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# JACOBIAN CONJECTURE, TWO-DIMENSIONAL CASE

**Abstract.** The Jacobian Conjecture was first formulated by O. Keller in 1939. In the modern form it supposes injectivity of the polynomial mapping $f$: $\mathbb{R}^n \to \mathbb{R}^n$ ($\mathbb{C}^n \to \mathbb{C}^n$) provided that jacobian $J_f \equiv \mathrm{const} \neq 0$. In this note we consider structure of polynomial mappings $f$ that provide $J_f \equiv \mathrm{const} \neq 0$.

**Key words:** *Jacobian conjecture*

**2010 Mathematical Subject Classification:** *14R15*

**Introduction.** Denote the set of all polynomials in $\mathbb{R}^n$ (or $\mathbb{C}^n$) of degree not higher than $m$ by $\mathcal{P}_m$. Let $P_m$ be the set of all polynomial mappings $F = (F_1, \ldots, F_n)$: $\mathbb{R}^n \to \mathbb{R}^n$ (or $\mathbb{C}^n \to \mathbb{C}^n$), $F_k \in \mathcal{P}_m(k = 1, \ldots, n)$ of degree $degF \leq m$. The Jacobi matrix and the jacobian of mapping $F$ are denoted by $DF$ and $J_F$, respectivly. In the complex case both $DF$ and $J_F$ are complex. The *Jacobian Conjecture (JC)* formulated by Keller [1] in 1939 in its modern form is:

*if* $F \in P_m$ *and* $J_F \equiv \mathrm{const} \neq 0$ *then* $F$ *is injective in* $\mathbb{R}^n$ *(*$\mathbb{C}^n$*).*

Proof of the conjecture would allow to use it widely in a number of branches of mathematics. Beside the one given above, also other equivalent formulations exist. Many publications are devoted to this conjecture: see e.g. [2–6]. In particular, in [7] the conjecture is proved for $F \in P_2$ for any $n$; in [8] it is checked for $n = 2$ and $F \in P_{100}$. However, it has not been proved neither to be true nor to be false for any $n$. It is included in the list of "Mathematical Problems for the Next Century" [9].

In the note we consider the question of structure of mappings $F \in P_m$ with $J_F \equiv \mathrm{const} \neq 0$. It seems to be the most important for proof or rejection of (JC). Solving this problem and applying criteria or sufficient conditions of injectivity of mappings would help to proceed in (JC). We here obtain results only for $n = 2$ and $m = 2, 3$. However, they serve as a starting point for results in the general case ($n, m \geq 3$) in our future article with S. Ponnusamy. Let us pass to results.

**Theorem 1**. *Let $F(x, y) = (\tilde{U}(x, y), \tilde{V}(x, y))$ be a polynomial mapping, $F(0, 0) = 0$, $\tilde{U}(x, y), \tilde{V}(x, y) \in \mathcal{P}_2$. Then $J_F \equiv \mathrm{const} \neq 0$ iff $F = \mathcal{A} \circ f \circ \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are linear homogeneous nondegenerate mappings, $f(x, y) = (u(x, y), v(x, y))$,*

$$u(x, y) = x + \alpha_2(x + y)^2, \quad v(x, y) = y - \alpha_2(x + y)^2, \tag{1}$$

*$\alpha_2$ is an arbitrary fixed constant.*

Case $m = 3$ is considered using Theorem 1.

**Theorem 2**. *Let $F(x, y) = (\tilde{U}(x, y), \tilde{V}(x, y))$ be a polynomial mapping, $F(0, 0) = 0$, $\tilde{U}(x, y), \tilde{V}(x, y) \in \mathcal{P}_3$. Then $J_F \equiv \mathrm{const} \neq 0$ iff $F = \mathcal{A} \circ f \circ \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are linear homogeneous nondegenerate mappings, $f(x, y) = (u(x, y), v(x, y))$,*

$$u(x, y) = x + \alpha_2(x + y)^2 + \alpha_3(x + y)^3, \; v(x, y) = y - \alpha_2(x + y)^2 - \alpha_3(x + y)^3,$$

