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UNIVALENCE OF HARMONIC FUNCTIONS, THE PROBLEM OF PONNUSAMY AND SAIRAM, AND CONSTRUCTIONS OF UNIVALENT POLYNOMIALS

Abstract. The criterion of the univalence of a harmonic mapping is obtained in this paper. Particularly, it permits to formulate the conjecture of coincidence of the harmonic function classes $S_H^0 = S_H^0(S)$ (the problem of Ponnusamy and Sairam), in analytic form. The method of construction of the univalent harmonic polynomials with desired properties, according to a given harmonic function, is obtained by means of the univalence criteria.

Key words: *harmonic functions, criteria of the univalence, harmonic univalent polynomials*

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1. Introduction. Let S be the class of all analytic and univalent functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The problem of determining a necessary coefficient condition in this class was set up by Bieberbach [1]. It consisted of validity of the inequality $|a_n| \leq n$ for each $n \in \mathbb{N}$ (equality for the Koebe function $k(z) = z/(1-z)^2$ and its rotations $k_\theta(z) = e^{-i\theta}k(ze^{i\theta})$). The Bieberbach hypothesis contributed largely to the origin and development of a great number of ideas and methods in complex analysis. The full solution of this problem was finally given by de Branges [2].

The theory of univalent harmonic functions began its active development since the eighties of the previous century. Here the main object of research is the class S_H of harmonic univalent and sense-preserving

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functions f in Δ given by $f(z) = h(z) + \overline{g(z)}$, where

$$h(z) = \sum_{n=1}^{\infty} a_n z^n \text{ and } \overline{g(z)} = \sum_{n=1}^{\infty} a_{-n} \bar{z}^n$$

S with $a_1 = 1$ (see, for example, [3]). The class S_H is an analog of the class S. For obtaining information about the functions S_H it is often convenient to have such information about the functions of the subclass $S_H^0 \subset \subset S_H$, where $S_H^0 = \{f \in S_H : a_{-1} = 0\}$. This circumstance explains the interest in studying S_H^0. In [4] Clunie and Sheil-Small formulated the following conjecture (the problem about coefficients in S_H^0): for all $f \in S_H^0$ and $n \in \mathbb{N}$, the inequalities

$$|a_n| \le \frac{(2n+1)(n+1)}{6}, \quad |a_{-n}| \le \frac{(2n-1)(n-1)}{6}, \quad |a_n - a_{-n}| \le n \quad (1)$$

are true. A great deal of papers are dedicated to this conjecture. Particularly, in the paper of Ponnusamy and Sairam Kaliraj [5] this conjecture, together with some other results was proved for the subclass

$$S_H^0(S) = \{f = h + \bar{g} \in S_H^0 : h + e^{i\phi} g \in S \text{ for some } \phi \in \mathbb{R}\}$$

of S_H^0. Besides, in this paper the authors conjectured that $S_H^0(S) = S_H^0$, whose prove would permit us to obtain the full solution of the coefficients problems (1) of Clunie and Sheil-Small. In this paper the criterion of univalency of harmonic functions (Theorem 1) is obtained. With the aid of this, the criterion for functions belonging to $S_H^0(S)$ (Theorem 2) is obtained and several examples are exhibited. Theorem 3 permits to construct harmonic univalent polynomials for a given $f \in S_H$.

2. The univalence criterion and the conjecture of hypothesis of Ponnusamy and Sairam Kaliraj. The univalence criterion of an arbitrary harmonic function in Δ of the form

$$f(z) = \sum_{n=1}^{\infty} (a_n z^n + a_{-n} \bar{z}^n) \quad (2)$$

will be obtained by analogy to Bazilevich's [6] univalence criterion for analytic functions:

Theorem A. [6] *An analytic function* $f(z) = \sum_{n=1}^{\infty} a_n z^n$ *in* Δ *is univalent in* Δ *if and only if for each* $z \in \Delta$ *and each* $t \in [0, \pi/2]$,

$$\sum_{n=1}^{\infty} a_n \frac{\sin nt}{\sin t} z^{n-1} \neq 0, \quad \left(\frac{\sin nt}{\sin t}\right)\Big|_{t=0} = n.$$

