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COEFFICIENT INEQUALITY FOR MULTIVALENT BOUNDED TURNING FUNCTIONS OF ORDER α

Abstract. The objective of this paper is to obtain the sharp upper bound to the $H_2(p+1)$, second Hankel determinant for p-valent (multivalent) analytic bounded turning functions (also called functions whose derivatives have positive real parts) of order α $(0 \leq \alpha < 1)$, using Toeplitz determinants. The result presented here includes three known results as their special cases.

Key words: p-*valent analytic function, bounded turning function, upper bound, Hankel determinant, positive real function, Toeplitz determinants*

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1. Introduction. Let A_p denote the class of functions f of the form

$$f(z) = z^p + a_{p+1}z^{p+1} + \cdots \tag{1}$$

in the open unit disc $E = \{z : |z| < 1\}$ with $p \in \mathbb{N} = \{1, 2, 3, ...\}$. Let S be the subclass of $A_1 = A$, consisting of univalent functions.

In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture, i.e., for a univalent function its n^{th} coefficient is bounded by n (see [1]). The bounds for the coefficients of these functions give information about their geometric properties. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [2] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{2}$$

This determinant has been considered by several authors in the literature. For example, Noonan and Thomas [3] studied the second Hankel determinant of areally mean p-valent functions. Noor [4] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions in S with bounded boundary rotation. The Hankel transform of an integer sequence and some of its properties were discussed by Layman [5]. One can easily observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with real μ and $f \in S$. Ali [6] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f for $p = 1$, given in (1.1), defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$, when $f \in \widetilde{ST}(\alpha)$, the class of strongly starlike functions of order α $(0 < \alpha \leq 1)$. Further sharp bounds for the functional $|a_2a_4 - a_3^2|$, the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant (functional), given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2, \tag{3}$$

were obtained for various subclasses of univalent and multivalent analytic functions by several authors in the literature. Janteng et al. [7] have considered the functional $|a_2a_4 - a_3^2|$ and found a sharp upper bound for the function f in the subclass $\mathcal{R}$ of S, consisting of functions whose derivative has a positive real part (also called bounded turning functions) studied by MacGregor [8]. In their work, they have shown that if $f \in \mathcal{R}$ then $|a_2a_4 - a_3^2| \leq \frac{4}{9}$. Motivated by this result, in this paper we consider the Hankel determinant in the case of $q = 2$ and $n = p + 1$, denoted by $H_2(p + 1)$, given by

$$H_2(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1}a_{p+3} - a_{p+2}^2. \tag{4}$$

Further, we seek a sharp upper bound to the functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for the functions belonging to the certain subclass of p-valent analytic functions, defined as follows.

Definition 1. *A function $f(z) \in A_p$ is said to be in the class $\mathcal{R}_p(\alpha)$ $(0 \leq \alpha < 1)$ if it satisfies the condition*

$$Re \frac{f'(z)}{pz^{p-1}} > \alpha, \quad \forall \;\; z \in E. \tag{5}$$

1) If $p = 1$, we obtain $\mathcal{R}_1(\alpha) = \mathcal{R}(\alpha)$, the class of bounded turning functions of order α.
2) Choosing $\alpha = 0$, we get $\mathcal{R}_p(0) = \mathcal{R}_p$, the class of p-valent bounded turning functions.
3) Selecting $p = 1$ and $\alpha = 0$, we have $\mathcal{R}_1(0) = \mathcal{R}$.

In the next section we give some preliminary Lemmas required for proving our result.

2. Preliminary Results. Let $\mathcal{P}$ denote the class of functions consisting of g such that

$$g(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{6}$$

which are regular in the open unit disc E and satisfy $\mathrm{Re}\, g(z) > 0$ for any $z \in E$. Here $g(z)$ is called a Caratheòdory function [9].

