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ON THE SCHWARZIAN NORM OF HARMONIC MAPPINGS

Abstract. We obtain estimations of the pre-Schwarzian and Schwarzian derivatives in terms of the order of family in linear and affine invariant families $\mathcal{L}$ of sense preserving harmonic mappings of the unit disk $\mathbb{D}$. As the converse result the order of family $\mathcal{L}$ is estimated in terms of suprema of Schwarzian and pre-Schwarzian norms over the family $\mathcal{L}$. Main results are obtained by means of theory of linear invariant families.

Key words: *harmonic mappings, linear invariant families, Schwarzian derivative*

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1. Preliminaries. Every sense preserving harmonic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ function $f(z)$ admits representation $f(z) = h(z) + \overline{g(z)}$, where h and g are analytic in $\mathbb{D}$,

$$h(z) = \sum_{k=0}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \tag{1}$$

the dilatation $\omega(z) = g'(z)/h'(z)$ is analytic in $\mathbb{D}$ and $|\omega(z)| < 1$ for all $z \in \mathbb{D}$.

The *pre-Schwarzian* and *Schwarzian derivatives* of a locally univalent analytic function h are given (cf. [1]) by

$$Ph(z) = \frac{h''(z)}{h'(z)} \quad \text{and} \quad Sh(z) = \left(\frac{h''(z)}{h'(z)}\right)' - \frac{1}{2}\left(\frac{h''(z)}{h'(z)}\right)^2$$

respectively. The important role of the Schwarzian derivative in theory of univalent analytic functions is well known. The following theorem was proved by W. Krauss [2] and rediscovered later by Z. Nehari [3].

Theorem A. *If h is a univalent analytic function in $\mathbb{D}$, then*

$$|Sh(z)| \leq \frac{6}{(1-|z|^2)^2}$$

for any $z \in \mathbb{D}$.

The same inequality with the constant 2 instead of 6 in the numerator was found by Nehari as the sufficient condition of univalence of an analytic function. L. Ahlfors and G. Weill [4] expanded this result by establishing the condition under which a univalent analytic in $\mathbb{D}$ function has a quasiconformal extension onto the whole Riemann sphere.

In 2003 the notion of Schwarzian derivative was generalized by P. Duren, B. Osgood and M. Chuaqui [5] onto the case of locally univalent harmonic functions $f = h + \overline{g}$ in the disk $\mathbb{D}$ with dilatation $\omega = g'/h' = q^2$, where q is some analytic function in $\mathbb{D}$. Later R. Hernández and M. J. Martín [6] proposed a modified definition of pre-Schwarzian P_f and Schwarzian S_f derivatives that are valid for the whole family of sense preserving harmonic mappings. This definitions preserve the main properties of classical Schwarzian derivative and are the following:

$$P_f(z) = Ph(z) - \frac{\omega'(z)\overline{\omega(z)}}{1-|\omega(z)|^2},$$
$$S_f(z) = Sh(z) + \frac{\overline{\omega(z)}}{1-|\omega(z)|^2}\left(\frac{h''(z)}{h'(z)}\omega'(z) - \omega''(z)\right) - \frac{3}{2}\left(\frac{\omega'(z)\overline{\omega(z)}}{1-|\omega(z)|^2}\right)^2.$$

So as in the analytic case we have

$$S_f(z) = \left(P_f(z)\right)_z - \frac{1}{2}\left(P_f(z)\right)^2.$$

The pre-Schwarzian and Schwarzian derivatives of harmonic function have *the chain rule* property (cf., [6]) exactly in the same form as in the analytic case: if f is a sense preserving harmonic function and φ is a locally univalent analytic function for which the composition $f \circ \varphi$ is defined, then

$$\begin{aligned} P_{f\circ\varphi}(z) &= P_f \circ \varphi(z) \cdot \varphi'(z) + P_\varphi(z), \\ S_{f\circ\varphi}(z) &= S_f \circ \varphi(z) \cdot \left(\varphi'(z)\right)^2 + S_\varphi(z). \end{aligned} \tag{2}$$

Both pre-Schwarzian and Schwarzian derivatives are invariant under affine transformation of a harmonic function f: if $A(w) = aw + b\overline{w} + c$, $|a| > |b|$, then

$$P_{A\circ f}(z) \equiv P_f(z) \quad \text{and} \quad S_{A\circ f}(z) \equiv S_f(z). \tag{3}$$

The *pre-Schwarzian* and *Schwarzian norms* of a harmonic function f are defined as

$$||P_f|| = \sup_{z\in\mathbb{D}} |P_f(z)|(1-|z|^2),$$
$$||S_f|| = \sup_{z\in\mathbb{D}} |S_f(z)|(1-|z|^2)^2.$$

In the current paper we deal with estimations of this quantities and concerned functionals in the linear and affine invariant families of harmonic functions.

