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ON REGULARITY THEOREMS FOR LINEARLY INVARIANT FAMILIES OF HARMONIC FUNCTIONS

Abstract. The classical theorem of growth regularity in the class S of analytic and univalent in the unit disc Δ functions f describes the growth character of different functionals of $f \in S$ and $z \in \Delta$ as z tends to $\partial\Delta$. Earlier the authors proved the theorems of growth and decrease regularity for harmonic and sense-preserving in Δ functions which generalized the classical result for the class S. In the presented paper we establish new properties of harmonic sense-preserving functions, connected with the regularity theorems. The effects both common for analytic and harmonic case and specific for harmonic functions are displayed.

Key words: *regularity theorem, linearly invariant family, harmonic function*

2010 Mathematical Subject Classification: *30C55*

1. Introduction. For a function $u(z)$, continuous in the disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, we denote

$$M(r,u) = \max_{|z|\le r} |u(z)| \quad \text{and} \quad m(r,u) = \min_{|z|\le r} |u(z)|.$$

Let S be the class of all univalent analytic functions $f(z) = z + \dots$ in Δ. The theorem of growth regularity asserts that functions having the maximal growth in the given class, grows smoothly (regularly).

Theorem A. [1], [2], [3, pp. 104, 105], [4, pp. 8–9] *Let $f \in S$. Then there exist a $\delta_0 \in [0,1]$ with*

$$\lim_{r\to 1-} \left[M(r,f)\frac{(1-r)^2}{r} \right] = \lim_{r\to 1-} \left[M(r,f')\frac{(1-r)^3}{1+r} \right] = \delta^0,$$

$\delta^0 = 1$ *iff* $f(z) = z(1 - ze^{-i\theta})^{-2}$. *If* $\delta^0 \neq 1$, *then the functions under the sign of the limit increase on* r.

If $\delta^0 \neq 0$, *then there exists* $\varphi^0 \in [0; 2\pi)$ *such that*

$$\lim_{r\to 1-}\left[|f(re^{i\varphi})|\frac{(1-r)^2}{r}\right] = \lim_{r\to 1-}\left[|f'(re^{i\varphi})|\frac{(1-r)^3}{1+r}\right] = \begin{cases} \delta^0, & \varphi = \varphi^0 \\ 0, & \varphi \neq \varphi^0. \end{cases}$$

Here the functions under the sign of the limit are also increasing on $r \in$ $\in (0, 1)$.

In [5], Ch. Pommerenke showed that many properties of functions from the class S can be extended to linearly invariant families (LIFs) of locally univalent analytic functions in Δ of finite order. In [6] and [7], the theorem of growth regularity was obtained for such LIFs.

In [8], [9], the authors introduced the notion of LIF for complex-valued harmonic functions f in Δ. Every such function can be presented, using analytic functions h and g in Δ in the following way:

$$f(z) = h(z) + \overline{g(z)}, \tag{1}$$

where

$$h(z) = z + \sum_{n=2}^{\infty} a_n(f)z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} \overline{a_{-n}(f)}z^n.$$

As in [5], L. E. Shaubroek considered locally univalent functions in Δ. Moreover, these functions are sense-preserving in Δ, i. e. the Jacobian $J_f(z)$ satisfies

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0 \quad \forall z \in \Delta.$$

Definition 1. [8], [9] *A set* $\mathfrak{M}_H$ *of harmonic sense-preserving functions* f *in* Δ *of form (1) is called the linearly invariant family (LIF) if for all* $f \in \mathfrak{M}_H$ *and for any conformal automorphism* $\phi(z) = \frac{z+a}{1+\bar{a}z}$, $a \in \Delta$, *the function* $e^{-i\theta} f_a(ze^{i\theta})$ *belongs to* $\mathfrak{M}_H$, *where*

$$f_a(z) = \frac{f(\varphi(z)) - f(\varphi(0))}{h'(\varphi(0))\varphi'(0)}. \tag{2}$$

It is assumed that the *order* of a family $\mathfrak{M}_H$

$$\operatorname{ord}\mathfrak{M}_H = \sup_{f\in\mathfrak{M}_H} |a_2(f)|$$

is finite.

In the analytic case (when $g(z) \equiv 0$), the definitions of LIF and $\operatorname{ord} \mathfrak{M}_H$ coincide with the definitions of Pommerenke [5].

In [10], for LIFs of harmonic functions, the *strong order*

$$\overline{\operatorname{ord}}\, \mathfrak{M}_H = \sup_{f \in \mathfrak{M}_H} \frac{|a_2(f) - a_{-1}(f)\overline{a_{-2}(f)}|}{1 - |a_{-1}(f)|^2}$$

was defined. The strong order proved to be convenient for investigation of LIFs, because it is not necessary to assume the affine invariance of a family. Moreover, for an affine LIF $\mathfrak{M}_H$ the strong order does not exceed the old order:

$$\operatorname{ord} \mathfrak{M}_H - \frac{1}{2} \leq \overline{\operatorname{ord}}\, \mathfrak{M}_H \leq \operatorname{ord} \mathfrak{M}_H.$$

This fact allows to describe properties of affine LIFs more precisely. For a LIF $\mathfrak{M}$ of analytic functions, $\operatorname{ord} \mathfrak{M}_H = \overline{\operatorname{ord}}\, \mathfrak{M}_H$. Analogously to the analytic case in [10] the *universal* LIF $\mathcal{U}_\alpha^H$ was introduced and studied. The family $\mathcal{U}_\alpha^H$ is defined as the union of all LIFs $\mathfrak{M}_H$ such that $\overline{\operatorname{ord}}\, \mathfrak{M}_H \leq \leq \alpha$. Equivalently, $\mathcal{U}_\alpha^H$ is the set of all harmonic sense-preserving functions f in Δ of the form (1) such that

$$\overline{\operatorname{ord}}\, f \stackrel{def}{=} \overline{\operatorname{ord}}\, \{e^{-i\theta} f_a(ze^{i\theta}) : a \in \Delta,\ \theta \in \mathbb{R}\} \leq \alpha.$$

It was shown in [10] that $\overline{\operatorname{ord}}\, \mathcal{U}_\alpha^H \geq 1$.

