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ON WEIGHTED GENERALIZED FUNCTIONS ASSOCIATED WITH QUADRATIC FORMS

Abstract. In this article we consider certain types of weighted generalized functions associated with nondegenerate quadratic forms. Such functions and their derivatives are used for constructing fundamental solutions of iterated ultra-hyperbolic equations with the Bessel operator and for constructing negative real powers of ultra-hyperbolic operators with the Bessel operator.

Key words: *weighted generalized function, quadratic form, ultra-hyperbolic operator, Bessel operator*

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1. Introduction and main definitions. The weighted generalized functions associated with nondegenerate indefinite quadratic forms considered in this article are necessary for construction of the ultra-hyperbolic Riezs potential with the Bessel operator. Riezs potential with the Bessel operator and other operators with the Bessel differential operator are very interesting subjects with many applications (see, for example, [1]–[9]).

We deal with the part of the Euclidean space

$$\mathbb{R}_n^+=\{x=(x_1,\ldots,x_n)\in\mathbb{R}_n,\ \ x_1>0,\ldots,x_n>0\}.$$

Let Ω be finite or infinite open set in $\mathbb{R}_n$, symmetric with respect to each hyperplane x_i=0, $i=1,...,n$, $\Omega_+=\Omega\cap\mathbb{R}^{\,+}_n$ and $\overline{\Omega}_+=\Omega\cap\overline{\mathbb{R}}^{\,+}_n$ where

$$\overline{\mathbb{R}}^{\,+}_n=\{x=(x_1,\ldots,x_n)\in\mathbb{R}_n,\ \ x_1\geq 0,\ldots,x_n\geq 0\}.$$

We have $\Omega_+\subseteq\mathbb{R}^{\,+}_n$ and $\overline{\Omega}_+\subseteq\overline{\mathbb{R}}^{\,+}_n$.

We consider the class $C^\infty(\Omega_+)$ consisting of infinitely differentiable on Ω_+ functions. We denote the subset of functions from $C^\infty(\Omega_+)$ such that all derivatives of these functions with respect to x_i for any $i=1,...,n$ are continuous up to x_i=0 by $C^\infty(\overline{\Omega}_+)$. A function $f \in C^\infty(\overline{\Omega}_+)$ will be called *even with respect to* x_i, $i = 1,...,n$ if $\left.\frac{\partial^{2k+1} f}{\partial x_i^{2k+1}}\right|_{x=0} = 0$ for all nonnegative integer k (see [10], p. 21). Class $C^\infty_{ev}(\overline{\Omega}_+)$ consists of functions from $C^\infty(\overline{\Omega}_+)$, even with respect to each variable x_i, $i = 1,...,n$. Let $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+)$ be the space of all functions $f \in C^\infty(\overline{\Omega}_+)$ with a compact support. We will call elements of $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+)$ *test functions* and use the notation $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+) = \mathcal{D}_+(\overline{\Omega}_+)$.

We define K as an arbitrary compact in $\mathbb{R}_n$ symmetric with respect to each hyperplane x_i=0, $i = 1,...,n$, $K_+ = K \cap \overline{\mathbb{R}}{}^+_n$. A *distribution* u on $\overline{\Omega}_+$ is a linear form on $\mathcal{D}_+(\overline{\Omega}_+)$ such that for all compacts $K_+ \subset \overline{\Omega}_+$, constants C and k exist and

$$|u(f)| \leq C \sum_{|\alpha| \leq k} \sup |D^\alpha f|, \qquad f \in \overset{\circ}{C}{}^\infty_{ev}(K_+),$$

where $D^\alpha = D^{\alpha_1}_{x_1}...D^{\alpha_n}_{x_n}$, $\alpha = (\alpha_1,...,\alpha_n)$, $\alpha_1,...,\alpha_n$ are integer nonnegative numbers, $D_{x_j} = i\frac{\partial}{\partial x_j}$, i is imaginary unit, $j = 1,...,n$. The set of all distributions on the set $\overline{\Omega}_+$ is denoted by $\mathcal{D}'_+(\overline{\Omega}_+)$ (see [10], p. 11 and [11], p. 34).

Multiindex γ=$(\gamma_1,...,\gamma_n)$ consists of positive fixed reals $\gamma_i > 0$, i=1,...,n and $|\gamma|$=γ_1+...+γ_n. Let $L^\gamma_p(\Omega_+)$, $1 \leq p < \infty$ be the space of all measurable in Ω_+ functions even with respect to each variable x_i, $i = 1,...,n$ such that

$$\int\limits_{\Omega_+} |f(x)|^p x^\gamma dx < \infty, \qquad x^\gamma = \prod_{i=1}^n x_i^{\gamma_i}.$$

For a real number $p \geq 1$, the $L^\gamma_p(\Omega_+)$–norm of f is defined by

$$||f||_{L^\gamma_p(\Omega_+)} = \left(\int\limits_{\Omega_+} |f(x)|^p x^\gamma dx \right)^{1/p}.$$

Weighted measure of Ω_+ is denoted by $\mathrm{mes}_\gamma(\Omega)$ and is defined by formula

$$\mathrm{mes}_\gamma(\Omega_+) = \int\limits_{\Omega_+} x^\gamma dx.$$

For every measurable function $f(x)$ defined on $\mathbb{R}_n^+$ we consider

$$\mu_\gamma(f,t)=\text{mes}_\gamma\{x\in\mathbb{R}_n^+ : |f(x)|>t\}=\int\limits_{\{x:\ |f(x)|>t\}^+} x^\gamma dx$$

where $\{x:\ |f(x)|>t\}^+ = \{x\in\mathbb{R}_n^+ : |f(x)|>t\}$. We will call the function $\mu_\gamma=\mu_\gamma(f,t)$ a *weighted distribution function* $|f(x)|$.

