Abstract. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is $(m_1, m_2)$-convex (concave) if
\[ f(m_1(x, y)) \leq (\leq) m_2(f(x), f(y)) \]
for all $x, y \in \mathbb{R}_+ = (0, \infty)$ and $m_1$ and $m_2$ are two mean functions. Anderson et al. \[1\] studies the dependence of $(m_1, m_2)$-convexity (concavity) on $m_1$ and $m_2$ and gave the sufficient conditions of $(m_1, m_2)$-convexity and concavity of a function defined by Maclaurin series. In this paper, we make a contribution to the topic and study the $(m_1, m_2)$-convexity and concavity of a function where $m_1$ and $m_2$ are identric and Alzer mean. As well, we prove a conjecture posed by Bruce Ebanks in \[2\].

Key words: logarithmic mean, identric mean, power mean, Alzer mean, convexity and concavity property, Ebanks’ conjecture

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1. Introduction. A function $M : (0, \infty) \times (0, \infty) \to (0, \infty)$ is called a Mean function if
   
   1) $M(x, y) = M(y, x),$
   2) $M(x, x) = x,$
   3) $x < M(x, y) < y,$ whenever $x < y,$
   4) $M(ax, ay) = a M(x, y)$ for all $a > 0,$

Some examples of mean functions of two distinct positive real numbers are given below:

\begin{align*}
\text{Arithmetic mean:} & \quad A = A(x, y) = \frac{x + y}{2}, \\
\text{Geometric mean:} & \quad G = G(x, y) = \sqrt{xy},
\end{align*}

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Harmonic mean: \( H = H(x, y) = \frac{1}{A(1/x, 1/y)} \),

Logarithmic mean: \( L = L(x, y) = \frac{x - y}{\log(x) - \log(y)} \),

Identric mean: \( I = I(x, y) = \frac{1}{e} \left( \frac{x^x}{y^y} \right)^{1/(x-y)} \),

Alzer mean: \( J_p = J_p(x, y) = \frac{p}{p + 1} \frac{x^{p+1} - y^{p+1}}{x^p - y^p} \), \( p \neq 0, -1 \),

Power mean: \( M_t = M_t(x, y) = \begin{cases} \left( \frac{x^t + y^t}{2} \right)^{1/t} & t \neq 0, \\ \sqrt[2]{xy} & t = 0. \end{cases} \)

It is easy to observe that \( J_1(x, y) = A(x, y) \), \( J_0(x, y) = L(x, y) \), \( J_{-2}(x, y) = H(x, y) \). For the historical background of these means we refer the reader to see [3]–[7] and the bibliography of these papers.

Before we introduce the earlier results from the literature we recall the following definition, see [1, 8].

**Definition 1.** Let \( f : I_0 \rightarrow (0, \infty) \) be continuous, where \( I_0 \) is a sub-interval of \((0, \infty)\). Let \( M \) and \( N \) be two any mean functions. We say that the function \( f \) is \( MN \)-convex (concave) if

\[ f(M(x, y)) \leq (\geq) N(f(x), f(y)) \quad \text{for all} \quad x, y \in I_0. \]

Throughout the paper, the notion \( I_0 \) is reserved for the sub-interval of \((0, \infty)\).

In [1], Anderson, Vamanamurthy and Vuorinen studied the convexity and concavity of a function \( f \) with respect two mean values, and gave the following detailed result:

**Lemma 1.** [1, Theorem 2.4] Let \( f : I_0 \rightarrow (0, \infty) \) be a differentiable function. In items (4)–(9), let \( I_0 = (0, b) \), \( 0 < b < \infty \). Then

1) \( f \) is AA-convex (concave) if and only if \( f'(x) \) is increasing (decreasing),

2) \( f \) is AG-convex (concave) if and only if \( f'(x)/f(x) \) is increasing (decreasing),

3) \( f \) is AH-convex (concave) if and only if \( f'(x)/f(x)^2 \) is increasing (decreasing),
4) $f$ is $GA$-convex (concave) if and only if $xf'(x)$ is increasing (decreasing),
5) $f$ is $GG$-convex (concave) if and only if $xf'(x)/f(x)$ is increasing (decreasing),
6) $f$ is $GH$-convex (concave) if and only if $xf'(x)/f(x)^2$ is increasing (decreasing),
7) $f$ is $HA$-convex (concave) if and only if $x^2f'(x)$ is increasing (decreasing),
8) $f$ is $HG$-convex (concave) if and only if $x^2f'(x)/f(x)^2$ is increasing (decreasing),
9) $f$ is $HH$-convex (concave) if and only if $x^2f'(x)/f(x)^2$ is increasing (decreasing).