In [11] and [12], the following regularity theorems for harmonic functions were proved:

Theorem B. (regularity of growth) *Let $f \in \mathcal{U}_\alpha^H$. Set*

$$\Phi_1(r) = \int_0^r M(\rho, J_f)\, d\rho, \quad \Psi_1(r, \varphi) = \int_0^r J_f(\rho e^{i\varphi})\, d\rho, \quad \text{and}$$

$$F_1(r) = \int_0^r \frac{(1+\rho)^{2\alpha-2}}{(1-\rho)^{2\alpha+2}}\, d\rho.$$

For each $n \geq 2$ successively denote

$$\Phi_n(r) = \int_0^r \Phi_{n-1}(\rho)\, d\rho, \quad \Psi_n(r, \varphi) = \int_0^r \Psi_{n-1}(\rho, \varphi)\, d\rho, \quad \text{and}$$

$$F_n(r) = \int\limits_0^r F_{n-1}(\rho)\, d\rho.$$

Then

a) for every $\varphi \in [0; 2\pi)$ *and* $n \in \mathbb{N}$, *the functions*

$$J_f(re^{i\varphi})\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}, \quad M(r, J_f)\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}},$$

$$\frac{\Phi_n(r)}{F_n(r)}, \quad \frac{\Psi_n(r,\varphi)}{F_n(r)}, \quad \text{and} \quad \frac{\max\limits_{\varphi}\Psi_n(r,\varphi)}{F_n(r)}$$

are non-increasing on $r \in (0; 1)$;

b) there exist constants $\delta^0 \in [0; 1]$ *and* $\varphi^0 \in [0; 2\pi)$ *such that for* $1 \le n \le 2\alpha + 2$,

$$\delta^0 = \lim_{r\to 1-}\left[\frac{M(r, J_f)}{J_f(0)}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right] = \lim_{r\to 1-}\left[\frac{J_f(re^{i\varphi^0})}{J_f(0)}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right] =$$

$$= \lim_{r\to 1-}\left[\frac{M(r, \frac{\partial}{\partial r}J_f)}{J_f(0)4(\alpha+1)}\frac{(1-r)^{2\alpha+3}}{(1+r)^{2\alpha-3}}\right] =$$

$$= \overline{\lim_{r\to 1-}}\left[\frac{\left|\frac{\partial}{\partial r}J_f(re^{i\varphi^0})\right|}{J_f(0)4(\alpha+1)}\frac{(1-r)^{2\alpha+3}}{(1+r)^{2\alpha-3}}\right] =$$

$$= \lim_{r\to 1-}\left[\frac{\int_0^r M(\rho, \frac{\partial}{\partial\rho}J_f)\, d\rho}{J_f(0)}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right] =$$

$$= \lim_{r\to 1-}\left[\frac{\int_0^r \left|\frac{\partial}{\partial\rho}J_f(\rho e^{i\varphi^0})\right|\, d\rho}{J_f(0)}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right] =$$

$$= \lim_{r\to 1-}\frac{\Phi_n(r)}{J_f(0)F_n(r)} = \lim_{r\to 1-}\frac{\Psi_n(r,\varphi^0)}{J_f(0)F_n(r)} = \lim_{r\to 1-}\frac{\max\limits_{\varphi}\Psi_n(r,\varphi)}{J_f(0)F_n(r)};$$

c) $\delta^0 = 1$ *for functions* $q_\theta(z) = e^{i\theta}k_\alpha(ze^{-i\theta}) + \sigma e^{i\theta}\overline{k_\alpha(ze^{-i\theta})}$, *where* $\sigma \in \Delta$, $\theta \in \mathbb{R}$, *and*

$$k_\alpha(z) = \frac{1}{2\alpha}\left[\left(\frac{1+z}{1-z}\right)^\alpha - 1\right]. \tag{3}$$

Theorem C. (regularity of decrease) *Let $f \in \mathcal{U}_\alpha^H$. Set*

$$Q_1(r) = \int_r^1 m(\rho, J_f)\, d\rho, \quad E_1(r) = \int_r^1 \frac{(1-\rho)^{2\alpha-2}}{(1+\rho)^{2\alpha+2}}\, d\rho.$$

For each $n \geq 2$ successively denote

$$Q_n(r) = \int_r^1 Q_{n-1}(\rho)\, d\rho, \quad \text{and} \quad E_n(r) = \int_r^1 E_{n-1}(\rho)\, d\rho.$$

Then

a) for every $\varphi \in [0; 2\pi)$ and $n \in \mathbb{N}$ the functions

$$J_f(re^{i\varphi})\frac{(1+r)^{2\alpha+2}}{(1-r)^{2\alpha-2}}, \quad m(r, J_f)\frac{(1+r)^{2\alpha+2}}{(1-r)^{2\alpha-2}}, \quad \text{and} \quad \frac{Q_n(r)}{E_n(r)}$$

are non-decreasing on $r \in (0; 1)$;

b) there exist constants $\delta_0 \in [1; \infty]$ and $\varphi_0 \in [0; 2\pi)$ such that

$$\delta_0 = \lim_{r\to 1-}\left[\frac{m(r, J_f)}{J_f(0)}\frac{(1+r)^{2\alpha+2}}{(1-r)^{2\alpha-2}}\right] = \lim_{r\to 1-}\left[\frac{J_f(re^{i\varphi_0})}{J_f(0)}\frac{(1+r)^{2\alpha+2}}{(1-r)^{2\alpha-2}}\right] =$$

$$= \lim_{r\to 1-}\frac{Q_n(r)}{J_f(0)E_n(r)};$$

c) for $\varphi \in [0; 2\pi)$ denote

$$R_1(r, \varphi) = \int_r^1 J_f(\rho e^{i\varphi})\, d\rho,$$

and for $n \geq 2$, set

$$R_n(r, \varphi) = \int_r^1 R_{n-1}(\rho, \varphi)\, d\rho$$

(under the assumptions of Theorem C the integrals converge). If $\delta_0 < \infty$ then for $n \geq 1$ the function $\frac{R_n(r,\varphi_0)}{E_n(r)}$ is non-decreasing on $r \in (0; 1)$. Moreover,

$$\delta_0 = \lim_{r\to 1-}\frac{R_n(r, \varphi_0)}{J_f(0)E_n(r)};$$

d) if $J_f(z)$ is bounded in Δ, then for every $n \in \mathbb{N}$ and every $\varphi \in [0; 2\pi)$, the functions

$$\frac{R_n(r,\varphi)}{E_n(r)} \quad \text{and} \quad \frac{\min\limits_{\varphi} R_n(r,\varphi)}{E_n(r)}$$

are non-decreasing on $r \in (0;1)$ and

$$\delta_0 = \lim_{r\to 1-} \frac{\min\limits_{\varphi} R_n(r,\varphi)}{J_f(0)E_n(r)};$$

e) $\delta_0 = 1$ for functions $q_\theta(z) = e^{i\theta}k_\alpha(ze^{-i\theta}) + \sigma e^{i\theta}\overline{k_\alpha(ze^{-i\theta})}$, where $\sigma \in \Delta$, $\theta \in \mathbb{R}$, and $k_\alpha(z)$ is the function defined by (3).