A space $L_\infty^\gamma(\Omega_+)$ is defined as a set of measurable on Ω_+ and even with respect to each variable functions $f(x)$ such as

$$||f||_{L_\infty^\gamma(\Omega_+)}=\operatorname*{ess\,sup_\gamma}_{x\in\Omega_+}|f(x)|=\inf_{a\in\Omega_+}\{\mu_\gamma(f,a)=0\}<\infty.$$

For $1\leq p\leq\infty$ the $L_{p,loc}^\gamma(\Omega_+)$ is the set of functions $u(x)$ defined almost everywhere in Ω_+ such that $uf\in L_p^\gamma(\Omega_+)$ for any $f\in \overset{\circ}{C}{}_{ev}^\infty(\overline{\Omega}_+)$. Each function $u(x)\in L_{1,loc}^\gamma(\Omega_+)$ will be identified with the functional $u\in$ $\in\mathcal{D}'_+(\overline{\Omega}_+)$ acting according to the formula

$$(u,f)_\gamma=\int\limits_{\mathbb{R}_n^+} u(x)\,f(x)\,x^\gamma\,dx,\quad x^\gamma=\prod_{i=1}^n x_i^{\gamma_i},\quad f\in\overset{\circ}{C}{}_{ev}^\infty(\overline{\mathbb{R}}{}_n^+). \tag{1}$$

Functionals $u\in\mathcal{D}'_+(\overline{\Omega}_+)$ acting by the formula (1) will be called *regular weighted functionals*. All other functionals $u\in\mathcal{D}'_+(\overline{\Omega}_+)$ will be called *singular weighted functionals*.

2.Weighted generalized functions concentrated on the part of the cone. In this section we consider weighted generalized functions $\delta_\gamma(P)$ concentrated on the part of the cone and give formulas for their derivatives.

Generalized function δ_γ is defined by equality (by analogy with [12] p. 247)

$$(\delta_\gamma,\varphi)_\gamma=\varphi(0),\quad \varphi(x)\in K^+.$$

For convenience we will write

$$(\delta_\gamma,\varphi)_\gamma=\int\limits_{\mathbb{R}_n^+}\delta_\gamma(x)\varphi(x)x^\gamma dx=\varphi(0).$$

Let $p,q\in\mathbb{N}$, $n=p+q$ and

$$P=|x'|^2-|x''|^2=x_1^2+...+x_p^2-x_{p+1}^2-...-x_{p+q}^2,$$

where $x=(x_1,...,x_n)=(x',x'')\in\mathbb{R}_n^+$, $x'=(x_1,...,x_p)$, $x''=(x_{p+1},...,x_{p+q})$.

Definition 1. *Let $\varphi\in\mathcal{D}_+(\overline{\mathbb{R}}_n^+)$ vanishes at the origin. For such φ we define generalized function $\delta_\gamma(P)$ concentrated on the part of the cone $P=0$ belonging to $\mathbb{R}_n^+$ by the formula*

$$(\delta_\gamma(P),\varphi)_\gamma=\int\limits_{\mathbb{R}_n^+}\delta_\gamma(|x'|^2-|x''|^2)\varphi(x)x^\gamma dx. \tag{2}$$

If the function $\varphi\in\mathcal{D}_+(\overline{\mathbb{R}}_n^+)$ does not vanish at the origin then $(\delta_\gamma(P),\varphi)_\gamma$ is defined by regularizing the integral.

Lemma 1. *Let $\varphi\in\mathcal{D}_+(\overline{\mathbb{R}}_n^+)$ vanishes at the origin, $p>1$ and $q>1$. For $\delta_\gamma(P)$ the representation*

$$(\delta_\gamma(P),\varphi)_\gamma=\frac{1}{2}\int\limits_0^\infty\int\limits_{S_p^+}\int\limits_{S_q^+}\varphi(s\,\omega)s^{n+|\gamma|-3}\omega^\gamma dS_p dS_q ds \tag{3}$$

holds true. In (3) $\omega=(\omega',\omega'')$, $\omega'=(\omega_1,...,\omega_p)\in\mathbb{R}_p^+$, $\omega''=(\omega_{p+1},...,\omega_{p+q})\in$ $\in\mathbb{R}_q^+$, $n=p+q$, $|\omega'|=|\omega''|=1$, $\omega^\gamma=\prod\limits_{i=1}^n\omega_i^{\gamma_i}$, dS_p and dS_q are elements of surface area on the part of the unit sphere

$$S_p^+=\{\omega'\in\mathbb{R}_p^+:|\omega'|=1\}\qquad S_q^+=\{\omega''\in\mathbb{R}_q^+:|\omega''|=1\},$$

respectively. For the k-th derivative ($k\in\mathbb{N}$) of $\delta_\gamma(P)$ we have

$$(\delta_\gamma^{(k)}(P),\varphi)_\gamma=\int\limits_0^\infty\left[\left(\frac{1}{2s}\frac{\partial}{\partial s}\right)^k\psi(r,s)s^{q+|\gamma''|-2}\right]_{s=r}r^{p+|\gamma'|-1}dr, \tag{4}$$

where

$$\psi(r,s)=\frac{1}{2}\int\limits_{S_p^+}\int\limits_{S_q^+}\varphi(r\omega',s\omega'')\omega^\gamma dS_p dS_q. \tag{5}$$

Proof. Let us transform (2) to bipolar coordinates defined by

$$x_1=r\omega_1,...,x_p=r\omega_p,\ x_{p+1}=s\omega_{p+1},...,x_{p+q}=s\omega_{p+q}, \tag{6}$$

where

$$r=\sqrt{x_1^2+...+x_p^2},\quad s=\sqrt{x_{p+1}^2+...+x_{p+q}^2},$$

$$|\omega'|=\sqrt{\omega_1^2+...+\omega_p^2}=1,\quad |\omega''|=\sqrt{\omega_{p+1}^2+...+\omega_{p+q}^2}=1.$$

We obtain

$$(\delta_\gamma(P),\varphi(x))_\gamma=$$

$$=\int\limits_0^\infty\int\limits_0^\infty\int\limits_{S_p^+}\int\limits_{S_q^+}\delta_\gamma(r^2-s^2)\varphi(r\omega',s\omega'')r^{p+|\gamma'|-1}s^{q+|\gamma''|-1}\omega^\gamma dS_1^p dS_1^q dr ds.$$