Theorem 1. *Harmonic sense-preserving function in* Δ, *determined by the formula* (2), *is univalent in* Δ *if and only if for each* $z \in \Delta \setminus \{0\}$ *and each* $t \in (0, \pi/2]$,

$$\sum_{n=1}^{\infty} \left[(a_n z^n - a_{-n}\bar{z}^n)\frac{\sin nt}{\sin t}\right] \neq 0. \tag{3}$$

Proof. Let $f \in S_H$. Then, for $z_1, z_2 (z_1 \neq z_2)$ from Δ, we have

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} \neq 0.$$

Particularly, for $z_1 = re^{i\theta_1} \neq re^{i\theta_2} = z_2$, $r \in (0,1)$, $\theta_k \in \mathbb{R}$, this is equivalent to

$$\frac{f(re^{i\theta_2}) - f(re^{i\theta_1})}{re^{i\theta_2} - re^{i\theta_1}} = \tag{4}$$

$$= \sum_{n=1}^{\infty} r^{n-1}\left(a_n \frac{e^{in\theta_2} - e^{in\theta_1}}{e^{i\theta_2} - e^{i\theta_1}} + a_{-n}\frac{e^{-in\theta_2} - e^{-in\theta_1}}{e^{i\theta_2} - e^{i\theta_1}}\right) \neq 0.$$

Without loss of generality, we may assume that $\theta_1 < \theta_2 \leq \theta_1 + \pi$. Let

$$t = \frac{\theta_2 - \theta_1}{2} \in (0, \pi/2] \text{ and } \theta = \frac{\theta_2 + \theta_1}{2} \in \mathbb{R}.$$

Then

$$\frac{e^{in\theta_2} - e^{in\theta_1}}{e^{i\theta_2} - e^{i\theta_1}} = e^{i(n-1)\frac{\theta_2+\theta_1}{2}} \frac{e^{in\frac{\theta_2-\theta_1}{2}} - e^{-in\frac{\theta_2-\theta_1}{2}}}{e^{i\frac{\theta_2-\theta_1}{2}} - e^{-i\frac{\theta_2-\theta_1}{2}}} = e^{i(n-1)\theta}\frac{e^{int} - e^{-int}}{e^{it} - e^{-it}},$$

and

$$\frac{e^{-in\theta_2} - e^{-in\theta_1}}{e^{i\theta_2} - e^{i\theta_1}} =$$

$$= \overline{\left(\frac{e^{in\theta_2} - e^{in\theta_1}}{e^{i\theta_2} - e^{i\theta_1}}\right)} \frac{e^{-i\theta_2} - e^{-i\theta_1}}{e^{i\theta_2} - e^{i\theta_1}} = e^{-i(n-1)\theta} \frac{e^{int} - e^{-int}}{e^{it} - e^{-it}} (-e^{-2i\theta}).$$

Hence (4) may be represented as

$$\sum_{n=1}^{\infty} \left[(a_n r^{n-1} e^{i(n-1)\theta} - a_{-n} r^{n-1} e^{-i(n-1)\theta} e^{-2i\theta}) \frac{\sin nt}{\sin t} \right] \neq 0$$

which is equivalent to

$$\sum_{n=1}^{\infty} \left[(a_n z^{n-1} - a_{-n} \bar{z}^{n-1} e^{-2i\theta}) \frac{\sin nt}{\sin t} \right] \neq 0, \tag{5}$$

where $z = re^{i\theta} \in \Delta$, $\theta = \arg z$, $t \in (0, \pi/2]$. But (5) $\Longleftrightarrow$ (3). Let us note, that (5) is fulfilled for $z = 0$ as well, because $|a_1| > |a_{-1}|$, since f is sense-preserving.

Let us next prove the inverse proposition. Suppose a harmonic function (2) is sense-preserving in Δ and condition (3) is fulfilled. According to the accepted designations this is equivalent to fulfilling of the condition (4), i. e.

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} \neq 0 \quad \forall z_1 = re^{i\theta_1} \neq re^{i\theta_2} = z_2,\ r \in (0, 1),\ \theta_k \in \mathbb{R}.$$

Thus f is univalent on any circle $\{z \in \mathbb{C} : |z| = r\}$.

The local univalence of the function f implies that $\partial f(\Delta_r) \subset f(\partial \Delta_r)$, where $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$. Then the assumption that f is not univalent (but locally univalent) in Δ implies the existence of a disk Δ_R (a disk of f univalence), in which f is univalent, but on $\partial \Delta_R$ there exist points $z_1 \neq z_2$ such that $f(z_1) = f(z_2)$. This contradicts the univalence of f on the circle $\{z \in \mathbb{C} : |z| = R\}$. This contradiction proves Theorem 1. $\square$

The following theorem represents the criterion of belonging of a function to the class $S_H^0(S)$.