Definition 2. *We say that the constant φ^0 from Theorem B is a direction of maximal growth (d.m.g.) of a function $f(z)$. The constant φ_0 from Theorem C is a direction of maximal decrease (d.m.d.) of $f(z)$.*

Definition 3. *The numbers δ^0 from Theorem B and δ_0 from Theorem C are called the Hayman numbers of a function $f(z)$.*

In the presented paper we establish new properties of $\mathcal{U}_\alpha^H$, connected with the regularity theorems.

2. Main results. For fixed $c \in [0; 1)$ introduce the class $\mathcal{U}_{\alpha,c}^H$, consisting of all functions $f = h + \overline{g} \in \mathcal{U}_\alpha^H$ such that $|g'(0)| \le c$. That is, $J_f(0) \ge 1 - c^2 > 0$ for all $f \in \mathcal{U}_{\alpha,c}^H$. The class $\mathcal{U}_{\alpha,c}^H$ is not a LIF. Note that the family $\mathcal{U}_\alpha^H$ is not compact in the topology inducted by locally uniform convergence in Δ, but for $\mathcal{U}_{\alpha,c}^H$ the following theorem takes place.

Theorem 1. *The family $\mathcal{U}_{\alpha,c}^H$ is compact in the topology inducted by locally uniform convergence in Δ.*

Proof. Let $f_n \in \mathcal{U}_{\alpha,c}^H$, $f_n = h_n + \bar{g}_n$, $n \in \mathbb{N}$, h_n and g_n be analytic functions in Δ. By A_α denote the set of all analytic functions h in Δ such that there exists an analytic function g in Δ and $f = h + \bar{g} \in \mathcal{U}_\alpha^H$. In other words, A_α is the set of analytic parts of functions $f \in \mathcal{U}_\alpha^H$. The linearly invariance of $\mathcal{U}_\alpha^H$ implies that A_α is a LIF of analytic functions. But for LIFs of analytic functions $\overline{\operatorname{ord}}\, A_\alpha = \operatorname{ord} A_\alpha$. Therefore for all $h \in A_\alpha$

$$|h'(z)| \le \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}, \quad |z| = r,$$

see [5]. Since $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$ for all $z \in \Delta$ and all $f \in \mathcal{U}_\alpha^H$, we have

$$|g'(z)| \leq \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}},$$

for all $f = h + \bar{g} \in \mathcal{U}_\alpha^H$ and $z \in \Delta$, $|z| = r$. Consequently, $\mathcal{U}_{\alpha,c}^H \subset \subset \mathcal{U}_\alpha^H$ is uniformly bounded on compact subsets of Δ. According to the compactness principle, there exists a subsequence of f_n (let us save the notation) which converges locally uniformly in Δ to a harmonic function f_0. Let us show that $f_0 \in \mathcal{U}_{\alpha,c}^H$.

For $f \in \mathcal{U}_\alpha^H$ the following inequality holds (see [10])

$$\frac{(1-r)^{2\alpha-2}}{(1+r)^{2\alpha+2}} \leq \frac{J_f(z)}{J_f(0)} \leq \frac{(1+r)^{2\alpha-2}}{(1-r)^{2\alpha+2}}, \quad |z| = r.$$

Therefore for $f_n \in \mathcal{U}_{\alpha,c}^H$ we have

$$J_{f_n}(z) \geq \frac{(1-r)^{2\alpha-2}}{(1+r)^{2\alpha+2}}(1-c^2) > 0.$$

This implies $J_{f_0}(z) > 0$ for all $z \in \Delta$. This means that the harmonic in Δ function f_0 is sense-preserving.

Next, we prove that $\overline{\operatorname{ord}}\, f_0 \leq \alpha$. Suppose not. Then, we may let $\overline{\operatorname{ord}}\, f_0 = \beta > \alpha$. Then, by the definition of the strong order, there exist a conformal automorphism $\varphi(z) = \frac{z+a}{1+\bar{a}z}$ of Δ and $\theta \in \mathbb{R}$ such that for harmonic function

$$e^{-i\theta}(f_0)_a(ze^{i\theta}) = \frac{f_0(\varphi(ze^{i\theta})) - f_0(\varphi(0))}{h_0'(\varphi(0))\varphi'(0)e^{i\theta}} = \sum_{k=1}^{\infty}(A_k z^k + A_{-k}\bar{z}^k),$$

$(A_1 = 1,\ f_0 = h_0 + \overline{g_0})$ the inequality

$$\frac{\left|A_2 - A_{-1}\overline{A_{-2}}\right|}{1 - |A_{-1}|^2} > \alpha + \frac{\beta - \alpha}{2} \tag{4}$$

is valid.

For the automorphism φ and the number θ denote

$$e^{-i\theta}(f_n)_a(ze^{i\theta}) = \sum_{k=1}^{\infty}(A_k^{(n)} z^k + A_{-k}^{(n)}\bar{z}^k), \quad (A_1^{(n)} = 1).$$

From locally uniform convergence of f_n to f_0, the Weierstrass theorem on series of analytic functions, and inequality (4) it follows that for sufficiently large $n > N$

$$\frac{\left|A_2^{(n)} - A_{-1}^{(n)}\overline{A_{-2}^{(n)}}\right|}{1 - |A_{-1}^{(n)}|^2} > \alpha + \frac{\beta - \alpha}{2}.$$

Hence if $n > N$ we have $\overline{\mathrm{ord}}\, f_n > \alpha + \frac{\beta-\alpha}{2}$ and $f_n \notin \mathcal{U}^H_{\alpha,c}$. This contradiction proves the theorem. $\square$

In claim c) of Theorem B and claim e) of Theorem C some set of functions with the Hayman number $\delta^0 = 1$ (or $\delta_0 = 1$ for the theorem of decrease regularity) is described. These claims differ from the analytic case. In the analytic case $\delta^0 = 1$ and $\delta_0 = 1$ only for the functions $e^{i\theta}k_\alpha(ze^{-i\theta})$, where $\theta \in \mathbb{R}$, $k_\alpha(z)$ is the function defined by (3) [7], [13], [14]. The following example shows that in the harmonic case this set has more complicated structure. We construct the example of functions f of arbitrary strong order $\beta \geq 3/2$ with $\delta^0 = 1$. These functions are not equal to the function $q_\theta(z)$ from Theorem B. We use the Clunie and Sheil-Small shear construction [15] (see also [16, ch. 3.4]) to give our example. Let us note that our construction is not stable. As one can show, if we multiply the coanalytic part g of the function from our example by constant $k \in (0,1)$, then the strong order of the function changes stepwise and $\delta^0 \neq 1$ for this function.