Now let us choose the coordinates to be $r^2=u$, $s^2=v$. In these coordinates we have

$$(\delta_\gamma(P),\varphi)_\gamma=\frac{1}{4}\int\limits_0^\infty\int\limits_0^\infty\int\limits_{S_p^+}\int\limits_{S_q^+}\delta_\gamma(u-v)\varphi(\sqrt{u}\omega',\sqrt{v}\omega'')u^{\frac{p+|\gamma'|}{2}-1}\times$$

$$\times v^{\frac{q+|\gamma''|}{2}-1}\omega^\gamma dS_p dS_q du dv=\frac{1}{4}\int\limits_0^\infty\int\limits_{S_p^+}\int\limits_{S_q^+}\varphi(\sqrt{v}\omega)v^{\frac{n+|\gamma|}{2}-2}\omega^\gamma dS_p dS_q dv.$$

Returning to variable s by the formula $v=s^2$, we obtain (3).

Now we prove the formula (4). After the change of variables by (6) and $r^2=u$, $s^2=v$ in $(\delta_\gamma^{(k)}(P),\varphi)_\gamma$ we get

$$(\delta_\gamma^{(k)}(P),\varphi)_\gamma=\frac{1}{4}\int\limits_0^\infty\int\limits_0^\infty\int\limits_{S_p^+}\int\limits_{S_q^+}\frac{\partial^k}{\partial v^k}[\delta_\gamma(v-u)]\varphi(\sqrt{u}\omega',\sqrt{v}\omega'')\times$$

$$\times u^{\frac{p+|\gamma'|}{2}-1}v^{\frac{q+|\gamma''|}{2}-1}\omega^\gamma dS_p dS_q du dv=\int\limits_0^\infty\int\limits_0^\infty\int\limits_{S_p^+}\int\limits_{S_q^+}\delta_\gamma(v-u)\times$$

$$\times\frac{(-1)^k}{4}\frac{\partial^k}{\partial v^k}\left[\varphi(\sqrt{u}\omega',\sqrt{v}\omega'')v^{\frac{q+|\gamma''|}{2}-1}\right]u^{\frac{p+|\gamma'|}{2}-1}\omega^\gamma dS_p dS_q du dv=$$

$$=\frac{(-1)^k}{4}\int\limits_0^\infty\int\limits_{S_p^+}\int\limits_{S_q^+}u^{\frac{p+|\gamma'|}{2}-1}\omega^\gamma\left[\frac{\partial^k}{\partial v^k}\varphi(\sqrt{u}\omega',\sqrt{v}\omega'')v^{\frac{q+|\gamma''|}{2}-1}\right]_{v=u}dS_p dS_q du.$$

Returning to variables r, s and using notation (5) we obtain (4). This completes the proof of Lemma 1. □

Remark 1. *Similarly, we can get the formula*

$$(\delta_\gamma^{(k)}(P),\varphi)_\gamma=(-1)^k\int\limits_0^\infty\left[\left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^k\psi(r,s)r^{p+|\gamma'|-2}\right]_{r=s}s^{q+|\gamma''|-1}ds. \quad (7)$$

Remark 2. *Noticing that when k=0 formulas* (4) *and* (7) *are equivalent to the formula* (3) *we will examine intergals* (4) *and* (7) *at $k\in\mathbb{N}\cup\{0\}$.*

Let $\varphi\in\mathcal{D}_+(\overline{\mathbb{R}}_n^+)$. Assuming that the function φ vanishes at the origin we have that integrals (4) and (7) converge for all $k\in\mathbb{N}\cup\{0\}$. If the function φ does not vanish at the origin then integrals (4) and (7) converge only for $k<\frac{p+q+|\gamma|-2}{2}$. In this case for $k\geq\frac{p+q+|\gamma|-2}{2}$ we will consider the regularization of (4) and (7) denoting them $\delta_{\gamma,1}^{(k)}(P)$ and $\delta_{\gamma,2}^{(k)}(P)$, respectively. So using the expression (5) for $p>1$, $q>1$ and $k\in\mathbb{N}\cup\{0\}$ we have

$$(\delta_{\gamma,1}^{(k)}(P),\varphi)_\gamma=\int\limits_0^{+\infty}\left[\left(\frac{1}{2s}\frac{\partial}{\partial s}\right)^k\psi(r,s)s^{q+|\gamma''|-2}\right]\Bigg|_{s=r}r^{p+|\gamma'|-1}dr, \quad (8)$$

$$(\delta_{\gamma,2}^{(k)}(P),\varphi)_\gamma=(-1)^k\int\limits_0^{+\infty}\left[\left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^k\psi(r,s)r^{p+|\gamma'|-2}\right]\Bigg|_{r=s}s^{q+|\gamma''|-1}ds. \quad (9)$$

The integrals (8) and (9) converge and coincide for $k<\frac{p+q+|\gamma|-2}{2}$ and for $k\geq\frac{p+q+|\gamma|-2}{2}$ these integrals must be understood in the sense of their regularizations.

2. Weighted generalized function $P_{\gamma,+}^\lambda$. Let n=p+q, p>1, q>1 and $P(x)=x_1^2+...+x_p^2-x_{p+1}^2-...-x_{p+q}^2$. Here and further let $\varphi\in\mathcal{D}_+(\overline{\mathbb{R}}_n^+)$. We define the weighted generalized function $P_{\gamma,+}^\lambda$ by

$$(P_{\gamma,+}^\lambda,\varphi)_\gamma=\int\limits_{\{P(x)>0\}^+}P^\lambda(x)\varphi(x)x^\gamma dx, \quad (10)$$

where $\{P(x)>0\}^+=\{x\in\mathbb{R}_n^+:P(x)>0\}$, $\lambda\in\mathbb{C}$.

