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ON THE GENERALIZED CONVEXITY AND CONCAVITY

Abstract. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is (m_1, m_2)-convex (concave) if $f(m_1(x, y)) \leq (\geq)\, m_2(f(x), f(y))$ for all $x, y \in \mathbb{R}_+ = (0, \infty)$ and m_1 and m_2 are two mean functions. Anderson et al. [1] studies the dependence of (m_1, m_2)-convexity (concavity) on m_1 and m_2 and gave the sufficient conditions of (m_1, m_2)-convexity and concavity of a function defined by Maclaurin series. In this paper, we make a contribution to the topic and study the (m_1, m_2)-convexity and concavity of a function where m_1 and m_2 are identric and Alzer mean. As well, we prove a conjecture posed by Bruce Ebanks in [2].

Key words: *logarithmic mean, identric mean, power mean, Alzer mean, convexity and concavity property, Ebanks' conjecture*

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1. Introduction. A function $M : (0, \infty) \times (0, \infty) \to (0, \infty)$ is called a Mean function if

1) $M(x, y) = M(y, x)$,
2) $M(x, x) = x$,
3) $x < M(x, y) < y$, whenever $x < y$,
4) $M(a\,x, a\,y) = a\,M(x, y)$ for all $a > 0$,

Some examples of mean functions of two distinct positive real numbers are given below:

$$\textit{Arithmetic mean:} \qquad A = A(x, y) = \frac{x + y}{2},$$
$$\textit{Geometric mean:} \qquad G = G(x, y) = \sqrt{xy},$$

$$\begin{aligned}
&\textit{Harmonic mean:} && H = H(x,y) = \frac{1}{A(1/x,1/y)},\\
&\textit{Logarithmic mean:} && L = L(x,y) = \frac{x-y}{\log(x)-\log(y)},\\
&\textit{Identric mean:} && I = I(x,y) = \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{1/(x-y)},\\
&\textit{Alzer mean:} && J_p = J_p(x,y) = \frac{p}{p+1}\,\frac{x^{p+1}-y^{p+1}}{x^p-y^p}, \quad p\neq 0,-1,\\
&\textit{Power mean:} && M_t = M_t(x,y) = \begin{cases} \left(\dfrac{x^t+y^t}{2}\right)^{1/t}, & t\neq 0,\\ \sqrt{x\,y}, & t=0\,. \end{cases}
\end{aligned}$$

It is easy to observe that $J_1(x,y) = A(x,y)$, $J_0(x,y) = L(x,y)$, $J_{-2}(x,y) = H(x,y)$. For the historical background of these means we refer the reader to see [3]–[7] and the bibliography of these papers.

Before we introduce the earlier results from the literature we recall the following definition, see [1, 8].

Definition 1. *Let $f : I_0 \to (0,\infty)$ be continuous, where I_0 is a subinterval of $(0,\infty)$. Let M and N be two any mean functions. We say that the function f is MN-convex (concave) if*

$$f(M(x,y)) \leq (\geq) N(f(x),f(y)) \quad \textit{for} \quad \textit{all } x,y\in I_0\,.$$

Throughout the paper, the notion I_0 is reserved for the sub-internal of $(0,\infty)$.

In [1], Anderson, Vamanamurthy and Vuorinen studied the convexity and concavity of a function f with respect two mean values, and gave the following detailed result:

Lemma 1. [1, Theorem 2.4] *Let $f : I_0 \to (0,\infty)$ be a differentiable function. In items (4)–(9), let $I_0 = (0,b)$, $0 < b < \infty$. Then*

1) *f is AA-convex (concave) if and only if $f'(x)$ is increasing (decreasing),*
2) *f is AG-convex (concave) if and only if $f'(x)/f(x)$ is increasing (decreasing),*
3) *f is AH-convex (concave) if and only if $f'(x)/f(x)^2$ is increasing (decreasing),*

4) *f is GA-convex (concave) if and only if $xf'(x)$ is increasing (decreasing),*
5) *f is GG-convex (concave) if and only if $xf'(x)/f(x)$ is increasing (decreasing),*
6) *f is GH-convex (concave) if and only if $xf'(x)/f(x)^2$ is increasing (decreasing),*
7) *f is HA-convex (concave) if and only if $x^2f'(x)$ is increasing (decreasing),*
8) *f is HG-convex (concave) if and only if $x^2f'(x)/f(x)$ is increasing (decreasing),*
9) *f is HH-convex (concave) if and only if $x^2f'(x)/f(x)^2$ is increasing (decreasing).*

After the publication ([1]), many authors have studied generalized convexity. For a partial survey of the recent results, see [9].