Lemma 1. [10, 11] *If $g \in \mathcal{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2. [12] *The power series for $g(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ given in (6) converges in the open unit disc E to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3 \ldots$$

and $c_{-k} = \overline{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z)$, with $\sum_{k=1}^{m} \rho_k = 1$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in [12] is due to Caratheòdory and Toeplitz. We may assume without restriction that $c_1 > 0$. From Lemma 2, for $n = 2$ we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix}.$$

On expanding the determinant, we get

$$D_2 = 8 + 2Re\{c_1^2 c_2\} - 2|c_2|^2 - 4|c_1|^2 \geq 0.$$

Applying the fundamental principles of complex numbers, the above expression is equivalent to

$$2c_2 = c_1^2 + y(4 - c_1^2). \tag{7}$$

In the same way,

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix}.$$

Then $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{8}$$

Simplifying relations (7) and (8), we obtain

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)y - c_1(4 - c_1^2)y^2 + 2(4 - c_1^2)(1 - |y|^2)\zeta\} \tag{9}$$

for some complex valued y with $|y| \leq 1$ and for some complex valued ζ with $|\zeta| \leq 1$. To obtain our result, we refer to the classical method devised by Libera and Zlotkiewicz [13], used by several authors in literature.

3. Main Result

Theorem 1. *If $f(z) \in \mathcal{R}_p(\alpha)$ $(0 \leq \alpha < 1)$ with $p \in \mathbb{N}$ then*

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{2p(1-\alpha)}{p+2}\right]^2$$

and the inequality is sharp.

For the function $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in \mathcal{R}_p(\alpha)$, by virtue of Definition 1, there exists an analytic function $g \in \mathcal{P}$ in the open unit disc E with $g(0) = 1$ and $\operatorname{Re} g(z) > 0$ such that

$$\frac{f'(z) - p\alpha z^{p-1}}{p(1-\alpha)z^{p-1}} = g(z) \Leftrightarrow f'(z) - p\alpha z^{p-1} = p(1-\alpha)z^{p-1}g(z). \tag{10}$$

Replacing $f'(z)$ and $g(z)$ with their equivalent series expressions in (10), we have

$$pz^{p-1} + \sum_{n=p+1}^{\infty} na_n z^{n-1} - p\alpha z^{p-1} = p(1-\alpha)z^{p-1}\left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}.$$

Upon simplification, we obtain

$$p(1-\alpha)z^{p-1} + (p+1)a_{p+1}z^p + (p+2)a_{p+2}z^{p+1} + (p+3)a_{p+3}z^{p+2} + ... = \\ = p(1-\alpha)z^{p-1}[1 + c_1 z + c_2 z^2 + c_3 z^3 + ...]. \quad (11)$$

Equating the coefficients of same powers of z^p, z^{p+1} and z^{p+2} in (11), we have

$$a_{p+1} = \frac{p(1-\alpha)c_1}{p+1}, \quad a_{p+2} = \frac{p(1-\alpha)c_2}{p+2} \quad \text{and} \quad a_{p+3} = \frac{p(1-\alpha)c_3}{p+3}. \quad (12)$$

Substituting the values of $a_{p+1}, a_{p+2},$ and a_{p+3} from (12) in the functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$, after simplifying we get

$$|a_{p+1}a_{p+3} - a_{p+2}^2| =$$

$$= \frac{p^2(1-\alpha)^2}{(p+1)(p+2)^2(p+3)} \left|(p+2)^2 c_1 c_3 - (p+1)(p+3)c_2^2\right|.$$

The above expression is equivalent to

$$|a_{p+1}a_{p+4} - a_{p+2}^2| = t\left|d_1 c_1 c_3 + d_2 c_2^2\right|, \quad (13)$$

where

$$t = \frac{p^2(1-\alpha)^2}{(p+1)(p+2)^2(p+3)}, d_1 = (p+2)^2 \text{ and } d_2 = -(p+1)(p+3). \quad (14)$$