*$\alpha_2$ and $\alpha_3$ are arbitrary fixed constants.*

It is natural to assume that statements similar to Theorems 1 and 2 hold in $P_m$ for any $m > 3$, i.e., to the *conjecture:*

*If $F(x, y) = (\tilde{U}(x, y), \tilde{V}(x, y)) \in P_m$, $F(0, 0) = 0$, then $J_F \equiv \mathrm{const} \neq 0$ iff $F = \mathcal{A} \circ f \circ \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are linear homogeneous nondegenerate mappings, $f(x, y) = (u(x, y), v(x, y))$,*

$$u(x, y) = x + \alpha_2(x + y)^2 + \ldots + \alpha_m(x + y)^m,$$

$$v(x, y) = y - \alpha_2(x + y)^2 - \ldots - \alpha_m(x + y)^m,$$

*where $\alpha_2, \ldots, \alpha_m$ are arbitrary fixed constants.*

This conjecture is implicitly supported by our (with S. Ponnusamy)

**Theorem A.** *The Jacobian Conjecture is true for mappings $F(X) = (\mathcal{A} \circ f \circ \mathcal{B})(X)$, where $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\mathcal{A}$ and $\mathcal{B}$ are linear such that $\det \mathcal{A}, \det \mathcal{B} \neq 0$, $f = (u_1, \ldots, u_n)$, for $k = 1, \ldots, n$*

$$u_k(X) = x_k + \gamma_k[\alpha_2(x_1 + \ldots + x_n)^2 + \alpha_3(x_1 + \ldots + x_n)^3 + \ldots + \\ + \alpha_m(x_1 + \ldots + x_n)^m],$$

*$\alpha_j, \gamma_k \in \mathbb{R}$ with $\sum_{k=1}^n \gamma_k = 0$.*

**Remark.** *All formulated theorems hold both in real and in complex case.*

**Proof of Theorem 1.** Let $\Phi(x,y) = (DF)^{-1}(0,0)F(x,y) =$
$= (U(x,y), V(x,y))$; then

$$U(x,y) = x + A_2x^2 + A_1xy + A_0y^2 = x + L(x,y),$$

$$V(x,y) = y + a_2x^2 + a_1xy + a_0y^2 = y + l(x,y),$$

Therefore $1 \equiv J_\Phi(x,y) = 1 + I + II$, where

$$I = L_x + l_y \equiv 0, \quad II = L_xl_y - l_xL_y \equiv 0,$$

as $I$ and $II$ are homogeneous polynomials of first and second degree, respectively.

$$II \equiv 0 \Longleftrightarrow \frac{2A_2x + A_1y}{2a_2x + a_1y} \equiv \frac{A_1x + 2A_0y}{a_1x + 2a_0y} \Longleftrightarrow \frac{A_2}{a_2} = \frac{A_1}{a_1} = \frac{A_0}{a_0},$$

i.e. polynomials $L$ and $l$ are mutually proportional. Therefore such $\xi, \eta \in$
$\in \mathbb{R}$ exist that

$$\xi L + \eta l \equiv 0, \quad |\xi| + |\eta| \neq 0. \tag{2}$$

Then

$$I \equiv 0 \Longleftrightarrow \xi l_y = \eta l_x \Longleftrightarrow \xi(a_1x + 2a_0y) = \eta(2a_2x + a_1y), \tag{3}$$

i.e. $\xi a_1 = 2\eta a_2$, $2a_0\xi = \eta a_1$. Let us use the following designation: $a_0 :=$
$:= \eta^2 r, r \in \mathbb{R}$. Then $a_1 = 2\xi\eta r$, $a_2 = \xi^2 r$ and $l = r(\xi x + \eta y)^2$.