Theorem 2. *Let $f \in S_H^0$. Define*

$$A = A(z, t) = \sum_{n=1}^{\infty} a_n \frac{\sin nt}{\sin t} z^n, \quad B = B(z, t) = \sum_{n=1}^{\infty} \overline{a_{-n}} \frac{\sin nt}{\sin t} z^n,$$

and

$$E = \{(z, t) \in (\Delta \setminus \{0\}) \times (0, \pi/2] : |A(z, t)| = |B(z, t)|\}.$$

Then $f \in S_H^0(S)$ *if and only if there exists a* $\phi \in [0, 2\pi)$ *such that*

$$A(z,t) \neq -e^{i\phi} B(z,t) \quad \forall\ (z,t) \in E.$$

Proof. Let $f = h + \bar{g} \in S_H^0(S)$ and let it be determined by (2). According to the definition, $S_H^0(S) \ni f$ if and only if there exists a $\phi \in [0, 2\pi)$ such that $h + e^{i\phi} g \in S$. Theorem A implies that

$$\begin{aligned} \sum_{n=1}^{\infty} (a_n + e^{i\phi}\overline{a_{-n}}) \frac{\sin nt}{\sin t} z^n &\neq 0, \quad \text{for } z \in \Delta \setminus \{0\},\ t \in (0, \pi/2], \\ \Longleftrightarrow A(z,t) &\neq -e^{i\phi} B(z,t) \quad \forall\ (z,t) \in E. \end{aligned}$$

This completes the proof. □

Corollary 1. *Let* $f \in S_H^0$ *and* E *be the set defined in Theorem 2. If* $E = \emptyset$*, then* $f \in S_H^0(S)$*.*

Remark 1. *Denote* $q(z) = \sum_{n=1}^{\infty} \frac{\sin nt}{\sin t} z^n$*. Then*

$$A(z,t) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} h\left(\frac{z}{\zeta}\right) q(\zeta) \frac{d\zeta}{\zeta}, \ B(z,t) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} g\left(\frac{z}{\zeta}\right) q(\zeta) \frac{d\zeta}{\zeta}.$$

Hence the statement of Theorem 2 may be represented by means of these integrals.

Example. For a fixed $R \in (0,1)$, consider the harmonic function $f = h + \overline{g}$ in Δ, where

$$h(z) = \frac{z}{(1 - Rz)^2} \quad \text{and } g(z) = kz^2, \quad k \in \mathbb{R}.$$

By means of Theorem 1 we determine for which values of k and R, $f \in S_H^0$. Applying Theorem 2, let us show that for these values of parameters, the function $f \in S_H^0(S)$.

Firstly we need to determine, for which values of the parameters, the function f is sense-preserving in Δ. The condition of sense-preservation means the validity for any $z \in \Delta$ of the inequality

$$|h'| - |g'| = \left| \frac{1 + Rz}{(1 - Rz)^3} \right| - |2kz| > 0$$

which holds if

$$\min_{|z|=r}\left|\frac{1+Rz}{(1-Rz)^3}\right| = \frac{1-Rr}{(1+Rr)^3} > 2|k|r,\ \forall\ r\in[0,1).$$

Since the function $\dfrac{1-Rr}{r(1+Rr)^3}$ decreases with respect to r, the latter inequality gives

$$2|k| \le \frac{1-R}{(1+R)^3}. \tag{6}$$

The values of parameters for which the function f is sense-preserving in Δ is determined from (6). Further, let the condition (6) be valid. According to Theorem 1, f is univalent if and only if $A(z,t)\neq\overline{B(z,t)}$ (designations from Theorem 2) in $(\Delta\setminus\{0\})\times(0,\pi/2]$. Let us show that the equality $A(z,t)=\overline{B(z,t)}$ is not possible. We see that

$$\begin{aligned} A(z,t) &= \sum_{n=1}^{\infty} nR^{n-1}z^n\frac{e^{int}-e^{-int}}{2i\sin t} = \\ &= \frac{1}{2i\sin t}\left[\frac{ze^{it}}{(1-Rze^{it})^2}-\frac{ze^{-it}}{(1-Rze^{-it})^2}\right] = \\ &= \frac{z(1-R^2z^2)}{(1-2Rz\cos t+R^2z^2)^2} \end{aligned}$$