Example. Put $h'(z) = \frac{(1+z)^{\alpha-1}}{(1-z)^{\alpha+2}}$, $g'(z) = zh'(z)$, $z \in \Delta$. Let $\alpha \in [1,\infty)$ be fixed. If $\varphi(z) = \frac{z+a}{1+\bar{a}z}$, $a \in \Delta$, is an automorphism of Δ, then for $f = h + \bar{g}$ we have

$$f_a(z) =: F(z) = H(z) + \overline{G(z)} = \frac{h(\varphi(z)) - h(\varphi(0))}{h'(\varphi(0))\varphi'(0)} + \overline{\left(\frac{g(\varphi(z)) - g(\varphi(0))}{h'(\varphi(0))\varphi'(0)}\right)},$$

where H and G are functions analytic in Δ,

$$H'(z) = \frac{h'(\varphi(z))\varphi'(z)}{h'(\varphi(0))\varphi'(0)} \quad \text{and}$$

$$G'(z) = \frac{g'(\varphi(z))\varphi'(z)}{h'(\varphi(0))\varphi'(0)} = \frac{\varphi(z)h'(\varphi(z))\varphi'(z)}{h'(\varphi(0))\varphi'(0)}.$$

Note that

$$J_F(z) = |H'(z)|^2 - |G'(z)|^2 = \frac{|h'(\varphi(z))|^2|\varphi'(z)|^2(1-|\varphi(z)|^2)}{|h'(\varphi(0))|^2|\varphi'(0)|^2},$$

and, in particular,

$$J_F(0) = 1 - |\varphi(0)|^2.$$

Therefore,

$$\frac{J_F(z)}{J_F(0)} = \frac{|1+\frac{z+a}{1+\bar{a}z}|^{2\alpha-2}}{|1+a|^{2\alpha-2}} \cdot \left|\frac{1-a}{1-\frac{z+a}{1+\bar{a}z}}\right|^{2\alpha+4} \cdot \frac{(1-|a|^2)^2}{|1+\bar{a}z|^4} \times$$

$$\times \left(1 - \left|\frac{z+a}{1+\bar{a}z}\right|^2\right) \frac{1}{(1-|a|^2)^3} =$$

$$= \frac{\left|1+z\frac{1+\bar{a}}{1+a}\right|^{2\alpha-2}}{\left|1-z\frac{1-\bar{a}}{1-a}\right|^{2\alpha+4}} \cdot \frac{|1+\bar{a}z|^2 - |z+a|^2}{1-|a|^2} = \frac{\left|1+z\frac{1+\bar{a}}{1+a}\right|^{2\alpha-2}}{\left|1-z\frac{1-\bar{a}}{1-a}\right|^{2\alpha+4}}(1-|z|^2),$$

by generalized Schwarz's lemma. Consequently, for $r \in (0,1)$

$$\sup_{\substack{a\in\Delta,\\ |z|=r}} \frac{J_F(z)}{J_F(0)} = \frac{(1+r)^{2\alpha-1}}{(1-r)^{2\alpha+3}}.$$

Therefore for $\beta = \alpha + \frac{1}{2}$, all $a \in \Delta$, and $|z| = r$ we get

$$\frac{J_F(z)}{J_F(0)} \le \frac{(1+r)^{2\beta-2}}{(1-r)^{2\beta+2}}. \tag{5}$$

In [10] it was shown that for functions f harmonic and sense-preserving in Δ,

$$\overline{\operatorname{ord}}\, f = \inf\left\{\beta : \frac{J_F(z)}{J_F(0)} \le \frac{(1+|z|)^{2\beta-2}}{(1-|z|)^{2\beta+2}}, \quad \forall F = f_a, \forall z \in \Delta\right\}. \tag{6}$$

From (5) and (6) we conclude that $\overline{\operatorname{ord}}\, f \le \beta = \alpha + \frac{1}{2}$. From Theorem B it follows that if for a function f harmonic and sense-preserving in Δ

$$\lim_{r\to 1-} \left[\frac{J_f(z)}{J_f(0)} \frac{(1-r)^{2\beta+2}}{(1+r)^{2\beta-2}}\right] > 0, \tag{7}$$

then $\overline{\operatorname{ord}}\, f \geq \beta$. For the considered function f the limit in (7) equals 1. Therefore, $\overline{\operatorname{ord}}\, f = \beta$ and

$$\delta^0 = \lim_{r\to 1-}\left[\frac{J_f(r)}{J_f(0)}\frac{(1-r)^{2\beta+2}}{(1+r)^{2\beta-2}}\right] = 1.$$

It is interesting to find out if there exist functions with $\delta^0 = 1$ which are not equal to the function from the example and the functions $q_\theta(z)$.

Definition 4. *A direction of intensive growth (d.i.g.) of a function $f(z)$ is a constant $\varphi \in [0; 2\pi)$ such that*

$$\lim_{r\to 1-}\left[\frac{J_f(re^{i\varphi})}{J_f(0)}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right] = \delta(f,\varphi) > 0.$$

A direction of intensive decrease (d.i.d) of a function $f(z)$ is a constant $\varphi \in [0; 2\pi)$ such that

$$\lim_{r\to 1-}\left[\frac{J_f(re^{i\varphi})}{J_f(0)}\frac{(1+r)^{2\alpha+2}}{(1-r)^{2\alpha-2}}\right] = \delta'(f,\varphi) < \infty.$$

Since we study LIFs, it is important to know how d.i.g.-'s and d.i.d.-'s of a function $f(z)$ are changed under the transformation $e^{-i\theta}f_a(ze^{i\theta})$. The case $a = 0$ is trivial: a d.i.g. (d.i.d.) $\varphi - \theta$ of the function $e^{-i\theta}f(ze^{i\theta})$ corresponds to the d.i.g. (d.i.d.) φ of $f(z)$. In this situation $\delta(f(z),\varphi) = \delta(f(ze^{i\theta}),\varphi-\theta)$ (and $\delta'(f(z),\varphi) = \delta'(f(ze^{i\theta}),\varphi-\theta)$). It is also interesting to find out the relationship between the Hayman numbers of the functions f and f_a in general case. The following theorem concerns the non-obvious case $a \neq 0$.