First we consider the case $\xi \neq 0$. Then from equality (2) we receive

$$L = -\frac{\eta}{\xi} l = -\frac{\eta}{\xi} r(\xi x + \eta y)^2$$

and we denote $r := \rho\xi, \rho \in \mathbb{R}$. Then

$$l = \rho\xi(\xi x + \eta y)^2, \quad L = -\rho\eta(\xi x + \eta y)^2$$

and

$$\Phi = (U, V), \quad U(x,y) = x - \rho\eta(\xi x + \eta y)^2, V(x,y) = y + \rho\xi(\xi x + \eta y)^2. \tag{4}$$

Note that for a fixed $\xi, \eta \neq 0$ and for a linear mappings with matrix

$$\mathcal{A} = \begin{pmatrix} 1/\xi & 0 \\ 0 & 1/\eta \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}$$

the polynomial mapping

$$\mathcal{A} \circ f \circ \mathcal{B} = (x + \frac{\alpha_2}{\xi}(\xi x + \eta y)^2, y - \frac{\alpha_2}{\eta}(\xi x + \eta y)^2),$$

($f$ is from the formulation of Theorem 1). We put here $\alpha_2 := -\rho\xi\eta$, then the parameter $\alpha_2$ is any fixed number since $\rho$ is arbitrary. We have $\mathcal{A} \circ f \circ \mathcal{B} = (U, V) = \Phi$. Thus polynomial mapping (4) coinsides with (1) up to linear transformations with nondegenerate matrices.

In case $\xi = 0$ we have, from (2), $l \equiv 0$ and equality $I = 0$ implies $L_x = 0$. Thus $L = A_0 y^2$ and $\Phi(x, y) = (x + A_0 y^2, y)$. Let $\mathcal{A}$ and $\mathcal{B}$ are linear homogeneous mappings with matrixes

$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \text{ and } \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

respectively. Then

$$f(x, y) = \mathcal{A} \circ \Phi(x, y) \circ \mathcal{B} = (x + A_0(x + y)^2, y - A_0(x + y)^2).$$

Sufficiency in Theorem 1 is checked by direct calculations. The proof is complete. □

**Proof of Theorem 2.** We will take the Jacobi matrix $DF(0, 0)$ as matrix $A$. Then for the polynomial mapping $\Phi(x, y) := DF^{-1}(0, 0) \circ F(x, y) = (U(x, y), V(x, y))$ we have

$$U(x, y) = x + L(x, y) + W(x, y), V(x, y) = y + l(x, y) + w(x, y),$$

where $L$ and $l$ are homogeneous polynomials of degree 2, $W$ and $w$ are those of degree 3, $J_\Phi(x, y) \equiv 1$. Therefore

$$J_\Phi(x, y) \equiv 1 = 1 + (L_x + l_y) + (L_x l_y - l_x L_y + W_x + w_y) +$$

$$+ [L_x w_y - l_x W_y + W_x l_y - w_x L_y] + \begin{vmatrix} W_x & W_y \\ w_x & w_y \end{vmatrix}. \quad (5)$$

The last term of (5) (and only it) is a homogeneous polynomial from $\mathcal{P}_4$, so it equals 0. Thus lines of the determinant in (5) are proportional, i. e. if we denote

$$W := Ax^3 + Bx^2 y + Cxy^2 + Dy^3, \quad w := ax^3 + bx^2 y + cxy^2 + dy^3,$$

then

$$\frac{W_x}{w_x} = \frac{W_y}{w_y} \Longleftrightarrow \frac{3Ax^2 + 2Bxy + Cy^2}{3ax^2 + 2bxy + cy^2} = \frac{Bx^2 + 2Cxy + 3Dy^2}{bx^2 + 2cxy + 3dy^2}$$

$$\Longrightarrow \frac{A}{a} = \frac{B}{b}, \quad \frac{C}{c} = \frac{D}{d}. \tag{6}$$