and

$$B(z,t) = 2\bar{k}z^2\cos t.$$

Now we show that for $t\in(0,\pi/2)$ and $z\in\Delta\setminus\{0\}$, the equation

$$\frac{z(1-R^2z^2)}{(1-2Rz\cos t+R^2z^2)^2} = 2k\bar{z}^2\cos t \tag{7}$$

has no solution. It is sufficient to show that in (7) the absolute values of the left hand and right hand sides are not equal. If

$$|2k| = \left|\frac{1-R^2z^2}{z\cos t(1-2R\cos tz+R^2z^2)^2}\right| = L(z,t),$$

then for some t,

$$|2k| \ge \min_{0<|z|\le 1} L(z,t) = \min_{|z|=1} L(z,t) = L(-1,t) =$$

$$= \frac{1-R^2}{\cos t(1+2R\cos t+R^2)^2} > \frac{1-R}{(1+R)^3}$$

for values of t under consideration. Thus a contradiction with (6) is obtained. Therefore, if the condition (6) is fulfilled, then all functions f from this example are univalent. As shown above, $|A(z,t)| \neq |B(z,t)|$, in $(\Delta \setminus \{0\}) \times (0, \pi/2]$ and therefore the set E defined in Theorem 2 is empty. Hence, for parameters' values satisfying the inequality (6), $f \in S_H^0(S)$.

3. Univalent harmonic polynomials are functions of the form $P = h + \overline{g}$, where h and g are classic polynomials in z. Generally speaking, there is not much information about univalent harmonic polynomials than about other functions from S_H (here we speak about univalence in Δ) (see [7]).

For example, in a survey paper [8] the authors note: "Finding a method of constructing sense-preserving univalent harmonic polynomials is another important problem". In the analytic case, Bazilevich [6] proposed a method of construction of univalent polynomials, associated with a given function from S. Further, his idea has been transferred to harmonic case in Theorem 3. Moreover, unlike with analytic case, the proof will be constructive. Thus, Theorem 3 gives an opportunity to construct harmonic univalent polynomials of sufficiently high power for any function $f \in S_H^0$.

Lemma 1. *If $f \in S_H^0, r \in (0,1), \quad t, \phi \in \mathbb{R}$, then*

$$\left| \frac{f(re^{it}) - f(re^{i\phi})}{re^{it} - re^{i\phi}} \right| \geq \frac{1-r}{4\alpha r} \left(\frac{1-r}{1+r} \right)^{\alpha} \left[1 - \left(\frac{1-r}{1+r} \right)^{2\alpha} \right],$$

where $\alpha (= ord S_H) \stackrel{def}{=} \sup_{f \in S_H} |a_2|$.

Proof. If $f = h + \bar{g} \in S_H^0$, then the linear invariance of the class S_H (see [9]) implies that

$$F(z) = \frac{f\left(\frac{z+a}{1+\bar{a}z}\right) - f(a)}{h'(a)(1-|a|^2)} \in S_H \quad \forall a \in \Delta.$$

Let

$$a = re^{i\phi}, \quad \frac{z+a}{1+\bar{a}z} = re^{it}, \quad \text{i. e.} \quad z = \frac{re^{it} - a}{1 - \bar{a}re^{it}}.$$

From the affine invariance (see. [9]) of the class S_H, it follows that the function

$$\psi(z)=\frac{F(z)-a_{-1}\overline{F(z)}}{1-|a_{-1}|^2}\in S_H^0,\quad \text{where } a_{-1}=\frac{\partial F}{\partial\bar z}(0)=\frac{\overline{g'(a)}}{h'(a)}.$$