After the publication ([1]), many authors have studied generalized convexity. For a partial survey of the recent results, see [9].

In [10], the following inequalities were studied:

**Lemma 2.** Let $f : I_0 \to (0, \infty)$ be a continuous function, then

1) $f$ is $LL$-convex (concave) if $f$ is increasing and log-convex (concave),
2) $f$ is $AL$-convex (concave) if $f$ is increasing and log-convex (concave).

Recently, Baricz [11] took one step further and studied the $MN$-convexity (concavity) of a function $f$ in a generalized way, and gave the following result:

**Lemma 3.** [11, Lemma 3] Let $p, q \in \mathbb{R}$ and let $f : [a, b] \to (0, \infty)$ be a differentiable function for $a, b \in (0, \infty)$. The function $f$ is $(p, q)$-convex ($(p, q)$-concave) if and only if $x \mapsto x^{1-p}f'(x)(f(x))^{q-1}$ is an increasing (decreasing) function.

It can be observed easily that $(1, 1)$-convexity means the $AA$-convexity, $(1, 0)$-convexity means the $AG$-convexity, and $(0, 0)$-convexity means $GG$-convexity.

**Lemma 4.** [11, Theorem 7] Let $a, b \in (0, \infty)$ and $f : [a, b] \to (0, \infty)$ be a differentiable function. Denote $g(x) = \int_1^x f(t) \, dt$ and $h(x) = \int_x^b f(t) \, dt$. Then

(a) If the function $x \mapsto x^{1-p}f(x)$ is increasing (decreasing), then $g$ is $(p, q)$-convex ($h$ is $(p, q)$-convex) for all $p \in \mathbb{R}$ and $q \geq 1$. 
(b) If the function \( x \mapsto x^{1-p} f(x) \) is increasing (decreasing), then \( g \) is \((p, q)\)-convex \((h \text{ is } (p, q)\)-convex) for all \( p \neq (0, 1) \) and \( q < 0 \).

2. Main results. In this paper we make a contribution to the subject by giving the following theorems, which could be natural questions to ask after reading the above literature. These results are the extension of [11][10].

**Theorem 1.** Let \( f : I_0 \rightarrow (0, \infty) \) be a continuously differentiable, increasing and log-convex \((\text{concave})\) function. Then
\[
I(f(x), f(y)) \geq (\leq) f(I(x, y)).
\]

**Theorem 2.** Let \( f \) be a continuous real-valued function on \((0, \infty)\). If \( f \) is strictly increasing and convex, then
\[
P_f(x, y) \leq R_f(x, y) \tag{1}
\]
where
\[
P_f(x, y) = f \left( (xy)^{1/4} \left( \frac{x + y}{2} \right)^{1/2} \right)
\]
and
\[
R_f(x, y) = \frac{1}{y - x} \int_x^y f(t)dt.
\]

**Remark 1.** In [2], Ebanks defined \( P_f(x, y) \) and \( R_f(x, y) \), and proposed a problem for a continuous and strictly monotonic real-valued function \( f \) on \((0, \infty)\) as follows:
Problem. Does strictly increasing and a convexity of \( f \) (or \( f'' > 0 \)) imply that \( P_f \leq R_f \)?

It is obvious that the Theorem 2 gives an affirmative answer to the Ebanks’ question.

**Theorem 3.** Let \( f : I_0 \rightarrow (0, \infty) \).

1. If \( f(x) \) is continuously differentiable, strictly increasing\( (\text{decreasing})\) and convex \((\text{concave})\) and \( f^{p-1}(x)f'(x) \) is increasing on \((0, 1)\), then
\[
J_p(f(x), f(y)) \geq f(J_p(x, y)) \tag{5}
\]
\[
J_p(f(x), f(y)) \leq f(A(x, y))
\]
for \( p \leq 1 \).
(2) If \( f(x) \) is continuously differentiable, strictly decreasing/increasing and convex/concave and \( f^{p-1}(x)f'(x) \) is decreasing on \((0, 1)\), then
\[
J_p(f(x), f(y)) \geq f(J_p(x, y))
\]
\[
J_p(f(x), f(y)) \leq f(A(x, y))
\]
for \( p > 1 \).

3. Lemmas and proofs. We recall the following lemmas which will be used in the proofs of the theorems.

Lemma 5. \[12\] Let \( f, g : [a, b] \to \mathbb{R} \) be integrable functions, both increasing or both decreasing. Furthermore, let \( p : [a, b] \to \mathbb{R} \) be a positive, integrable function. Then
\[
\int_a^b p(x)f(x)dx \cdot \int_a^b p(x)g(x)dx \leq \int_a^b p(x)dx \cdot \int_a^b p(x)f(x)g(x)dx. \tag{2}
\]
If one of the functions \( f \) or \( g \) is non-increasing and the other non-decreasing, then the inequality \((2)\) is reversed.