Substituting the values of c_2 and c_3 from (7) and (9) respectively from Lemma 2 in the expression $\left|d_1 c_1 c_3 + d_2 c_2^2\right|$, which is on the right-hand side of (13), we have

$$\left|d_1 c_1 c_3 + d_2 c_2^2\right| = \left|d_1 c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)y - c_1(4-c_1^2)y^2 + \right. \\ \left. + 2(4-c_1^2)(1-|y|^2)\zeta\} + d_2 \times \frac{1}{4}\{c_1^2 + y(4-c_1^2)\}^2\right|;$$

$$4\left|d_1c_1c_3+d_2c_2^2\right| = \left|(d_1+d_2)c_1^4+2d_1c_1(4-c_1^2)\zeta+2(d_1+d_2)c_1^2(4-c_1^2)y- \right.$$
$$\left. -\left\{d_1c_1^2y^2+2d_1c_1|y|^2\zeta-d_2(4-c_1^2)y^2\right\}(4-c_1^2)\right|;$$

$$4\left|d_1c_1c_3+d_2c_2^2\right| = \left|(d_1+d_2)c_1^4+2d_1c_1(4-c_1^2)\zeta+2(d_1+d_2)c_1^2(4-c_1^2)y- \right.$$
$$\left. -\left\{(d_1+d_2)c_1^2y^2+2d_1c_1|y|^2\zeta-4d_2y^2\right\}(4-c_1^2)\right|.$$

Applying the triangle inequality, we get

$$4\left|d_1c_1c_3+d_2c_2^2\right| \leq \left|(d_1+d_2)c_1^4+2d_1c_1(4-c_1^2)|\zeta|+2(d_1+d_2)c_1^2(4-c_1^2)|y|+ \right.$$
$$\left. +\left\{(d_1+d_2)c_1^2|y|^2+2d_1c_1|y|^2|\zeta|-4d_2|y|^2\right\}(4-c_1^2)\right|.$$

Using the fact that $|\zeta| < 1$ in the above iequality, we obtain

$$4\left|d_1c_1c_3+d_2c_2^2\right| \leq \left|(d_1+d_2)c_1^4+2d_1c_1(4-c_1^2)+2(d_1+d_2)c_1^2(4-c_1^2)|y|+ \right.$$
$$\left. +\left\{(d_1+d_2)c_1^2+2d_1c_1-4d_2\right\}(4-c_1^2)|y|^2\right|. \quad (15)$$

Using the values of d_1, d_2 given in (14), we can write

$$d_1+d_2=1 \quad \text{and} \quad \left\{(d_1+d_2)c_1^2+2d_1c_1-4d_2\right\} =$$
$$= c_1^2+2(p+2)^2c_1+4(p+1)(p+3). \quad (16)$$

Substituting the values from (16) and value of d_1 from (14) to the right-hand side of (15), we have

$$4\left|d_1c_1c_3+d_2c_2^2\right| \leq \left|c_1^4+2(p+2)^2c_1(4-c_1^2)+2c_1^2(4-c_1^2)|y|+ \right.$$
$$\left. +\left\{c_1^2+2(p+2)^2c_1+4(p+1)(p+3)\right\}(4-c_1^2)|y|^2\right|. \quad (17)$$

Consider $\left\{c_1^2+2(p+2)^2c_1+4(p+1)(p+3)\right\} =$

$$= \left[\left\{c_1+(p+2)^2\right\}^2-\left(\sqrt{p^4+8p^3+20p^2+16p+4}\right)^2\right] =$$
$$= \left[c_1+\left\{(p+2)^2+\left(\sqrt{p^4+8p^3+20p^2+16p+4}\right)\right\}\right] \times$$
$$\times \left[c_1+\left\{(p+2)^2-\left(\sqrt{p^4+8p^3+20p^2+16p+4}\right)\right\}\right].$$

Noting that $(c_1+a)(c_1+b) \geq (c_1-a)(c_1-b)$, where $a, b \geq 0$, and $c_1 \in [0,2]$ in the above expression, we obtain

$$\left\{c_1^2 + 2(p+2)^2 c_1 + 4(p+1)(p+3)\right\} \geq \\ \geq \left\{c_1^2 - 2(p+2)^2 c_1 + 4(p+1)(p+3)\right\}. \quad (18)$$