Hence,

$$|F(z)|(1+|a_{-1}|)\geq|F(z)-a_{-1}\overline{F(z)}|\geq|\psi(z)|(1-|a_{-1}|^2),$$

so that $|F(z)|\geq|\psi(z)|(1-|a_{-1}|)$. Since $f\in S_H^0$, then $\frac{\partial f}{\partial\bar z}(0)=0$, i.e. $g'(0)=0$. Since f is sense-preserving in Δ, we have $|g'(z)/h'(z)|<1$ in Δ. Therefore, according to Schwarz's lemma, $|a_{-1}|=|g'(a)/h'(a)|\leq r$ and

$$|F(z)|\geq|\psi(z)|(1-r). \tag{8}$$

For any function $\psi=H+\bar G\in S_H^0$ and any $z\in\Delta$, one has [9]:

$$\frac{1}{2\alpha}\left[1-\left(\frac{1-|z|}{1+|z|}\right)^{\alpha}\right]\leq|\psi(z)|\ \text{ and }\ \frac{(1-|z|)^{\alpha-1}}{(1+|z|)^{\alpha+1}}\leq|H'(z)|,$$

from which with regard to (8) we obtain

$$|f(re^{it})-f(re^{i\phi})|\geq\frac{1}{2\alpha}\left[1-\left(\frac{1-|z|}{1+|z|}\right)^{\alpha}\right](1-r)(1-r^2)|h'(re^{i\phi})|\geq$$

$$\geq\frac{1-r}{2\alpha}\left(\frac{1-r}{1+r}\right)^{\alpha}\left[1-\left(\frac{1-|z|}{1+|z|}\right)^{\alpha}\right], \tag{9}$$

where

$$z=\frac{r(e^{it}-e^{i\phi})}{1-r^2e^{i(t-\phi)}}.$$

Denote by $w=e^{is}=e^{i(t-\phi)}$ and $\zeta=\dfrac{1-w}{1-r^2w}$. Then

$$w=\frac{1-\zeta}{1-r^2\zeta},\quad 0\leq|z|=r|\zeta|\leq\frac{2r}{1+r^2},\quad |1-w|=\frac{|\zeta|(1-r^2)}{|1-r^2\zeta|}.$$

Represent (9) as follows

$$\left|\frac{f(re^{it})-f(re^{i\phi})}{re^{it}-re^{i\phi}}\right|\geq\frac{1-r}{2\alpha}\left(\frac{1-r}{1+r}\right)^{\alpha}\left[1-\left(\frac{1-r|\zeta|}{1+r|\zeta|}\right)^{\alpha}\right]\frac{|1-r^2\zeta|}{r|\zeta|(1-r^2)}\geq$$

$$\geq \frac{1}{2\alpha}\left(\frac{1-r}{1+r}\right)^{\alpha}\left[1-\left(\frac{1-x}{1+x}\right)^{\alpha}\right]\frac{1-rx}{x(1+r)}, \tag{10}$$

$x = r|\zeta| \in \big[0, 2r/(1+r^2)\big]$.
Define

$$u(x) = \frac{1}{x}\left[1-\left(\frac{1-x}{1+x}\right)^{\alpha}\right], \quad x \in \left[0, \frac{2r}{1+r^2}\right].$$

We show that u is decreasing on $(0, 2r/(1+r^2))$. Then,

$$x^2u'(x) \leq 0 \Longleftrightarrow \left(\frac{1-x}{1+x}\right)^{\alpha-1}\frac{2\alpha x}{(1+x)^2} - 1 + \left(\frac{1-x}{1+x}\right)^{\alpha} \leq 0$$

$$\Longleftrightarrow 2\alpha x + 1 - x^2 \leq (1-x^2)\left(\frac{1+x}{1-x}\right)^{\alpha} \Longleftrightarrow$$

$$\Longleftrightarrow u_1(x) \leq u_2(x), \tag{11}$$

where

$u_1(x) = \ln(1+2\alpha x-x^2)$ and $u_2(x) = \alpha(\ln(1+x)-\ln(1-x))+\ln(1-x^2)$.

Note that $u_1(0) = 0 = u_2(0)$ and

$$u_1'(x) = \frac{2(\alpha-x)}{1+2\alpha x-x^2} \leq \frac{2(\alpha-x)}{1-x^2} = u_2'(x),$$

since (see [9]) $3 \leq \alpha < 48,9$. This proves the validity of (11) and hence $u(x)$ is decreasing on $(0,\ 2r/(1+r^2))$ and hence

$$\min_{x\in[0,2r/(1+r^2)]} u(x) = u\left(\frac{2r}{1+r^2}\right) = \frac{1+r^2}{2r}\left[1-\left(\frac{1-r}{1+r}\right)^{2\alpha}\right].$$

Then from (10) we obtain the desired inequality in Lemma 1. The proof of the lemma is complete. □

In Lemma 2 below the estimates of coefficients of functions from S_H^0 will be obtained. The estimates are not exact, but they are sufficient to achieve the goal which we set up in this section (see Theorem 3).