Here and in the sequel we can assume that the coefficients $a, b, c, d$ are nonzero, because otherwise consider the polynomial mapping $\Phi^* := \mathcal{A}_\epsilon^{-1} \circ$ $\circ \Phi \circ \mathcal{A}_\epsilon$ with an appropriate homogeneous linear mapping $\mathcal{A}_\epsilon$ with matrix

$$\mathcal{A}_\epsilon = \begin{pmatrix} 1+\epsilon & \epsilon \\ -\epsilon & 1-\epsilon \end{pmatrix}$$

instead of $\Phi$, here $\epsilon$ is the independent variable (we do not consider the case $w \equiv 0 \equiv W$ that is reduced to Theorem 1). Indeed,

$$\Phi^* = (x + L^* + W^*, y + l^* + w^*), \quad (W^*, w^*) = \mathcal{A}_\epsilon^{-1} \circ (W, w) \circ \mathcal{A}_\epsilon,$$

where $L^*$ and $l^*$ are homogeneous polynomials of the second degree and $W^*$ and $w^*$ are that of the third,

$$w^* := a(\epsilon)x^3 + b(\epsilon)x^2y + c(\epsilon)xy^2 + d(\epsilon)y^3.$$

Also

$$a(\epsilon) = a(1+\epsilon)^4 - b(1+\epsilon)^3\epsilon + c(1+\epsilon)^2\epsilon^2 - d(1+\epsilon)\epsilon^3 +$$

$$+ A(1+\epsilon)^3\epsilon - B(1+\epsilon)^2\epsilon^2 + C(1+\epsilon)\epsilon^3 - D\epsilon^4,$$

and equality (6) easily implies $a(\epsilon) \not\equiv 0$; the same is true for other coefficients: $b(\epsilon), c(\epsilon), d(\epsilon) \not\equiv 0$.

So (6) implies

$$\frac{3Ax^2 + 2Bxy + Cy^2}{3ax^2 + 2bxy + cy^2} - \frac{A}{a} = \frac{Bx^2 + 2Cxy + 3Dy^2}{bx^2 + 2cxy + 3dy^2} - \frac{B}{b}.$$

Divide this equality on $y$ and pass to limit as $y \to 0$ to obtain

$$a(C - c\frac{B}{b}) = b(B - b\frac{A}{a}) = 0 \Longrightarrow \frac{C}{c} = \frac{B}{b}.$$

Therefore, $W = \lambda w$ for some constant $\lambda$. Note that this means that original forms $W$ and $w$ are proportional with transform $\mathcal{A}_\epsilon^{-1} \circ (W, w) \circ \mathcal{A}_\epsilon$ not taken into account.

Also in (5) the third degree is only in the last square bracket, so it equals 0 and

$$L_x w_y - \lambda l_x w_y + \lambda w_x l_y - w_x L_y \equiv 0,$$

i.e. for all $(x, y)$

$$(L - \lambda l)_x w_y = (L - \lambda l)_y w_x. \tag{7}$$

Note that the case $w_x \equiv 0 \equiv w_y$ suits conditions of Theorem 1, so we need not to consider it. We consider three cases.

1) $(L - \lambda l)_x \not\equiv 0 \not\equiv (L - \lambda l)_y$.

Let us show that such numbers $a$, $b$ ($|a| + |b| \neq 0$) exist, that

$$w_x = (L - \lambda l)_x (ax + by), \quad w_y = (L - \lambda l)_y (ax + by). \tag{8}$$

First assume that

$$\frac{(L - \lambda l)_x}{(L - \lambda l)_y} \not\equiv const.$$

Denote linear functions $I := (L - \lambda l)_x$, $II := (L - \lambda l)_y$. If $t := y/x$, $w_x$ can be decomposed in the field of complex numbers to product of linear factors $w_x = \xi x^2 (t - t_1)(t - t_2)$, $\xi = const$; analogously for $w_y$. Then in (7)

$$(L - \lambda l)_x w_y = (L - \lambda l)_y w_x = I \cdot II \cdot III,$$

where $III$ is also some linear factor. Thus

$$w_x = I \cdot III = (L - \lambda l)_x \cdot III, \quad w_y = II \cdot III = (L - \lambda l)_y \cdot III,$$

i.e. we obtain (8).