Weighted generalized function $P_{\gamma,+}^\lambda$ and its derivatives are used for constructing fundamental solutions of iterated B-ultra-hyperbolic equations of the form $L_B^k u=f(x)$, $\quad k\in\mathbb{N}, \quad x\in\mathbb{R}_n, \quad x_i>0, \quad i=1,...,n,$

where L_B is B-ultra-hyperbolic operator (see [9] and [13]–[15])

$$L_B = B_{x_1}+...+B_{x_p}-B_{x_{p+1}}-...-B_{x_n},$$

$B_{x_i}=\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$ is the Bessel operator, $\gamma_i>0$, $i=1,...,n$.

It should also be noted that negative real powers of an operator L_B called generalized B-hyperbolic potentials (see [16]) are constructed using function $P^\lambda_{\gamma,+}$. Let us find singularities of $(P^\lambda_{\gamma,+},\varphi)_\gamma$. For this purpose we transform (10) to bipolar coordinates (6) and using notation (5) for integral (10) we obtain

$$(P^\lambda_{\gamma,+},\varphi)_\gamma = \int\limits_0^\infty\int\limits_0^r (r^2-s^2)^\lambda\psi(r,s)r^{p+|\gamma'|-1}s^{q+|\gamma''|-1}drds. \tag{11}$$

We now make change of variables $u{=}r^2$, $v{=}s^2$ in (11):

$$(P^\lambda_{\gamma,+},\varphi)_\gamma = \frac{1}{4}\int\limits_0^\infty\int\limits_0^u (u-v)^\lambda\psi_1(u,v)u^{\frac{p+|\gamma'|}{2}-1}s^{\frac{q+|\gamma''|}{2}-1}dudv,$$

where $\psi_1(u,v){=}\psi(r,s)$ when $u{=}r^2$, $v{=}s^2$.

If we write $v{=}ut$ then we obtain

$$(P^\lambda_{\gamma,+},\varphi)_\gamma = \int\limits_0^\infty u^{\lambda+\frac{p+q+|\gamma|}{2}-1}\Phi(\lambda,u)du, \tag{12}$$

where

$$\Phi(\lambda,u) = \frac{1}{4}\int\limits_0^1 (1-t)^\lambda t^{\frac{q+|\gamma''|}{2}-1}\psi_1(u,tu)dt. \tag{13}$$

The formula (12) shows that $P^\lambda_{\gamma,+}$ has two sets of poles. The first consists of poles of $\Phi(\lambda,u)$. Namely for $t{=}1$ function $\Phi(\lambda,u)$ has singularity when

$$\lambda = -1,-2,...,-k,... \tag{14}$$

in which $\Phi(\lambda,u)$ has simple poles with residues

$$\operatorname*{res}_{\lambda=-k}\Phi(\lambda,u) = \frac{1}{4}\frac{(-1)^{k-1}}{(k-1)!}\frac{\partial^{k-1}}{\partial t^{k-1}}\left[t^{\frac{q+|\gamma''|-2}{2}}\psi_1(u,tu)\right]_{t=1}. \tag{15}$$

Moreover integral (12) has poles at the points

$$\lambda = -\frac{n+|\gamma|}{2},\ -\frac{n+|\gamma|}{2} - 1,\ ...,\ -\frac{n+|\gamma|}{2} - k,\ ..., \tag{16}$$

where $n = p + q$, $\gamma = (\gamma', \gamma'')$. Wherein

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k} (P^{\lambda}_{\gamma,+}, \varphi)_{\gamma} = \frac{1}{k!}\frac{\partial^k}{\partial u^k}\Phi\left(-\frac{n+|\gamma|}{2} - k, u\right)\bigg|_{u=0}. \tag{17}$$

We have three cases. The first case is when a singular point λ belongs to the first set (14), but not to the second (16). The second case is when singular point λ belongs to the second (16), but $\lambda{\neq}{-k}$, $k{\in}\mathbb{N}$. And the third case is when λ belongs both to the first set (14) and the second set (16). Let us now study each case separately in the following three theorems.

Theorem 1. *If $\lambda{=}{-k}$, $k \in \mathbb{N}$ and $n + |\gamma|{\in}\mathbb{R}\backslash\mathbb{N}$ or $n + |\gamma|{\in}\mathbb{N}$ and $n + |\gamma|{=}2k - 1$, $k{\in}\mathbb{N}$ and also if $n{+}|\gamma|$ is even and $k{<}\frac{n+|\gamma|}{2}$ the weighted generalized function $P^{\lambda}_{\gamma,+}$ has simple pole with residue*

$$\operatorname*{res}_{\lambda=-k} P^{\lambda}_{\gamma,+} = \frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}_{\gamma,1}(P). \tag{18}$$

Proof. Let us write $\Phi(\lambda, u)$ in the neighborhood of $\lambda = -k$ in the form

$$\Phi(\lambda, u) = \frac{\Phi_0(u)}{\lambda + k} + \Phi_1(\lambda, u), \quad \Phi_0(u) = \operatorname*{res}_{\lambda=-k} \Phi(\lambda, u),$$

where function $\Phi_1(\lambda, u)$ is regular at $\lambda{=}{-k}$. We obtain

$$(P^{\lambda}_{\gamma,+}, \varphi)_{\gamma}{=}\frac{1}{\lambda{+}k}\int\limits_0^{\infty} u^{\lambda+\frac{n+|\gamma|}{2}-1}\Phi_0(u)du{+}\int\limits_0^{\infty} u^{\lambda+\frac{n+|\gamma|}{2}-1}\Phi_1(\lambda, u)du. \tag{19}$$

The integrals in (19) are regular functions of λ at $\lambda{=}{-k}$. Therefore $(P^{\lambda}_{\gamma,+}, \varphi)_{\gamma}$ has a simple pole at such a point and using (15) we have

$$\operatorname*{res}_{\lambda=-k}(P^{\lambda}_{\gamma,+}, \varphi){=}\frac{(-1)^{k-1}}{4(k{-}1)!}\int\limits_0^{\infty} u^{\frac{n+|\gamma|}{2}-k-1}\frac{\partial^{k-1}}{\partial t^{k-1}}\left[t^{\frac{q+|\gamma''|}{2}-1}\psi_1(u, tu)\right]_{t=1} du. \tag{20}$$