Theorem 2. *Let $f \in \mathcal{U}_\alpha^H$. Denote*

$$R(r) = \left|\frac{re^{i\varphi}+a}{1+\bar{a}re^{i\varphi}}\right|, \quad \gamma(r) = \arg\frac{re^{i\varphi}+a}{1+\bar{a}re^{i\varphi}}, \quad a \in \Delta, \quad re^{i\varphi} \neq -a.$$

1) φ is a d.i.g. (d.i.d.) of the function $f_a(z)$ iff γ is a d.i.g. (d.i.d.) of $f(z)$ and

$$e^{i\varphi} = \frac{e^{i\gamma}-a}{1-\bar{a}e^{i\gamma}}; \tag{8}$$

2) for all $\gamma \in [0, 2\pi)$

$$\lim_{r\to 1-}\left[\frac{J_f(re^{i\gamma})}{J_f(0)}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right] = \lim_{r\to 1-}\left[\frac{J_f(R(r)e^{i\gamma(r)})}{J_f(0)}\frac{(1-R(r))^{2\alpha+2}}{(1+R(r))^{2\alpha-2}}\right],$$

and

$$\lim_{r\to 1-}\left[\frac{J_f(re^{i\gamma})}{J_f(0)}\frac{(1+r)^{2\alpha+2}}{(1-r)^{2\alpha-2}}\right] = \lim_{r\to 1-}\left[\frac{J_f(R(r)e^{i\gamma(r)})}{J_f(0)}\frac{(1+R(r))^{2\alpha+2}}{(1-R(r))^{2\alpha-2}}\right].$$

Here φ *and* γ *are connected by (8).*

3) if φ *is a d.i.g. of* $f_a(z)$, γ *is a d.i.g. of* $f(z)$, *and* φ *is connected with* γ *by (8), then*

$$\delta(f,\gamma) = \delta(f_a,\varphi)\frac{J_f(a)}{J_f(0)}\frac{(1-|a|^2)^{2\alpha+2}}{|1+\overline{a}e^{i\varphi}|^{4\alpha}};$$

if φ *is a d.i.d. of* $f_a(z)$, γ *is a d.i.d. of* $f(z)$, *and* φ *is connected with* γ *by (8), then*

$$\delta'(f,\gamma) = \delta'(f_a,\varphi)\frac{J_f(a)}{J_f(0)}\frac{|1+\overline{a}e^{i\varphi}|^{4\alpha}}{(1-|a|^2)^{2\alpha-2}}.$$

Proof. 1) Let φ be a d.i.g. of $f_a(z)$. This means that there exists the limit

$$\delta(f_a,\varphi) = \lim_{r\to 1-}\left[\frac{J_{f_a}(re^{i\varphi})}{J_{f_a}(0)}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right] > 0.$$

Note that

$$J_{f_a}(z) = \frac{J_f\left(\frac{z+a}{1+\overline{a}z}\right)}{|h'(a)|^2|1+\overline{a}z|^4}, \tag{9}$$

and

$$J_{f_a}(0) = \frac{J_f(a)}{|h'(a)|^2}. \tag{10}$$

Let us calculate the following limit, using (9) and (10),

$$\delta \stackrel{def}{=} \lim_{r\to 1-}\left[\frac{J_f(R(r)e^{i\gamma(r)})}{J_f(0)}\frac{(1-R(r))^{2\alpha+2}}{(1+R(r))^{2\alpha-2}}\right] =$$

$$= \lim_{r\to 1-}\left[\frac{J_{f_a}(re^{i\varphi})}{J_f(0)}|h'(a)|^2|1+\overline{a}re^{i\varphi}|^4\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\left(\frac{1-R(r)}{1-r}\right)^{2\alpha+2}\right].$$

We have

$$\lim_{r\to 1-}\frac{1-R(r)}{1-r}=\lim_{r\to 1-}R'(r)=\frac{1-|a|^2}{|1+\overline{a}e^{i\varphi}|^2}. \tag{11}$$

Using (11), we obtain

$$\delta=\delta(f_a,\varphi)\frac{J_f(a)}{J_f(0)}|1+\overline{a}e^{i\varphi}|^4\left(\frac{1-|a|^2}{|1+\overline{a}e^{i\varphi}|^2}\right)^{2\alpha+2}>0. \tag{12}$$

By (11), $\lim\limits_{r\to 1-}R'(r)>0$, therefore the function $R(r)$ increases on an interval $(r_0,1)$. By Theorem B, for $r_0<r<r_1<1$

$$\frac{J_f(R(r_1)e^{i\gamma(r_1)})}{J_f(0)}\frac{(1-R(r_1))^{2\alpha+2}}{(1+R(r_1))^{2\alpha-2}}\le\frac{J_f(R(r)e^{i\gamma(r_1)})}{J_f(0)}\frac{(1-R(r))^{2\alpha+2}}{(1+R(r))^{2\alpha-2}}.$$

Passing to the limit as $r_1\to 1-$ and using (8), we get

$$\delta\le\frac{J_f(R(r)e^{i\gamma})}{J_f(0)}\frac{(1-R(r))^{2\alpha+2}}{(1+R(r))^{2\alpha-2}}.$$

Thus,

$$\delta(f,\gamma)=\lim_{r\to 1-}\left[\frac{J_f(R(r)e^{i\gamma})}{J_f(0)}\frac{(1-R(r))^{2\alpha+2}}{(1+R(r))^{2\alpha-2}}\right]\ge\delta. \tag{13}$$

Taking into account (12), we conclude that γ is a d.i.g. of $f(z)$.