Now let

$$\frac{(L - \lambda l)_x}{(L - \lambda l)_y} \equiv c = const (\neq 0).$$

Decompose $w_y$ to linear factors, as in the previous paragraph:

$$w_y = (\alpha x + \beta y)(\gamma x + \delta y).$$

Assume that these factors are not mutually proportional, the case $\alpha = \gamma$, $\beta = \delta$ is considered in the similar way. From (7) we have $w_x = c w_y$. So

$$w_{xy} = c w_{yy} = c[\beta(\gamma x + \delta y) + \delta(\alpha x + \beta y)] = w_{yx} = \alpha(\gamma x + \delta y) + \gamma(\alpha x + \beta y)$$

$$\Longrightarrow (\gamma x + \delta y)(c\beta - \alpha) = (\alpha x + \beta y)(\gamma - c\delta).$$

Thus $c\beta - \alpha = 0 = \gamma - c\delta$, i.e. $c\beta = \alpha$, $\gamma = c\delta$. This implies

$$w_y = \beta\delta(cx + y)^2, \quad w_x = c\beta\delta(cx + y)^2.$$

Denote $(L-\lambda l)_y = qx+py$; then $(L-\lambda l)_x = c(qx+py)$ and $(L-\lambda l)_{xy} = = q = pc$. Therefore

$$(L - \lambda l)_x = pc(cx + y), \quad (L - \lambda l)_y = p(cx + y).$$

Assumption of case 1) implies $p \neq 0$. Now we have

$$w_y = (L - \lambda l)_y (cx + y)\frac{\beta\delta}{p}, \quad w_x = cw_y = (L - \lambda l)_x (cx + y)\frac{\beta\delta}{p}.$$

This completes the proof of (8).

Further we consider $b \neq 0$, the symmetric case $a \neq 0$ is similar. From (8) we have

$$w = \int w_x dx + C_1(y) = (ax + by)(L - \lambda l) - a\int (L - \lambda l)dx + C_1(y). \quad (9)$$

Denote $(L-\lambda l) = Ax^2+Bxy+Cy^2$. From this, (9) and the second equality in (8) we have

$$b(Ax^2 + Bxy + Cy^2) = a\left(\frac{B}{2}x^2 + 2Cyx\right) - C_1'(y)$$

$$\Longrightarrow C_1'(y) = -bCy^2, \quad B = \frac{2aC}{b}, \quad A = \frac{a^2C}{b^2}$$

and

$$L-\lambda l = C\left(\frac{a}{b}x + y\right)^2, (L-\lambda l)_y = \frac{2C\,(ax + by)}{b}, (L-\lambda l)_x = \frac{2aC\,(ax + by)}{b^2}.$$

From here and (7)

$$\frac{(L - \lambda l)_x}{(L - \lambda l)_y} = \frac{a}{b} = \frac{w_x}{w_y}$$

follows. Besides, (8) implies

$$w_x = \frac{2aC}{b^2}(ax + by)^2, \quad w_y = \frac{2C}{b}(ax + by)^2, \quad (10)$$

and

$$L_x = \lambda l_x + \frac{2aC}{b^2}(ax+by), \quad L_y = \lambda l_y + \frac{2C}{b}(ax+by). \tag{11}$$

Both parentheses in (5) equal zero due to identity (5).

$$L_x + l_y = 0, \tag{12}$$

$$L_x l_y - l_x L_y + \lambda w_x + w_y = 0. \tag{13}$$

Write down (12) using (11) in the form

$$\lambda l_x + \frac{2aC}{b^2}(ax+by) + l_y = 0. \tag{14}$$

Using (13), (11) and (10) get

$$al_y - bl_x + (\lambda a + b)(ax+by) = 0. \tag{15}$$

From this, taking (14) into account, obtain

$$(\lambda a + b) l_x = \left( (\lambda a + b) - \frac{2a^2C}{b^2} \right)(ax+by). \tag{16}$$

If $(\lambda a + b) = 0$, then from (16) it follows that $aC = 0$. Therefore, using the first equality from (11) we obtain $(L - \lambda l)_x = 0$. But this contradicts the assumption of case 1).