In [10], the following inequalities were studied:

Lemma 2. *Let $f : I_0 \to (0, \infty)$ be a continuous function, then*

1) *f is LL-convex (concave) if f is increasing and* log*-convex (concave),*
2) *f is AL-convex (concave) if f is increasing and* log*-convex (concave).*

Recently, Baricz [11] took one step further and studied the MN-convexity(concavity) of a function f in a generalized way, and gave the following result:

Lemma 3. [11, Lemma 3] *Let $p, q \in \mathbb{R}$ and let $f\colon [a, b] \to (0, \infty)$ be a differentiable function for $a, b \in (0, \infty)$. The function f is (p, q)-convex ((p, q)-concave) if and only if $x \mapsto x^{1-p}f'(x)(f(x))^{q-1}$ is an increasing (decreasing) function.*

It can be observed easily that $(1, 1)$-convexity means the AA-convexity, $(1, 0)$-convexity means the AG-convexity, and $(0, 0)$-convexity means GG-convexity.

Lemma 4. [11, Theorem 7] *Let $a, b \in (0, \infty)$ and $f\colon [a, b] \to (0, \infty)$ be a differentiable function. Denote $g(x) = \int_1^x f(t)\, dt$ and $h(x) = \int_x^b f(t)\, dt$. Then*
(a) If the function $x \mapsto x^{1-p}f(x)$ is increasing (decreasing), then g is (p, q)-convex (h is (p, q)-convex) for all $p \in \mathbb{R}$ and $q \geq 1$.

(b) If the function $x \mapsto x^{1-p}f(x)$ is increasing (decreasing), then g is (p,q)-convex (h is (p,q)-convex) for all $p \neq (0,1)$ and $q < 0$.

2. Main results. In this paper we make a contribution to the subject by giving the following theorems, which could be natural questions to ask after reading the above literature. These results are the extension of [1, 11, 10].

Theorem 1. *Let $f : I_0 \to (0,\infty)$ be a continuously differentiable, increasing and* log-*convex (concave) function. Then*

$$I(f(x), f(y)) \geq (\leq) f(I(x,y)).$$

Theorem 2. *Let f be a continuous real-valued function on $(0,\infty)$. If f is strictly increasing and convex, then*

$$P_f(x,y) \leq R_f(x,y) \tag{1}$$

where

$$P_f(x,y) = f\left((xy)^{1/4}\left(\frac{x+y}{2}\right)^{1/2}\right)$$

and

$$R_f(x,y) = \frac{1}{y-x}\int_x^y f(t)dt.$$

Remark 1. In [2], Ebanks defined $P_f(x,y)$ and $R_f(x,y)$, and proposed a problem for a continuous and strictly monotonic real-valued function f on $(0,\infty)$ as follows:
Problem. Does strictly increasing and a convexity of f (or $f'' > 0$) imply that $P_f \leq R_f$?

It is obvious that the Theorem 2 gives an affirmative answer to the Ebanks' question.

Theorem 3. *Let $f : I_0 \to (0,\infty)$.*
(1) If $f(x)$ is continuously differentiable, strictly increasing(decreasing) and convex (concave) and $f^{p-1}(x)f'(x)$ is increasing on $(0,1)$, then

$$J_p(f(x), f(y)) \geq f(J_p(x,y))$$
$$J_p(f(x), f(y)) \leq f(A(x,y))$$

for $p \leq 1$.

(2) If $f(x)$ is continuously differentiable, strictly decreasing(increasing) and convex(concave) and $f^{p-1}(x)f'(x)$ is decreasing on $(0,1)$, then

$$J_p(f(x), f(y)) \geq f(J_p(x,y))$$
$$J_p(f(x), f(y)) \leq f(A(x,y))$$

for $p > 1$.

3. Lemmas and proofs. We recall the following lemmas which will be used in the proofs of the theorems.

Lemma 5. [12] *Let $f, g : [a,b] \to R$ be integrable functions, both increasing or both decreasing. Furthermore, let $p : [a,b] \to R$ be a positive, integrable function. Then*

$$\int_a^b p(x)f(x)dx \cdot \int_a^b p(x)g(x)dx \leq \int_a^b p(x)dx \cdot \int_a^b p(x)f(x)g(x)dx. \quad (2)$$

If one of the functions f or g is non-increasing and the other non-decreasing, then the inequality (2) is reversed.