Lemma 2. *If $\alpha = \operatorname{ord} S_H$, $S_H^0 \ni f$ with series expansion (2), then the following inequality is true:*

$$|a_{\pm n}| < \frac{(2e^2)^{\alpha}}{2\alpha}n^{\alpha},\ n \in \mathbb{N}.$$

Proof. Sheil-Small [9] proved for $f \in S_H^0$ the inequality

$$|f(z)| \le \frac{1}{2\alpha}\left[\left(\frac{1+|z|}{1-|z|}\right)^{\alpha} - 1\right], \ z \in \triangle.$$

Hence for $n \in \mathbb{N}$ and $r \in (0,1)$,

$$|a_{-n}| = \left|\frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)d\bar{z}}{\bar{z}^{n+1}}\right| \le \frac{1}{2\alpha r^n}\left(\frac{1+r}{1-r}\right)^{\alpha} = \frac{\psi(r)}{2\alpha}.$$

The same estimate is also true for $|a_n|, \ n \in \mathbb{N}$. In order to find a minimum of the right hand side of inequality we find the point of minimum of the function $\ln\psi(r)$:

$$(\ln\psi(r))' = \frac{2\alpha}{1-r^2} - \frac{n}{r} = 0 \Longleftrightarrow r^2 + \frac{2\alpha}{n}r - 1 = 0.$$

Therefore, $r_0 = \sqrt{\alpha^2/n^2+1} - \alpha/n$ is the point of minimum and

$$|a_n| \le \frac{\psi(r_0)}{2\alpha} \quad \text{for all integer} \quad n,$$

where

$$\psi(r_0) =$$

$$= \left(\sqrt{\frac{\alpha^2}{n^2}+1} + \frac{\alpha}{n}\right)^n \left[\frac{n}{2\alpha}\left(1+\sqrt{\frac{\alpha^2}{n^2}+1}+\frac{\alpha}{n}\right)\left(1+\sqrt{\frac{\alpha^2}{n^2}+1}-\frac{\alpha}{n}\right)\right]^{\alpha}.$$

Using inequality $\sqrt{1+x} \le 1+\sqrt{x}, \ x>0$, we obtain

$$\psi(r_0) \le \left(\frac{n}{\alpha}\right)^{\alpha}\left[\left(1+\frac{2\alpha}{n}\right)^{\frac{n}{2\alpha}}\right]^{2\alpha}\left(2+\frac{\alpha}{n}\right)^{\alpha} =$$

$$= (2n)^{\alpha}\left(\frac{1}{2n}+\frac{1}{\alpha}\right)^{\alpha}\left[\left(1+\frac{2\alpha}{n}\right)^{n/(2\alpha)}\right]^{2\alpha}. \tag{12}$$

Introduce

$$\Psi\ (y) = y\ln\left(1+\frac{1}{y}\right).$$

Then $\sup_{y>0} \Psi(y) = 1$, since $\lim_{y\to+0} \Psi(y) < 1$, $\lim_{y\to+\infty} \Psi(y) = 1$, and, if $\Psi(y)$ has a maximum on the interval $(0,\ \infty)$ at the point y_0, then $\Psi'(y_0) = 0$. This gives

$$\ln\left(1+\frac{1}{y_0}\right) = \frac{1}{1+y_0}$$

which implies that

$$\Psi(y_0) = \frac{y_0}{1+y_0} < 1.$$

Hence from (12) we have

$$\psi(r_0) < (2n)^\alpha e^{2\alpha} \quad \text{and} \quad |a_{\pm n}| < \frac{(2e^2)^\alpha}{2\alpha} n^\alpha$$

and the proof of Lemma 2 is complete. □

Theorem 3. *Let $f \in S_H^0$ and have the series expansion (1), $\alpha = = \mathrm{ord} S_H$, $s = [\alpha+2]$, where $[.]$ denotes the integer part of a number. Let $m \in \mathbb{N}$, $\epsilon \in (0, (2e^2)^\alpha/(2\alpha))$ and*

$$r \in \left(\left[1 - \frac{2\alpha\epsilon}{(2e^2 m)^\alpha}\right]^{1/m}, 1\right) \tag{13}$$

let $N \in \mathbb{N}$ be so large that

$$\frac{2}{|\ln r|}\left[\alpha(2+\ln 2) + \ln(s!) - (s+1)\ln|\ln r| - \ln(1-r) - \alpha \ln\frac{1-r}{1+r} + \right.$$
$$\left. + \ln 4 - \ln\left[1 - \left(\frac{1-r}{1+r}\right)^{2\alpha}\right]\right] + 2 < N, \quad m < N, \quad \frac{\ln N}{N} \le \frac{|\ln r|}{2[\alpha+2]}.$$