If in (20) we get $tu=v$ then we may write

$$\operatorname*{res}_{\lambda=-k}(P^{\lambda}_{\gamma,+},\varphi)=\frac{(-1)^{k-1}}{4(k-1)!}\int\limits_0^\infty\frac{\partial^{k-1}}{\partial v^{k-1}}\left[v^{\frac{q+|\gamma''|}{2}-1}\psi_1(u,v)\right]_{v=u}u^{\frac{p+|\gamma'|}{2}-1}du,\tag{21}$$

where the integral is to be understood in the sense of its regularization for $k\geq\frac{n}{2}$. We now make the change of variables $u=r^2$ and $v=s^2$ in (8) and have

$$(\delta^{(k-1)}_{\gamma,1}(P),\varphi)_\gamma=\frac{1}{2}\int\limits_0^\infty\left[\frac{\partial^{k-1}}{\partial v^{k-1}}v^{\frac{q+|\gamma''|}{2}-1}\psi_1(u,v)\right]_{v=u}u^{\frac{p+|\gamma'|}{2}-1}du,\tag{22}$$

where

$$\psi_1(u,v)=\frac{1}{2}\int\limits_{S_p^+}\int\limits_{S_q^+}\varphi(\sqrt{u}\omega',\sqrt{v}\omega'')\omega^\gamma dS_p dS_q.$$

Formulas (21) and (22) imply (17). For $k\geq\frac{n}{2}$ integral in (22) is to be understood in the sense of its regularization. In the case when $n+|\gamma|\in\mathbb{R}\backslash\mathbb{N}$ or $n+|\gamma|\in\mathbb{N}$ and $n+|\gamma|=2k-1$, $k\in\mathbb{N}$ regularization of the integral in (22) is defined by analytic continuation. This proves the desired result. $\Box$

Now we study the case when the singular point λ is in the second set (16), but not in the first (14). If $\lambda=-\frac{n+|\gamma|}{2}-k$, $k=0,1,2,...$, and $n+|\gamma|\in\mathbb{R}\backslash\mathbb{N}$ or $n+|\gamma|\in\mathbb{N}$ and $n+|\gamma|=2k-1$, $k\in\mathbb{N}$, then function $\Phi(\lambda,u)$ is regular in the neighborhood of $\lambda=-\frac{n+|\gamma|}{2}-k$. Therefore function $(P^{\lambda}_{\gamma,+},\varphi)_\gamma$ will have a simple pole with residue given by (17).

Before proceeding to the expression of the residue $\operatorname*{res}\limits_{\lambda=-\frac{n+|\gamma|}{2}-k}(P^{\lambda}_{\gamma,+},\varphi)$ through derivatives of function $\varphi(x)$ at the origin we will obtain one useful formula. Consider the B-ultra-hyperbolic differential operator

$$L_B=B_{\gamma'_1}+...+B_{\gamma'_p}-B_{\gamma''_{p+1}}-B_{\gamma''_{p+q}},\qquad B_{\gamma_i}=\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}.$$

Applying an operator L_B to quadratic form

$$P(x)=x_1^2+...+x_p^2-x_{p+1}^2-...-x_{p+q}^2,\ n=p+q,\ p>1,\ q>1$$

we obtain

$$L_BP^{\lambda+1}(x)=4(\lambda+1)\left(\lambda+\frac{n+|\gamma|}{2}\right)P^\lambda(x).\tag{23}$$

Theorem 2. *Let $n+|\gamma|$ be not integer or $n+|\gamma|{\in}\mathbb{N}$ and $n+|\gamma|{=}2k-1$, $k{\in}\mathbb{N}$. When $p+|\gamma'|$ is not integer or $p{+}|\gamma'|{\in}\mathbb{N}$, $p{+}|\gamma'|{=}2m{-}1$, $m{\in}\mathbb{N}$ and $q{+}|\gamma''|$ is even weighted functional $P^{\lambda}_{\gamma,+}$ has simple poles at $\lambda{=}{-}\frac{n+|\gamma|}{2}{-}k$, $k{\in}\mathbb{N}{\cup}\{0\}$ with residues*

$$\underset{\lambda=-\frac{n+|\gamma|}{2}-k}{\operatorname{res}} P^{\lambda}_{\gamma,+}=\frac{(-1)^{\frac{q+|\gamma''|}{2}}}{2^{n+2k}k!}\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}+k\right)}L^k_B\delta_\gamma(x).$$

If $p{+}|\gamma'|$ is even then weighted functional $P^{\lambda}_{\gamma,+}$ is regular at $\lambda{=}{-}\frac{n+|\gamma|}{2}{-}k$, $k{\in}\mathbb{N}{\cup}\{0\}$.

Proof. We first consider $\lambda{=}{-}\frac{n+|\gamma|}{2}$. Using formula (17) we can write

$$\underset{\lambda=-\frac{n+|\gamma|}{2}}{\operatorname{res}}(P^{\lambda}_{\gamma,+},\varphi)_\gamma{=}\Phi\left(-\frac{n{+}|\gamma|}{2},0\right)={\frac{\psi_1(0,0)}{4}}\int\limits_0^1(1{-}t)^{-\frac{n+|\gamma|}{2}}t^{\frac{q+|\gamma''|}{2}}dt=$$

$$=\frac{1}{4}\,\psi_1(0,0)\,\frac{\Gamma\left(\frac{q+|\gamma''|}{2}\right)\Gamma\left(-\frac{n+|\gamma|}{2}+1\right)}{\Gamma\left(-\frac{p+|\gamma'|}{2}+1\right)}. \tag{24}$$

From the last formula we can see that if $p{+}|\gamma'|$ is even then

$$\underset{\lambda=-\frac{n+|\gamma|}{2}}{\operatorname{res}}(P^{\lambda}_{\gamma,+},\varphi){=}0.$$