Now let us consider the sets

$$A=\{e^{i\gamma}:\gamma \text{ is a d.i.g. of } f(z)\},$$

$$B=\left\{\frac{e^{i\varphi}+a}{1+\overline{a}e^{i\varphi}}:\varphi \text{ is a d.i.g. of } f_a(z)\right\},$$

$$C=\left\{e^{i\eta}:\eta \text{ is a d.i.g. of } [f_a]_{(-a)}(z)\right\}.$$

Here $[f_a]_{(-a)}(z)$ is the transformation (2) of the function f_a with the parameter $-a$. If η is a d.i.g. of $[f_a]_{(-a)}(z)$, then, as it was proved above,

$$e^{i\eta}=\frac{e^{i\varphi}+a}{1+\overline{a}e^{i\varphi}},$$

where φ is a d.i.g. of $f_a(z)$. This implies that $C\subset B$. Let φ be a d.i.g. of $f_a(z)$. Then

$$e^{i\gamma}=\frac{e^{i\varphi}+a}{1+\overline{a}e^{i\varphi}},$$

where γ is a d.i.g. of $f(z)$. Thus $B \subset A$. Since $[f_a]_{(-a)}(z) = f(z)$, we have $A = C$ and, consequently, $A = B$. This completes the proof of the statement about d.i.g.-'s.

The statement about d.i.d.-'s is proved analogously.

2) Let us prove the first equality. If γ is not a d.i.g. of $f(z)$, then

$$\lim_{r\to 1-}\left[\frac{J_f(re^{i\gamma})}{J_f(0)}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right] = 0.$$

Thus, by (13),

$$\delta \leq \lim_{r\to 1-}\left[\frac{J_f(R(r)e^{i\gamma})}{J_f(0)}\frac{(1-R(r))^{2\alpha+2}}{(1+R(r))^{2\alpha-2}}\right] = 0.$$

This implies $\delta = 0$.

Now let us consider the case when γ is a d.i.g. of $f(z)$. We have proved above that $\delta(f,\gamma) \geq \delta$ (see (13)). It remains to show that $\delta(f,\gamma) \leq \delta$.

Denote

$$R_1(r) = \left|\frac{re^{i\gamma}-a}{1-\bar{a}re^{i\gamma}}\right|.$$

Since $[f_a]_{(-a)}(z) = f(z)$, γ is a d.i.g. of $[f_a]_{(-a)}(z)$, i. e.

$$\delta([f_a]_{(-a)},\gamma) = \delta(f,\gamma) = \lim_{r\to 1-}\left[\frac{J_{[f_a]_{(-a)}}(re^{i\gamma})}{J_f(0)}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right] > 0.$$

Arguing as in the proof of claim 1), one can note that there exists

$$\delta^* \stackrel{def}{=} \lim_{r\to 1-}\left[\frac{J_{f_a}\left(\dfrac{re^{i\gamma}-a}{1-\bar{a}re^{i\gamma}}\right)}{J_{f_a}(0)}\frac{(1-R_1(r))^{2\alpha+2}}{(1+R_1(r))^{2\alpha-2}}\right].$$

Apply (13) to the function $f_a(z)$, using (9), (10), and (11):

$$\delta^* \leq \lim_{r\to 1-}\left[\frac{J_{f_a}(re^{i\varphi})}{J_{f_a}(0)}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right] =$$

$$= \lim_{r\to 1-}\left[\frac{J_f\left(\dfrac{re^{i\varphi}+a}{1+\bar{a}re^{i\varphi}}\right)}{J_f(a)|1+\bar{a}re^{i\varphi}|^4}\frac{(1-R(r))^{2\alpha+2}}{(1+R(r))^{2\alpha-2}}\right]\cdot\lim_{r\to 1-}\left(\frac{1-r}{1-R(r)}\right)^{2\alpha+2} =$$

$$= \frac{\delta J_f(0)}{J_f(a)|1+\overline{a}e^{i\varphi}|^4}\left(\frac{|1+\overline{a}e^{i\varphi}|^2}{1-|a|^2}\right)^{2\alpha+2} = \frac{\delta J_f(0)}{J_f(a)}\frac{|1+\overline{a}e^{i\varphi}|^{4\alpha}}{(1-|a|^2)^{2\alpha+2}}. \quad (14)$$

On the other hand, by (9),

$$J_{f_a}\left(\frac{z-a}{1-\overline{a}z}\right) = \frac{J_f(z)}{|h'(a)|^2\left|1+\overline{a}\dfrac{z-a}{1-\overline{a}z}\right|^4}.$$

Thus, using (8), (10), and (11), we can write δ^* in the form

$$\delta^* = \lim_{r\to 1-}\left[\frac{J_f(re^{i\gamma})}{J_f(a)\left|1+\overline{a}\frac{re^{i\gamma}-a}{1-\overline{a}re^{i\gamma}}\right|^4}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right]\times$$

$$\times\lim_{r\to 1-}\left(\frac{1-R_1(r)}{1-r}\right)^{2\alpha+2} =$$

$$= \delta(f,\gamma)\frac{J_f(0)}{J_f(a)|1+\overline{a}e^{i\varphi}|^4}\left(\frac{1-|a|^2}{|1-\overline{a}e^{i\gamma}|^2}\right)^{2\alpha+2} =$$

$$= \delta(f,\gamma)\frac{J_f(0)}{J_f(a)}\frac{|1+\overline{a}e^{i\varphi}|^{4\alpha}}{(1-|a|^2)^{2\alpha+2}}.$$

Substituting

$$\delta^* = \delta(f,\gamma)\frac{J_f(0)}{J_f(a)}\frac{|1+\overline{a}e^{i\varphi}|^{4\alpha}}{(1-|a|^2)^{2\alpha+2}}$$

in (14), we get $\delta(f,\gamma)\le\delta$. Therefore, $\delta(f,\gamma)=\delta$.

The second equality of claim 2) is proved analogously.

3) The formula, connected $\delta(f,\gamma)$ and $\delta(f_a,\varphi)$ is obtained from (12), using $\delta=\delta(f,\gamma)$.

The second equality is proved analogously. □

Theorem 2 implies the following

Remark. *Let $f\in\mathcal{U}_\alpha^H$. For every $\varphi\in[0;2\pi)$ there exist $\delta(f,\varphi)\in[0;1]$ and $\delta'(f,\varphi)\in[1;\infty]$ such that for any circle or straight line $\Gamma\subset\Delta$, orthogonal to $\partial\Delta$ at the point $e^{i\varphi}$, we have*

$$\lim_{\Gamma\ni z\to e^{i\varphi}}\left[\frac{J_f(z)}{J_f(0)}\frac{(1-|z|)^{2\alpha+2}}{(1+|z|)^{2\alpha-2}}\right] = \delta(f,\varphi),$$

$$\lim_{\Gamma\ni z\to e^{i\varphi}}\left[\frac{J_f(z)}{J_f(0)}\frac{(1+|z|)^{2\alpha+2}}{(1-|z|)^{2\alpha-2}}\right]=\delta'(f,\varphi),$$

and the constants $\delta(f,\varphi)$, $\delta'(f,\varphi)$ *do not depend on* Γ.