Then the harmonic polynomial

$$P(z) = \sum_{k=1}^{N} (c_k z^k + c_{-k}\bar{z}^k) = \sum_{k=1}^{N} (a_k r^k z^k + a_{-k} r^k \bar{z}^k)$$

is univalent in Δ and moreover $|a_{\pm k} - c_{\pm k}| < \epsilon$ for all $k = 1, \ldots, m$.

Proof. Let us note that $2\alpha\epsilon/(2e^2)^\alpha < 1$ for the indicated values ϵ. Hence $\dfrac{2\alpha\epsilon}{(2e^2 k)^\alpha} < 1$ for $k = 1, \ldots, m$.

The function $\phi(x) = (1 - x^\alpha(2\alpha\epsilon)/(2e^2)^\alpha)^x$ decreases on $(0,1]$, since $\dfrac{d\ln\phi(x)}{dx} < 0$ for $x \in (0,1]$. Therefore, $\phi(1/k) \leq \phi(1/m)$ for each $k = 1,\ldots,m$, and (13) implies the inequality $\phi(1/k) \leq \phi(1/m) < r$. Hence

$$(1 - r^k)\frac{(2e^2k)^\alpha}{2\alpha} < \epsilon \quad \text{for} \quad k = 1,\ldots,m.$$

From these inequalities and Lemma 2 we obtain inequalities for the first coefficients $c_{\pm k} = r^k a_{\pm k}$ of the polynomial $P(z)$:

$$|a_{\pm k} - c_{\pm k}| = |a_{\pm k} - r^k a_{\pm k}| < \epsilon, \quad k = 1,\ldots,m.$$

Let us verify, by the method of mathematical induction, that

$$|\sin nt| \leq n\sin t \;\forall\; t \in (0,\pi/2] \text{ and } \forall\; n \in \mathbb{N}.$$

The inequality is true for $n = 1$. Assume that the inequality is true for $(n-1)$. Then

$$|\sin nt| \leq |\sin(n-1)t|\cos t + |\cos(n-1)t|\sin t \leq$$

$$\leq \sin t[(n-1)\cos t + |\cos(n-1)t|] \leq n\sin t.$$

Let us show under the hypothesis of the theorem that the polynomial $P(z)$ is univalent in $\triangle$. For $z \in \Delta$, let us estimate the remainder R_N of the series. We have

$$|R_N| = \left|\sum_{n=N+1}^{\infty}\left[(a_n z^n - a_{-n}\bar{z}^n)r^n\frac{\sin nt}{\sin t}\right]\right| \leq \frac{(2e^2)^\alpha}{\alpha}\sum_{n=N+1}^{\infty} n^{\alpha+1}r^n$$

according to Lemma 2. By hypothesis, $N|\ln r| \geq 2s\ln N > \alpha + 1$, and therefore, $N > \dfrac{\alpha+1}{|\ln r|}$. But for $x \in \left[\dfrac{\alpha+1}{|\ln r|}, \infty\right)$, the function $T(x) = x^{\alpha+1}r^x$ decreases. Hence with the increasing values of n, the terms of the series $\sum\limits_{n=N}^{\infty} n^{\alpha+1}r^n$ decrease and therefore,

$$\sum_{n=N+1}^{\infty} n^{\alpha+1}r^n < \int_N^\infty T(x)dx \leq \int_N^\infty x^s r^x dx.$$

Integrating by parts successfully s times, we obtain

$$\int_N^\infty x^s r^x dx = \frac{N^s r^N}{|\ln r|} + \frac{sN^{s-1} r^N}{|\ln r|^2} + \frac{s(s-1)N^{s-2} r^N}{|\ln r|^3} + \dots + \frac{s! r^N}{|\ln r|^{s+1}}.$$