Now assume that $p{+}|\gamma'|$ is not integer or $p{+}|\gamma'|{\in}\mathbb{N}$ and $p{+}|\gamma'|{=}2k{-}1$, $k{\in}\mathbb{N}$ and $q{+}|\gamma''|$ is even. We have

$$\psi_1(0,0)=\psi(0,0)=\varphi(0)\int\limits_{S^+_p}\int\limits_{S^+_q}\omega^\gamma dS_pdS_q=\varphi(0)|S^+_1(p)|_{\gamma'}|S^+_1(q)|_{\gamma''}, \tag{25}$$

where

$$|S^+_1(p)|_{\gamma'}=\frac{\prod\limits_{i=1}^{p}\Gamma\left(\frac{\gamma'_i+1}{2}\right)}{2^{p-1}\Gamma\left(\frac{p+|\gamma'|}{2}\right)},\quad |S^+_1(q)|_{\gamma''}=\frac{\prod\limits_{i=1}^{q}\Gamma\left(\frac{\gamma''_i+1}{2}\right)}{2^{q-1}\Gamma\left(\frac{q+|\gamma''|}{2}\right)} \tag{26}$$

(see [1], p. 20, formula (1.2.5)). After some simple calculations, we obtain

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}}(P^{\lambda}_{\gamma,+},\varphi)_{\gamma}=\frac{(-1)^{\frac{q+|\gamma''|}{2}}}{2^n}\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)}\,\varphi(0).$$

Also we have

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}}P^{\lambda}_{\gamma,+}=\frac{(-1)^{\frac{q+|\gamma''|}{2}}}{2^n}\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)}\delta_{\gamma}(x). \tag{27}$$

Using Green's theorem and formula (23) we derive

$$\int\limits_{\{P(x)>0\}^+}\big(\varphi(x)[L_BP^{\lambda+1}(x)]-P^{\lambda+1}(x)[L_B\varphi(x)]\big)\,x^{\gamma}dx=0,$$

therefore

$$(P^{\lambda}_{\gamma,+},\varphi)_{\gamma}=\frac{1}{2(\lambda+1)(2\lambda+n+|\gamma|)}(P^{\lambda+1}_{\gamma,+},L_B\varphi)_{\gamma}. \tag{28}$$

Then k-fold iteration of (28) leads to

$$(P^{\lambda}_{\gamma,+},\varphi)_{\gamma}=\frac{(P^{\lambda+k}_{\gamma,+},L_B^k\varphi)_{\gamma}}{2^{2k}(\lambda+1)...(\lambda+k)\left(\lambda+\frac{n+|\gamma|}{2}\right)...\left(\lambda+\frac{n+|\gamma|}{2}+k-1\right)}. \tag{29}$$

Consequently

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k}(P^{\lambda}_{\gamma,+},\varphi)_{\gamma}=\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k}(P^{\lambda+k}_{\gamma,+},L_B^k\varphi)_{\gamma}\times$$

$$\times\left.\frac{1}{2^{2k}(\lambda+1)...(\lambda+k)\left(\lambda+\frac{n+|\gamma|}{2}\right)...\left(\lambda+\frac{n+|\gamma|}{2}+k-1\right)}\right|_{\lambda=-\frac{n+|\gamma|}{2}-k},$$

and

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k}(P^{\lambda+k}_{\gamma,+},L_B^k\varphi)_{\gamma}=\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}}(P^{\lambda}_{\gamma,+},L_B^k\varphi)_{\gamma}.$$

Therefore if $p+|\gamma'|$ is even this residue vanishes. If $p+|\gamma'|$ is not integer or $p+|\gamma'|\in\mathbb{N}$ and $p+|\gamma'|=2k-1$, $k\in\mathbb{N}$ then (27) gives

$$\underset{\lambda=-\frac{n+|\gamma|}{2}-k}{\operatorname{res}}(P^\lambda_{\gamma,+},\varphi)_\gamma=\frac{(-1)^{\frac{q+|\gamma''|}{2}}}{2^{n+2k}k!}\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}+k\right)}(L^k_B\delta_\gamma(x),\varphi)_\gamma.$$

This completes the proof of Theorem 2. □

Theorem 3. *If $n+|\gamma|$ is even and $p+|\gamma'|$ and $q+|\gamma''|$ are also even, $k\in\mathbb{N}\cup\{0\}$, then function $P^\lambda_{\gamma,+}$ has a simple pole in $\lambda=-\frac{n+|\gamma|}{2}-k$ with residue*

$$\underset{\lambda=-\frac{n+|\gamma|}{2}-k}{\operatorname{res}}P^\lambda_{\gamma,+}=\frac{1}{\Gamma\left(\frac{n+|\gamma|}{2}+k\right)}\left[(-1)^{\frac{n+|\gamma|}{2}+k-1}\delta^{\left(\frac{n+|\gamma|}{2}+k-1\right)}_{\gamma,1}(P)+\right.$$

$$\left.+\frac{(-1)^{\frac{q+|\gamma''|}{2}}}{2^{2k}k!}\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)L^k_B\delta_\gamma(x)\right].$$

If $p+|\gamma'|$ and $q+|\gamma''|$ are not integer or $p+|\gamma'|,q+|\gamma''|\in\mathbb{N}$ and $p+|\gamma'|=$ $=2m-1$, $q+|\gamma''|=2k-1$, $m,k\in\mathbb{N}$ then function $P^\lambda_{\gamma,+}$ a pole of order two at $\lambda=-\frac{n+|\gamma|}{2}-k$. Coefficients $c^{(k)}_{-2}$ and $c^{(k)}_{-1}$ of expansion of function $P^\lambda_{\gamma,+}$ in Laurent series at $\lambda=-\frac{n+|\gamma|}{2}-k$ are expressed by formulas

$$c^{(0)}_{-1}=\frac{1}{\Gamma\left(\frac{n+|\gamma|}{2}+k\right)}\left[(-1)^{\frac{n+|\gamma|}{2}+k-1}\delta^{\left(\frac{n+|\gamma|}{2}+k-1\right)}_{\gamma,1}(P)+\frac{(-1)^{\frac{n+|\gamma|}{2}-1}}{2^{2k}k!}\times\right.$$

$$\left.\times\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\sin\left(\frac{p+|\gamma'|}{2}\pi\right)\left(\psi\left(\frac{p+|\gamma'|}{2}\right)-\psi\left(\frac{n+|\gamma|}{2}\right)\right)L^k_B\delta_\gamma(x)\right],$$

$$c^{(k)}_{-2}=(-1)^{\frac{n+|\gamma|}{2}+1}\frac{\sin\frac{\pi(p+|\gamma'|)}{2}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n+2k}k!\pi\Gamma\left(\frac{n+|\gamma|+k}{2}\right)}L^k_B\delta_\gamma(x),$$

where $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$.

