.), Walter de Gruyter, Berlin, 1997, pp. 231–238.

[9] Anikiev A. N. *Plane domains with special cone condition.* Russian Mathematics (Izvestiya VUZ. Matematika), 2014, vol. 58, no. 2, pp. 62–63. DOI: 10.3103/S1066369X14020108.

[10] Anikiev A. N. *Plane domains with special cone condition.* Probl. Anal. Issues Anal., 2014, vol. 3 (21), no. 1, pp. 16–31. DOI: 10.15393/j3.art.2014.2609.

[11] Amozova K. F., Starkov V. V. *α–accessible domains, a nonsmooth case.* Izv. Sarat. Univ. N. S. Ser. Math. Mech. Inform., 2013, vol. 13, no. 3, pp. 3–8.

[12] Amozova K. F. *Sufficient conditions of α–accessibility of domain in nonsmooth case.* Probl. Anal. Issues Anal., 2013, vol. 2 (20), no. 1, pp. 3–13.

Received September 3, 2014.
In revised form, October 4, 2014.

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