By $\mathcal{U}_\alpha^H(\delta^0)$ denote the set of all functions from $\mathcal{U}_\alpha^H$ with the same Hayman number δ^0 from Theorem B.

Let $\mathcal{U}_\alpha^H(\delta_0)$ be the set of all functions, having the Hayman number δ_0 from Theorem C.

Theorem 3. *1) If* $f\in\mathcal{U}_\alpha^H(\delta^0)$, $\delta^0\in(0;1)$, *then for every* $\delta\in[\delta^0,1)$ *there exists* $a\in\Delta$ *such that* $f_a(z)\in\mathcal{U}_\alpha^H(\delta)$.

2) If $f\in\mathcal{U}_\alpha^H(\delta_0)$, $\delta_0\in(1;\infty)$, *then for every* $\delta'\in(1,\delta^0]$ *there exists* $a\in\Delta$ *such that* $f_a(z)\in\mathcal{U}_\alpha^H(\delta')$.

Proof. By Theorem B, for any $\varphi\in[0;2\pi)$ there exists

$$\lim_{r\to1-}\left[\frac{J_f(re^{i\varphi})}{J_f(0)}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right]=\delta(f,\varphi).$$

Let us fix $a\in\Delta$ $\varphi\in[0;2\pi)$. Denote $z=\dfrac{re^{i\varphi}-a}{1-\bar{a}re^{i\varphi}}$, $|z|=R(r)$ and consider the limit

$$\delta^*(\varphi)\stackrel{def}{=}\lim_{r\to1-}\left[\frac{J_{f_a}(z)}{J_{f_a}(0)}\frac{(1-R(r))^{2\alpha+2}}{(1+R(r))^{2\alpha-2}}\right].$$

Let us calculate $\delta^*(\varphi)$, using (9) and (10)

$$\delta^*(\varphi)=\lim_{r\to1-}\left[\frac{J_f(re^{i\varphi})}{J_f(a)\left|1+\bar{a}\frac{re^{i\varphi}-a}{1-\bar{a}re^{i\varphi}}\right|^4}\frac{(1-R(r))^{2\alpha+2}}{(1+R(r))^{2\alpha-2}}\right]=$$

$$=\lim_{r\to1-}\left[\frac{J_f(re^{i\varphi})}{J_f(0)}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\frac{J_f(0)}{J_f(a)}\left(\frac{1-R(r)}{1-r}\right)^{2\alpha+2}\right]\cdot\frac{1}{\left|1+\bar{a}\frac{re^{i\varphi}-a}{1-\bar{a}re^{i\varphi}}\right|}.$$

By (11),

$$\delta^*(\varphi)=\delta(f,\varphi)\frac{J_f(0)}{J_f(a)}\frac{(1-|a|^2)^{2\alpha+2}}{|1-\bar{a}e^{i\varphi}|^{4\alpha+4}}\frac{|1-\bar{a}e^{i\varphi}|^4}{(1-|a|^2)^4}=$$

$$=\delta(f,\varphi)\frac{J_f(0)}{J_f(a)}\frac{(1-|a|^2)^{2\alpha-2}}{|1-\bar{a}e^{i\varphi}|^{4\alpha}}\leq$$

$$\leq \lim_{R(r)\to 1-}\left[\frac{M(R(r),J_{f_a})}{J_{f_a}(0)}\frac{(1-R(r))^{2\alpha+2}}{(1+R(r))^{2\alpha-2}}\right]\stackrel{def}{=}\delta_a.$$

Let φ be equal to d.m.g. φ^0 of $f(z)$ and $a=\rho e^{i\varphi^0}$. Then $\delta(f,\varphi)=\delta^0$ and

$$\delta^0\frac{J_f(0)}{J_f(\rho e^{i\varphi^0})}\frac{(1-\rho^2)^{2\alpha-2}}{(1-\rho)^{4\alpha}}=\delta^0\frac{J_f(0)}{J_f(\rho e^{i\varphi^0})}\frac{(1+\rho)^{2\alpha-2}}{(1-\rho)^{2\alpha-2}}\leq\delta_a. \tag{15}$$

By Theorem B, there exists a d.m.g. $\varphi_1\in[0;2\pi)$ of $f_a(z)$ such that

$$\delta_a=\lim_{r\to 1-}\left[\frac{J_{f_a}(re^{i\varphi^0})}{J_{f_a}(0)}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right]=$$

$$=\lim_{r\to 1-}\left[\frac{J_f\left(\frac{re^{i\varphi_1}+a}{1+\bar{a}re^{i\varphi_1}}\right)}{J_f(a)|1+\bar{a}re^{i\varphi_1}|^4}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right].$$

Denote $R_1(r)e^{i\gamma_1(r)}=\dfrac{re^{i\varphi_1}+a}{1+\bar{a}re^{i\varphi_1}}$, where $\gamma_1(r)$ is a real-valued function. Then, using (11) for $R(r)=R_1(r)$, we obtain

$$\delta_a\leq\lim_{r\to 1-}\left[\frac{M(R_1(r),J_f)}{J_f(a)|1+\bar{a}re^{i\varphi_1}|^4}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right]=$$

$$=\lim_{r\to 1-}\left[\frac{M(R_1(r),J_f)}{J_f(0)}\frac{(1-R_1(r))^{2\alpha+2}}{(1+R_1(r))^{2\alpha-2}}\right]\times$$

$$\times\frac{J_f(0)}{J_f(a)}\frac{1}{|1+\bar{a}e^{i\varphi_1}|^4}\cdot\lim_{r\to 1-}\left(\frac{1-r}{1-R_1(r)}\right)^{2\alpha+2}=$$

$$=\delta^0\frac{J_f(0)}{J_f(a)}\frac{1}{|1+\bar{a}e^{i\varphi_1}|^4}\left(\frac{|1+\bar{a}e^{i\varphi_1}|^2}{1-|a|^2}\right)^{2\alpha+2}=\delta^0\frac{J_f(0)}{J_f(a)}\frac{|1+\bar{a}e^{i\varphi_1}|^{4\alpha}}{(1-|a|^2)^{2\alpha+2}}\leq$$

$$\leq\delta^0\frac{J_f(0)}{J_f(a)}\frac{(1+\rho)^{4\alpha}}{(1-\rho^2)^{2\alpha+2}}=\delta^0\frac{J_f(0)}{J_f(a)}\frac{(1+\rho)^{2\alpha-2}}{(1-\rho)^{2\alpha+2}}.$$