If $(\lambda a + b) \neq 0$, then

$$l_x = \left[ 1 - \frac{2a^2C}{b^2(\lambda a + b)} \right](ax+by).$$

If $a = 0$ then $l_x = by$ and from (11) implies

$$L_x = \lambda by \Longrightarrow (L - \lambda l)_x = 0;$$

this contradicts the assumption of case 1). Therefore $a \neq 0$. Then from (15):

$$al_y = -\left[ \frac{2a^2C}{b(\lambda a + b)} + \lambda a \right](ax+by)$$

and since $a \neq 0$ then

$$l_{yx} = -\left[ \frac{2a^2C}{b(\lambda a + b)} + \lambda a \right] = l_{xy} = b\left[ 1 - \frac{2a^2C}{b^2(\lambda a + b)} \right].$$

Consequently $b = -\lambda a$, this contradicts the assumption $(\lambda a + b) \neq 0$.

Thus, case 1) is not realised.

2) Let $(L - \lambda l)_x = (L - \lambda l)_y \equiv 0$. Then $L = \lambda l$ and from (13) we see that

$$\lambda w_x + w_y = 0. \tag{17}$$

Repeating the proof of Theorem 1 using equalities $L = \lambda l$ and (12) instead of (2) and (3), we obtain, similarly to conclusion of the proof of Theorem 1, that

$$\Phi(x, y) = (x + \alpha_2(x + y)^2 + W(x, y), y - \alpha_2(x + y)^2 + w(x, y))$$

up to linear mappings $\mathcal{A}$ and $\mathcal{B}$ from formulation of Theorem 2. In particular, this implies that constant $\lambda = -1$ in equality $L = \lambda l$. Then equality (17) becomes $w_y = w_x$. Denote $w(x, y) = \sum_{k=0}^{3} \beta_k x^k y^{3-k}$ and get

$$w_x = \sum_{k=0}^{3} k\beta_k x^{k-1} y^{3-k} = w_y = \sum_{k=0}^{3} (3 - k)\beta_k x^k y^{2-k}.$$

Compare coefficients of the same powers in this equality to obtain the recurrent equation $\beta_{j+1} = \beta_j \dfrac{3 - j}{1 + j}$, $j = 0, 1, 2$, i. e. $\beta_1 = 3\beta_0 = \beta_2$, $\beta_3 = \beta_0$. So,

$$w = \beta_0(x + y)^3, \quad W = \lambda w = -w = -\beta_0(x + y)^3.$$

This finishes the proof of case 2) of Theorem 2.

3) Let $(L - \lambda l)_x = 0$, $(L - \lambda l)_y \not\equiv 0$ (the symmetric case is considered similarly).

Then (7) implies $w_x \equiv 0$ (but $w_y \not\equiv 0$, otherwise we will come to situation from formulation of Theorem 1. These assumptions mean that $(L - \lambda l) = ry^2$, $w = sy^3$, $r \neq 0 \neq s$ are constants. So

$$L_y = \lambda l_y + 2ry, \quad L_x = \lambda l_x. \tag{18}$$

Then from (12) we have

$$\lambda l_x + l_y = 0, \tag{19}$$

from (13) and (18) $w_y = 2ryl_x$, therefore $l_x = \dfrac{3s}{2r} y$. Now (19) implies

$$l_y = -\lambda l_x = -\frac{3s\lambda}{2r} y \Longrightarrow l = \int l_x dx + C(y) = \frac{3s}{2r} yx + C(y),$$

$$l_y = \frac{3s}{2r}x + C'(y) = -\frac{3s\lambda}{2r}y,$$

i. e., $C'(y) = -\frac{3s}{2r}(\lambda y + x)$. The left-hand side of the last equality depends only on $y$, therefore this equality can be true only for $s = 0$; thus $w \equiv 0$ and $W = \lambda w \equiv 0$. And we have the case that is reduced to Theorem 1.

Sufficiency in Theorem 2 is checked by direct calculations. Theorem 2 is proved. □

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