Proof. Let $n{+}|\gamma|$ be even and $\lambda{=}-\frac{n+|\gamma|}{2}{-}k$, $k{\in}\mathbb{N}{\cup}\{0\}$. We express this $(P^\lambda_{\gamma,+},\varphi)_\gamma$ in the form

$$(P^\lambda_{\gamma,+},\varphi)_\gamma{=}\frac{1}{\lambda{+}k}\int\limits_0^\infty u^{\lambda+\frac{n+|\gamma|}{2}-1}\Phi_0(u)du{+}\int\limits_0^\infty u^{\lambda+\frac{n+|\gamma|}{2}-1}\Phi_1(\lambda,u)du, \quad (30)$$

where $\Phi_0(u){=}\underset{\lambda=-\frac{n+|\gamma|}{2}-k}{\operatorname{res}}\Phi(\lambda,u)$ and $\Phi_1(\lambda,u)$ is a regular at $\lambda{=}-\frac{n+|\gamma|}{2}{-}k$ function. By virtue of the proposal each integral in (30) may have at $\lambda{=}-\frac{n+|\gamma|}{2}{-}k$ a simple pole therefore function $(P^\lambda_{\gamma,+},\varphi)_\gamma$ may have a pole of order two at $\lambda{=}-\frac{n+|\gamma|}{2}{-}k$. In the neighborhood of such a point we may expand $P^\lambda_{\gamma,+}$ in the Laurent series

$$P^\lambda_{\gamma,+}=\frac{c^{(k)}_{-2}}{\left(\lambda+\frac{n+|\gamma|}{2}+k\right)^2}+\frac{c^{(k)}_{-1}}{\lambda+\frac{n+|\gamma|}{2}+k}+\ldots.$$

Let us find $c^{(k)}_{-1}$, $c^{(k)}_{-2}$. We have

$$(c^{(k)}_{-2},\varphi)_\gamma=\underset{\lambda=-\frac{n+|\gamma|}{2}-k}{\operatorname{res}}\int\limits_0^\infty u^{\lambda+\frac{n+|\gamma|}{2}-1}\Phi_0(u)du=\frac{1}{k!}\Phi_0^{(k)}(0).$$

If $k=0$ then $c^{(0)}_{-2}=\Phi_0(0)$. According to (13)

$$\Phi_0(0)=\frac{1}{4}\psi_1(0,0)\underset{\lambda=-\frac{n+|\gamma|}{2}}{\operatorname{res}}\int\limits_0^1(1-t)^\lambda t^{\frac{q+|\gamma''|-2}{2}}dt=$$

$$=\psi_1(0,0)\underset{\lambda=-\frac{n+|\gamma|}{2}}{\operatorname{res}}\frac{\Gamma\left(\frac{q+|\gamma''|}{2}\right)\Gamma(\lambda+1)}{4\Gamma\left(\lambda+\frac{q+|\gamma''|}{2}+1\right)}.$$

Considering that $\psi_1(0,0){=}\varphi(0)|S_1^+(p)|_{\gamma'}|S_1^+(q)|_{\gamma''}$ where $|S_1^+(p)|_{\gamma'}$ and $|S_1^+(q)|_{\gamma''}$ were determined in (26) we obtain

$$(c^{(0)}_{-2},\varphi)_\gamma=$$

$$=\frac{(-1)^{\frac{n+|\gamma|}{2}+1}B\left(\frac{p+|\gamma'|}{2},\frac{q+|\gamma''|}{2}\right)}{4\pi}\sin\frac{\pi(p+|\gamma'|)}{2}|S_1^+(p)|_{\gamma'}|S_1^+(q)|_{\gamma''}\varphi(0).$$

When $p+|\gamma'|$ is even (in this case $q+|\gamma''|$ is also even) we have $c_{-2}^{(k)}=0$ i.e. function $(P_{\gamma,+}^{\lambda},\varphi)_\gamma$ has a simple pole at $\lambda=-\frac{n+|\gamma|}{2}$. If $p+|\gamma'|$ is not integer or $p+|\gamma'|\in\mathbb{N}$ and $p+|\gamma'|=2k-1$, $k\in\mathbb{N}$ then

$$c_{-2}^{(0)}=(-1)^{\frac{n+|\gamma|}{2}+1}\frac{\sin\frac{\pi(p+|\gamma'|)}{2}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\pi\Gamma\left(\frac{n+|\gamma|}{2}\right)}\delta_\gamma(x).$$

As well as in Theorem 2 we obtain that if $p+|\gamma'|$ and $q+|\gamma''|$ are even then function $P_{\gamma,+}^{\lambda}$ has a simple pole at $\lambda=-\frac{n+|\gamma|}{2}-k$. If $p+|\gamma'|$ and $q+|\gamma''|$ are not integer or $p+|\gamma'|,q+|\gamma''|\in\mathbb{N}$ and $p+|\gamma'|=2m-1$, $q+|\gamma''|=2k-1$, $m,k\in\mathbb{N}$ then

$$c_{-2}^{(k)}=(-1)^{\frac{n+|\gamma|}{2}+1}\frac{\sin\frac{\pi(p+|\gamma'|)}{2}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n+2k}k!\pi\Gamma\left(\frac{n+|\gamma|+k}{2}\right)}L_B^k\delta_\gamma(x).$$