Let us represent the condition $N|\ln r| \geq 2s \ln N$ of the theorem as $r^{\frac{N}{2}} N^s \leq$ ≤ 1. Hence $r^N N^j \leq r^{N/2}$ for any $j = 1, \dots, s$. Taking into account these inequalities, we have the estimate

$$\int_N^\infty x^s r^x dx \leq$$

$$\leq \frac{s! r^{N/2}}{|\ln r|^{s+1}} \left[\frac{|\ln r|^s}{s!} + \frac{|\ln r|^{s-1}}{(s-1)!} + \ldots + \frac{|\ln r|}{1!} + r^{N/2} \right] < \frac{r^{N/2-1} s!}{|\ln r|^{s+1}}.$$

Hence

$$|R_N| \leq \frac{(2e^2)^\alpha}{\alpha} \frac{r^{N/2-1} s!}{|\ln r|^{s+1}}. \tag{14}$$

From (14), Theorem 1 and Lemma 1 we obtain

$$\left| \sum_{n=1}^{N} \left[(a_n z^n - a_{-n} \bar{z}^n) r^n \frac{\sin nt}{\sin t} \right] \right| \geq$$

$$\geq \left| \sum_{n=1}^{\infty} \left[(a_n z^n - a_{-n} \bar{z}^n) r^n \frac{\sin nt}{\sin t} \right] \right| - |R_N| \geq$$

$$\geq \frac{1-r}{4\alpha} \left(\frac{1-r}{1+r} \right)^\alpha \left[1 - \left(\frac{1-r}{1+r} \right)^{2\alpha} \right] - \frac{(2e^2)^\alpha}{\alpha} \frac{r^{N/2-1} s!}{|\ln r|^{s+1}} = Q.$$

If now $Q > 0$, then the univalency of the polynomial P follows from Theorem 1. The latter inequality is equivalent to

$$\frac{2}{|\ln r|} \left[\alpha(2 + \ln 2) + \ln(s!) - (s+1) \ln |\ln r| - \ln(1-r) - \alpha \ln \frac{1-r}{1+r} + \right.$$

$$\left. + \ln 4 - \ln \left[1 - \left(\frac{1-r}{1+r} \right)^{2\alpha} \right] \right] + 2 < N,$$

which proves Theorem 3. $\square$

The proof of Theorem 3 implies the following result.

Corollary 2. *Let $f \in S_H^0$ and f have the expansion (1), $\alpha = \mathrm{ord} S_H$, $r \in (0,1)$ and $0 \neq m \in \mathbb{Z}$. Then the m-th coefficient $c_m = a_m r^{|m|}$ of the univalent function*

$$f(rz) = \sum_{n=1}^{\infty} (c_n z^n + c_{-n} \bar{z}^n)$$

may be replaced by $c_m + \lambda$, where $\lambda \in \mathbb{C}$, and

$$|\lambda| < \frac{1-r}{4\alpha|m|} \left(\frac{1-r}{1+r}\right)^{\alpha} \left[1 - \left(\frac{1-r}{1+r}\right)^{2\alpha}\right]$$

without loss of univalence.

Proof. According to Lemma 1,

$$\left| \sum_{n=1}^{\infty} \left[(a_n z^n - a_{-n} \bar{z}^n) r^n \frac{\sin nt}{\sin t} \right] \right| \geq \frac{1-r}{4\alpha} \left(\frac{1-r}{1+r}\right)^{\alpha} \left[1 - \left(\frac{1-r}{1+r}\right)^{2\alpha}\right].$$

By Theorem 1, the condition of the univalence of the new function, obtained by the variation of the m-th coefficient c_m gives

$$\left| \sum_{n=1}^{\infty} \left[(a_n z^n - a_{-n} \bar{z}^n) r^n \frac{\sin nt}{\sin t} \right] + \sigma \lambda \rho^{|m|} \frac{\sin mt}{\sin t} \right| > 0, \ z \in \Delta \setminus 0, \quad \rho = |z|,$$

$t \in (0, \pi/2]$, where $|\sigma| = 1$. It is fulfilled provided that

$$\left| \lambda \frac{\sin mt}{\sin t} \right| < \frac{1-r}{4\alpha} \left(\frac{1-r}{1+r}\right)^{\alpha} \left[1 - \left(\frac{1-r}{1+r}\right)^{2\alpha}\right].$$

Hence it is certainly fulfilled if the inequality concerning $|\lambda|$, from the statement of Lemma, is true. □

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