Lemma 6. [13] *If $f(x)$ is a continuous and convex function on $[a,b]$, and $\varphi(x)$ is continuous on $[a,b]$, then*

$$f\left(\frac{1}{b-a}\int_a^b \varphi(x)dx\right) \leq \frac{1}{b-a}\int_a^b f(\varphi(x))\,dx. \quad (3)$$

If function $f(x)$ is continuous and concave on $[a,b]$, the inequality (3) is reversed.

Lemma 7. [4] *Fix two positive number a, b. Then*

$$L(a,b) \leq I(a,b) \leq A(a,b).$$

Lemma 8. [13] *The function $p \mapsto J_p(x,y)$ is strictly increasing on $\mathbb{R} \setminus \setminus\{0,-1\}$.*

Proof of Theorem 1. Since the proof of part (2) is similar to part (1), we only prove the part (1) here. Clearly

$$\ln I(f(x), f(y)) = \frac{f(x)\ln f(x) - f(y)\ln f(y)}{f(x) - f(y)} - 1.$$

An easy computation and substitution $t = f(u)$ yield

$$\ln I(f(x), f(y)) = \frac{\int_{f(y)}^{f(x)} \ln t \, dt}{\int_{f(y)}^{f(x)} 1 \, dt} = \frac{\int_y^x \ln f(u) f'(u) du}{\int_y^x f'(u) du}. \tag{4}$$

Since the functions $f(x)$ and $f'(x)$ are increasing on $I \subseteq (0, \infty)$ then, using Lemma 5 and assuming $x > y$, we have

$$\int_y^x 1 du \cdot \int_y^x \ln f(u) f'(u) du \geq \int_y^x f'(u) du \cdot \int_y^x \ln f(u) du. \tag{5}$$

Combining (4) and (5), we obtain

$$\ln I(f(x), f(y)) \geq \frac{\int_y^x \ln f(u) du}{y - x}, \tag{6}$$

where we assume that $x > y$. Using the inequality (6), Lemmas 6 and 7, and considering the log-convexity of the function $f(x)$, we get

$$I(f(x), f(y)) \geq \ln f\left(\frac{\int_y^x u du}{y - x}\right) = \ln f\left(\frac{x + y}{2}\right) \geq \ln f\left(I(x, y)\right).$$

This completes the proof. □

Proof of Theorem 2. Since f is a strictly increasing and convex function, then from Lemma 5 and the inequality $G(x, y) \leq A(x, y)$ we obtain

$$\begin{aligned} R_f(x, y) &\geq \frac{\int_x^y f(u) du}{y - x} \geq f\left(\frac{\int_x^y u du}{y - x}\right) = \\ &= f\left(\frac{x + y}{2}\right) \geq f\left((xy)^{1/4}\left(\frac{x + y}{2}\right)^{1/2}\right) = \\ &= P_f(x, y). \end{aligned}$$

This completes the proof. □

Proof of Theorem 3. For the proof of part (1), letting $t = f(u)$, we get

$$J_p(f(x), f(y)) = \frac{\int_{f(y)}^{f(x)} t^p dt}{\int_{f(y)}^{f(x)} t^{p-1} dt} = \frac{\int_y^x f^p(u) f'(u) du}{\int_y^x f^{p-1}(u) f'(u) du}.$$

By using Lemma 5, we obtain

$$J_p(f(x), f(y)) \geq \frac{\int_y^x f(u)du}{y-x}.$$

Considering convexity of the function $f(x)$ and using Lemmas 6 and 8, we get

$$J_p(f(x), f(y)) \geq f\left(\frac{\int_y^x udu}{y-x}\right) = f\left(\frac{x+y}{2}\right) \geq f\left(J_p(x,y)\right),$$

which implies (1). The proof of part (2) follows similarly. □

The convexity and concavity properties of a real-valued function were studied in [1, 11, 10, 14] in the sense of many classical means, i. e. arithmetic mean, geometric mean, logarithmic mean, harmonic mean etc. In this paper, we made a contribution to the topic, and studied the convexity and concavity properties of a real-valued function with respect to identric mean, Alzer mean, as well as proved the conjecture posed by Ebanks.

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