From expressions (17) and (18), we get

$$4\left|d_1 c_1 c_3 + d_2 c_2^2\right| \leq \left|c_1^4 + 2(p+2)^2 c_1(4-c_1^2) + 2c_1^2(4-c_1^2)|y| + \\ + \left\{c_1^2 - 2(p+2)^2 c_1 + 4(p+1)(p+3)\right\}(4-c_1^2)|y|^2\right|. \quad (19)$$

Choosing $c_1 = c \in [0,2]$, replacing $|y|$ by μ on the right-hand side of (19), we obtain

$$4\left|d_1 c_1 c_3 + d_2 c_2^2\right| \leq \left[c^4 + 2(p+2)^2 c(4-c^2) + 2c^2(4-c^2)\mu + \\ + \left\{c^2 - 2(p+2)^2 c + 4(p+1)(p+3)\right\}(4-c^2)\mu^2\right] = \\ = F(c,\mu)\ ,\ 0 \leq \mu = |y| \leq 1 \text{ and } 0 \leq c \leq 2. \quad (20)$$

Next, we maximize function $F(c,\mu)$ on the closed region $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ given in the right-hand side of (20) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = 2\left[c^2 + \left\{c^2 - 2(p+2)^2 c + 4(p+1)(p+3)\right\}\mu\right](4-c^2). \quad (21)$$

For $0 < \mu < 1$, for fixed c with $0 < c < 2$ and $p \in \mathbb{N}$, from (21), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ becomes an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed region $[0,2] \times [0,1]$. The maximum value of $F(c,\mu)$ occurs on the boundary i.e., when $\mu = 1$. Therefore, for fixed $c \in [0,2]$, we have

$$\max_{0 \leq \mu \leq 1} F(c,\mu) = F(c,1) = G(c). \quad (22)$$

Replacing μ by 1 in $F(c,\mu)$, it simplifies to

$$G(c) = -2c^4 - 4p(p+4)c^2 + 16(p+1)(p+3), \quad (23)$$

$$G'(c) = -8c^3 - 8p(p+4)c. \quad (24)$$

From (24), we observe that $G'(c) \leq 0$ for every $c \in [0, 2]$ with $p \in \mathbb{N}$. Consequently, $G(c)$ becomes a decreasing function of c, whose maximum value occurs at $c = 0$ only. From (23), the maximum value of $G(c)$ at $c = 0$ is obtained to be

$$G_{max} = G(0) = 16(p+1)(p+3). \tag{25}$$

Simplifying expressions (20) and (25), we get

$$\left| d_1 c_1 c_3 + d_2 c_2^2 \right| \leq 4(p+1)(p+3). \tag{26}$$

From relations (13) and (26), along with the value of t in (14), upon simplification, we obtain

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{2p(1-\alpha)}{p+2} \right]^2. \tag{27}$$

By setting $c_1 = c = 0$ and selecting $y = 1$ in the expressions (7) and (9), we find that $c_2 = 2$ and $c_3 = 0$, respectively. Substituting the values $c_2 = 2$, $c_3 = 0$, and $d_2 = -(p+1)(p+3)$ in (26), we observe that equality is attained, which shows that our result is sharp. For the values $c_2 = 2$ and $c_3 = 0$, from (6) we derive the extremal function given by

$$g(z) = 1 + 2z^2 + 2z^4 + ... = \frac{1+z^2}{1-z^2}.$$

So that from (10), we have

$$\frac{f'(z) - p\alpha z^{p-1}}{p(1-\alpha)z^{p-1}} = 1 + 2z^2 + 2z^4 + ... = \frac{1+z^2}{1-z^2}.$$

This completes the proof of our Theorem.

Remark 1. *If $p = 1$ and $\alpha = 0$ in (27) then $|a_2a_4 - a_3^2| \leq \frac{4}{9}$; this coincides with the result of Janteng et al. [7].*

Remark 2. *If $p = 1$ in (27) then $|a_2a_4 - a_3^2| \leq \frac{4(1-\alpha)^2}{9}$, this result is same as that of Vamshee Krishna and RamReddy [14].*

Remark 3. *If $\alpha = 0$ in (27) then $|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{2p}{p+2}\right]^2$, this result coincides with the result obtained by Vamshee Krishna and RamReddy [15].*

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