Taking into account inequality (15), we get

$$\delta^0\frac{J_f(0)}{J_f(\rho e^{i\varphi^0})}\frac{(1+\rho)^{2\alpha-2}}{(1-\rho)^{2\alpha+2}}=\delta_a.$$

Since the continuous function $\frac{J_f(0)}{J_f(\rho e^{i\varphi^0})}\frac{(1+\rho)^{2\alpha-2}}{(1-\rho)^{2\alpha+2}}$ decreases on ρ, equals 1 as $\rho = 0$, and tends to zero as $\rho \to 1-$, then we can find $\rho \in [0; 1)$ such that δ_a takes preassigned value from $[\delta^0; 1)$.

Claim 2 of the theorem is proved analogously. □

In [7] (see also [17], [14]) it was proved that the set of all d.i.g.-'s and d.i.d.-'s of a given analytic function is at most countable. The following theorem shows that this statement is true for set of d.i.g.-'s of harmonic function too. But we don't know whether this fact is true for set of d.i.d.-'s.

Theorem 4. *Let $f \in \mathcal{U}_\alpha^H$. Then the set of all d.i.g.-'s of f is at most countable.*

Proof. If $f = h + \bar{g} \in \mathcal{U}_\alpha^H$, then $\overline{\operatorname{ord}}\, h \le \alpha$. Since

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 \le |h'(z)|^2$$

for all $z \in \Delta$, then for $\varphi \in [0, 2\pi)$ and $r \in [0, 1)$

$$\frac{J_f(re^{i\varphi})}{J_f(0)}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}} \le \left[|h'(re^{i\varphi})|\frac{(1-r)^{\alpha+1}}{(1+r)^{\alpha-1}}\right]^2 \frac{1}{J_f(0)}. \tag{16}$$

By Theorem B and theorem of growth regularity from [7], there exist the limits

$$\delta(f, \varphi) = \lim_{r\to 1-}\left[\frac{J_f(re^{i\varphi})}{J_f(0)}\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right],$$

and

$$\tilde{\delta}(h, \varphi) = \lim_{r\to 1-}\left[|h'(re^{i\varphi})|\frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}\right].$$

From (16) we get $\delta(f, \varphi) \le \frac{\tilde{\delta}^2(h,\varphi)}{J_f(0)}$. If φ is a d.i.g. of f, then $\delta(f, \varphi) > 0$. Consequently, $\tilde{\delta}(h, \varphi) > 0$ and φ is a d.i.g. of h. Therefore the set V of all d.i.g.-'s of f is contained in the set W of all d.i.g.-'s of h. As it was proved in [7], W is at most countable. Hence V is at most countable too. □

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References

[1] Hayman W. K. *Some applications of the transfinite diameter to the theory of functions.* J. Anal. Math., 1951, no. 1, pp. 155–179.

[2] Krzyż J. *On the maximum modulus of univalent function.* Bull. Pol. Acad. Sci. Math., 1955, vol. CI, no. 3, pp. 203–206.

[3] Bieberbach L. *Einführung in die konforme Abbildung.* Sammlung Göschen, Band 768/786a, 1967.

[4] Hayman W. K. *Multivalent functions.* Cambridge University Press, 1994.

[5] Pommerenke Ch. *Linear-invariante Familien analytischer Funktionen. I.* Math. Ann., 1964, vol. 155, pp. 108–154. DOI: 10.1007/BF01344077.

[6] Campbell D. M. *Applications and proofs of a uniqueness theorems for linear invariant families of finite order.* Rocky Mountain J. Math., 1974, vol. 4, no. 4, pp. 621–634. DOI: 10.1216/RMJ-1974-4-4-621.

[7] Starkov V. V. *Regularity theorems for universal linearly invariant families of functions.* Serdica Math. J., 1985, vol. 11, no. 3, pp. 299–318. (in Russian).

[8] Schaubroeck L. E. *Subordination of planar harmonic functions.* Complex Variables, 2000, vol. 41, is. 2, pp. 163–178. DOI: 10.1080/17476930008815245

[9] Sheil-Small T. *Constants for planar harmonic mappings.* J. Lond. Math. Soc. (2), 1990, vol. 42, pp. 237–248. DOI: 10.1112%2Fjlms%2Fs2-42.2.237.

[10] Sobczak-Kneć M., Starkov V. V., Szynal J. *Old and new order of linear invariant family of harmonic mappings and the bound for Jacobian.* Ann. Univ. Mariae Curie-Sklodowska Sect. A, 2012, vol. LXV, no. 2, pp. 191–202. DOI: 10.2478/v10062-011-0024-3

[11] Ganenkova E. G., Starkov V. V. *Regularity theorems for harmonic functions.* J. Appl. Anal., 2014, vol. 21, is. 1, pp. 25-36. DOI: 10.1515/jaa-2015-0003.

[12] Graf S. Yu. *Regularity theorems for Jacobian in linearly and affine invariant families of harmonic mappings.* Application of the functional analysis in the approximation theory, 2014, pp. 10–21. (in Russian).

[13] Ganenkova E. G. *A theorem of decrease regularity in linearly invariant families of functions.* Tr. Petrozavodsk. Gos. Univ. Ser. Mat, 2006, vol. 13, pp. 46–59 (in Russian).

[14] Ganenkova E. G. *A theorem on the regularity of decrease in linearly invariant families of functions.* Russian Mathematics (Izvestiya VUZ. Matematika), 2007, vol. 51, no. 2, pp. 71-74. DOI: 10.3103/S1066369X07020090.

[15] Clunie J., Sheil-Small T. *Harmonic univalent functions.* Ann. Acad. Sci. Fenn. Math., 1984, vol. 9, pp. 3–25. DOI: 10.5186/aasfm.1984.0905#sthash.GEE2gFK0.dpuf.

[16] Duren P. *Harmonic mappings in the plane.* Cambridge university press, 2004.

[17] Godula J., Starkov V. V. *Linear invariant families.* Tr. Petrozavodsk. Gos. Univ. Ser. Mat., 1998, vol. 5, p. 3–96 (in Russian).

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