Let's find $c_{-1}^{(k)}$. We have

$$(c_{-1}^{(k)},\varphi)=\int\limits_0^\infty u^{-k-1}\Phi_0(u)du+$$

$$+\underset{\lambda=-\frac{n+|\gamma|}{2}-k}{\operatorname{res}}\int\limits_0^\infty u^{\lambda+\frac{n+|\gamma|}{2}-1}\Phi_1\left(-\frac{n+|\gamma|}{2}-k,u\right)du.$$

Since $\Phi_0(u)=\underset{\lambda=-k}{\operatorname{res}}\Phi(\lambda,u)$ then using the formulas (15) and (22) we obtain

$$\int\limits_0^\infty u^{-k-1}\Phi_0(u)du=\frac{(-1)^{\frac{n+|\gamma|}{2}+k-1}}{\Gamma\left(\frac{n+|\gamma|}{2}+k-1\right)}\left(\delta_{\gamma,1}^{(\frac{n+|\gamma|}{2}+k-1)}(P),\varphi\right)_\gamma.$$

Thus

$$\underset{\lambda=-\frac{n+|\gamma|}{2}-k}{\operatorname{res}}\int\limits_0^\infty u^{\lambda+\frac{n+|\gamma|}{2}-1}\Phi_1\left(-\frac{n+|\gamma|}{2}-k,u\right)du=$$

$$=\frac{1}{k!}\frac{\partial^k\Phi_1\left(-\frac{n+|\gamma|}{2}-k,u\right)}{\partial u^k}\bigg|_{u=0}=(\alpha_\gamma^{(k)},\varphi)_\gamma$$

and
$$c_{-1}^{(k)}=\frac{(-1)^{\frac{n+|\gamma|}{2}+k-1}}{\Gamma\left(\frac{n+|\gamma|}{2}+k-1\right)}\delta_{\gamma,1}^{(\frac{n+|\gamma|}{2}+k-1)}(P)+\alpha_\gamma^{(k)}.$$

For $k=0$ we obtain
$$(\alpha_\gamma^{(0)},\varphi)_\gamma=\Phi_1\left(-\frac{n+|\gamma|}{2},0\right).$$

In order to find $\Phi_1\left(-\frac{n+|\gamma|}{2},0\right)$ we consider $\Phi(\lambda,0)$. Using (24), (25) and (26) we obtain
$$\Phi(\lambda,0)=\varphi(0)\frac{\Gamma(\lambda+1)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{p+|\gamma'|}{2}\right)\Gamma\left(\lambda+\frac{q+|\gamma''|}{2}+1\right)}.$$

Taking into account the formula $\Gamma(1-x)\Gamma(x)=\frac{\pi}{\sin\pi x}$ we can write
$$\Phi(\lambda,0)=\frac{\sin\pi\left(\lambda+\frac{q+|\gamma''|}{2}\right)}{\sin\pi\lambda}\frac{\Gamma\left(-\lambda-\frac{q+|\gamma''|}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{p+|\gamma'|}{2}\right)\Gamma(-\lambda)}\varphi(0).$$

If $p{+}|\gamma'|$ and $q{+}|\gamma''|$ are even then
$$\lim_{\lambda\to-\frac{n+|\gamma|}{2}}\frac{\sin\pi\left(\lambda+\frac{q+|\gamma''|}{2}\right)}{\sin\pi\lambda}=(-1)^{\frac{q+|\gamma''|}{2}},$$

hence function $\Phi(\lambda,0)$ is regular at $\lambda{=}-\frac{n+|\gamma|}{2}$ and
$$\Phi_1\left(-\frac{n+|\gamma|}{2},0\right)=\Phi\left(-\frac{n+|\gamma|}{2}\right)$$

whence
$$(\alpha_\gamma^{(0)},\varphi)_\gamma=(-1)^{\frac{q+|\gamma''|}{2}}\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)}\varphi(0).$$

If $p{+}|\gamma'|$ and $q{+}|\gamma''|$ are not integer or $p{+}|\gamma'|,q{+}|\gamma''|{\in}\mathbb{N}$ and $p{+}|\gamma'|{=}2m{-}1$, $q{+}|\gamma''|{=}2k{-}1$, $m,k{\in}\mathbb{N}$ then $\Phi(\lambda,0)$ has a pole at $\lambda=-\frac{n+|\gamma|}{2}$. In this case
$$(\alpha_\gamma^{(0)},\varphi)_\gamma=\Phi_1\left(-\frac{n+|\gamma|}{2},0\right)=(-1)^{\frac{n+|\gamma|}{2}-1}\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\times$$

$$\times \frac{\sin\left(\frac{p+|\gamma'|}{2}\pi\right)\left(\psi\left(\frac{p+|\gamma'|}{2}\right)-\psi\left(\frac{n+|\gamma|}{2}\right)\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)}\varphi(0),$$

where $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$. We obtain

$$c_{-1}^{(0)}=\frac{1}{\Gamma\left(\frac{n+|\gamma|}{2}\right)}\left[(-1)^{\frac{n+|\gamma|}{2}-1}\delta_{\gamma,1}^{\left(\frac{n+|\gamma|}{2}-1\right)}(P)+\theta\delta_\gamma(x)\right],$$

with a value

$$\theta{=}(-1)^{\frac{q+|\gamma''|}{2}}\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)$$

if $p{+}|\gamma'|$ and $q{+}|\gamma''|$ are even. If $p{+}|\gamma'|$ and $q{+}|\gamma''|$ are not integer or $p{+}|\gamma'|, q{+}|\gamma''|{\in}\mathbb{N}$ and $p{+}|\gamma'|{=}2m{-}1$, $q{+}|\gamma''|{=}2k{-}1$, $m,k{\in}\mathbb{N}$ then

$$\theta=(-1)^{\frac{n+|\gamma|}{2}-1}\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i{+}1}{2}\right)\sin\left(\frac{p{+}|\gamma'|}{2}\pi\right)\times$$

$$\times\left(\psi\left(\frac{p+|\gamma'|}{2}\right)-\psi\left(\frac{n+|\gamma|}{2}\right)\right).$$

Finally, in order to obtain $c_{-1}^{(k)}$ for arbitrary k, we again use the formula (29). This proves the desired result. □

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