Lemma 6. \[13\] If \( f(x) \) is a continuous and convex function on \([a, b]\), and \( \varphi(x) \) is continuous on \([a, b]\), then
\[
f\left( \frac{1}{b-a} \int_a^b \varphi(x)dx \right) \leq \frac{1}{b-a} \int_a^b f(\varphi(x))dx. \tag{3}
\]
If function \( f(x) \) is continuous and concave on \([a, b]\), the inequality \((3)\) is reversed.

Lemma 7. \[4\] Fix two positive number \( a, b \). Then
\[
L(a, b) \leq I(a, b) \leq A(a, b).
\]

Lemma 8. \[13\] The function \( p \mapsto J_p(x, y) \) is strictly increasing on \( \mathbb{R} \setminus \{0, -1\} \).

Proof of Theorem \[1\] Since the proof of part (2) is similar to part (1), we only prove the part (1) here. Clearly
\[
\ln I(f(x), f(y)) = \frac{f(x)\ln f(x) - f(y)\ln f(y)}{f(x) - f(y)} - 1.
\]
An easy computation and substitution \( t = f(u) \) yield
\[
\ln I(f(x), f(y)) = \frac{\int_{f(y)}^{f(x)} t \, dt}{\int_{f(y)}^{f(x)} 1 \, dt} = \frac{\int_{y}^{x} \ln f(u)f'(u)du}{\int_{y}^{x} f'(u)du}.
\] (4)

Since the functions \( f(x) \) and \( f'(x) \) are increasing on \( I \subseteq (0, \infty) \) then, using Lemma 5 and assuming \( x > y \), we have
\[
\int_{y}^{x} 1du \cdot \int_{y}^{x} \ln f(u)f'(u)du \geq \int_{y}^{x} f'(u)du \cdot \int_{y}^{x} \ln f(u)du.
\] (5)

Combining (4) and (5), we obtain
\[
\ln I(f(x), f(y)) \geq \frac{\int_{y}^{x} \ln f(u)du}{y - x},
\] (6)

where we assume that \( x > y \). Using the inequality (6), Lemmas 6 and 7 and considering the log-convexity of the function \( f(x) \), we get
\[
I(f(x), f(y)) \geq \ln f \left( \frac{\int_{y}^{x} udu}{y - x} \right) = \ln f \left( \frac{x + y}{2} \right) \geq \ln f (I(x, y)).
\]

This completes the proof.

\[\square\]

**Proof of Theorem 2.** Since \( f \) is a strictly increasing and convex function, then from Lemma 5 and the inequality \( G(x, y) \leq A(x, y) \) we obtain
\[
R_f(x, y) \geq \frac{\int_{x}^{y} f(u)du}{y - x} \geq f \left( \frac{\int_{x}^{y} udu}{y - x} \right) = f \left( \frac{x + y}{2} \right) \geq f \left( (xy)^{1/4} \left( \frac{x + y}{2} \right)^{1/2} \right) = P_f(x, y).
\]

This completes the proof.

\[\square\]

**Proof of Theorem 3.** For the proof of part (1), letting \( t = f(u) \), we get
\[
J_p(f(x), f(y)) = \frac{\int_{f(y)}^{f(x)} t^p dt}{\int_{f(y)}^{f(x)} t^{p-1} dt} = \frac{\int_{y}^{x} f^p(u)f'(u)du}{\int_{y}^{x} f^{p-1}(u)f'(u)du}.
\]
By using Lemma 5, we obtain
\[ J_p(f(x), f(y)) \geq \frac{\int_x^y f(u)du}{y - x}. \]

Considering convexity of the function \( f(x) \) and using Lemmas 6 and 8, we get
\[ J_p(f(x), f(y)) \geq f \left( \frac{\int_x^y udu}{y - x} \right) = f \left( \frac{x + y}{2} \right) \geq f (J_p(x, y)), \]
which implies (1). The proof of part (2) follows similarly. \( \square \)

The convexity and concavity properties of a real-valued function were studied in [1, 11, 10, 14] in the sense of many classical means, i.e. arithmetic mean, geometric mean, logarithmic mean, harmonic mean etc. In this paper, we made a contribution to the topic, and studied the convexity and concavity properties of a real-valued function with respect to identric mean, Alzer mean, as well as proved the conjecture posed by Ebanks.

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References


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