A family $\mathcal{L}$ of sense preserving harmonic functions $f = h + \overline{g}$, where h and g have form (1) with $a_0 = a_1 - 1 = 0$, is said to be *linear invariant family* (l.i.f.) if for any $f \in \mathcal{L}$

$$L_\phi[f](z) = \frac{f(\phi(z)) - f(\phi(0))}{f_z(\phi(0))\phi'(0)} \in \mathcal{L} \ \ \forall\ \phi(z) = e^{i\theta}\frac{z+a}{1+\overline{a}z},\ a \in \mathbb{D},\ \theta \in \mathbb{R},$$

and $\mathcal{L}$ is called *affine invariant family* (a.i.f.) if the property

$$A_\varepsilon[f](z) = \frac{f(z) + \varepsilon\overline{f(z)}}{1 + \varepsilon\overline{f}_z(0)} \in \mathcal{L} \ \ \forall\ \varepsilon \in \mathbb{D}$$

holds. The family $\mathcal{L}$ is called affine and linear invariant (a.l.i.f.) if it is l.i.f and a.i.f simultaneously. It is remarkable that many important properties of a.l.i.f. depend on supremum of the absolute value of the second coefficients a_2 over the family $\mathcal{L}$. The notion of *order of family* $\mathcal{L}$ was defined by T. Sheil-Small [7] as $\alpha = \sup_{f\in\mathcal{L}} |a_2|$ (exactly in the same form as in the analytic case, cf., [8]). This paper of Sheil-Small on linear and affine invariant families of harmonic functions attracted attention of many mathematicians (cf., [9, 10, 11]). In 2004 V. Starkov ([12], see also [10] for details) gave definition of *new order* of l.i.f. as

$$\overline{\alpha} = \sup_{f\in\mathcal{L}} \frac{|a_2 - \overline{b_1}b_2|}{1-|b_1|^2},$$

such that $\overline{\alpha} \le \alpha \le \overline{\alpha} + 1/2$. In 2007, the author [13] introduced the notion of *specified order* of a.l.i.f. $\mathcal{L}$ as

$$\alpha_0 = \sup_{f\in\mathcal{L}^0} |a_2|$$

where $\mathcal{L}^0 = \{f \in \mathcal{L} : b_1 = 0\}$. In [14], it was established that the new order $\overline{\alpha}$ of l.i.f. $\mathcal{L}$ coincides with the specified order α_0 of affine hull of $\mathcal{L}$. Let $\beta_0 = \sup_{f\in\mathcal{L}^0} |b_2|$. So $\alpha_0 \le \alpha \le \alpha_0 + 1/2$ and $\beta_0 \le 1/2$ for any a.l.i.f. $\mathcal{L}$ (cf., [15]).

At present the orders α and α_0 are known only for several subclasses of the sense preserving univalent harmonic mappings $f = h + \overline{g}$ in $\mathbb{D}$ with h, g of the form (1), where $a_0 = a_1 - 1 = 0$. Particularly for linear and affine invariant families $\mathcal{K}_H$ and $\mathcal{C}_H$ of convex and close-to-convex harmonic mappings the order α_0 is proved to be $3/2$ and $5/2$ respectively (see [16] and [17]), while $\alpha = \alpha_0 + 1/2$ and $\beta_0 = 1/2$ in both cases. Recall that the function f is called close-to-convex if the range $D = f(\mathbb{D})$ under the mapping f is a close-to-convex domain i.e. $\mathbb{C} \setminus D$ can be represented as union of disjoint lines or half-lines intersecting only in the boundary of D.

2. Analogue of the Krauss theorem for harmonic functions. The main results of the present paper are the estimations for pre-Schwarzian and Schwarzian derivatives and their norms in the a.l.i.f. of locally univalent harmonic functions. We start with estimation of pre-Schwarzian derivative in the spirit of well-known inequality for $|h''/h'|$ by L. Bieberbach for the class $\mathcal{S}$ of univalent conformal mappings h.

Theorem 1. *For any locally univalent harmonic mapping $f \in \mathcal{L}$ and for all $z \in \mathbb{D}$*

$$|P_f(z)| \le 2\frac{\alpha_0 + |z|}{1 - |z|^2}. \tag{4}$$

The estimation is sharp for example in a.l.i.f.s $\mathcal{K}_H$ and $\mathcal{C}_H$ of univalent convex and close-to-convex harmonic functions. Particularly

$$||P_f|| \le 2(\alpha_0 + 1).$$

Proof. Let $f = h+\overline{g} \in \mathcal{L}$ and $z \in \mathbb{D}$ be fixed. Using the property of linear invariance of $\mathcal{L}$, define the linear transformation $\varphi(\zeta) = (\zeta + z)/(\overline{1 + \overline{z}\zeta})$ such that $\varphi(0) = z$, $P_\varphi(0) = -2\overline{z}$ and consider $F(\zeta) = H(\zeta) + \overline{G(\zeta)} = L_\varphi[f](z) \in \mathcal{L}$. Then in view of the chain rule (2) for pre-Schwarzian derivative we have

$$P_f(z) = \frac{P_F(0) + 2\overline{z}}{1 - |z|^2}.$$

Now, applying the affine invariance of $\mathcal{L}$, let $F_0(\zeta) = H_0(\zeta) + \overline{G_0(\zeta)} = A_\varepsilon[F](\zeta) \in \mathcal{L}$, where $\varepsilon = -\overline{G'(0)}$. Then $G_0'(0) = (G'(0) + \overline{\varepsilon}H'(0))/(1-$

$-|\varepsilon|^2) = 0$ and hence $F_0 \in \mathcal{L}^0$ so $\omega_0(0) = G_0'(0)/H_0'(0) = 0$. Therefore, from (3), we conclude that $P_F(0) = P_{F_0}(0) = H_0''(0)$ and

$$P_f(z) = \frac{H_0''(0) + 2\overline{z}}{1 - |z|^2}.$$

But $|H_0''(0)| \leq 2\alpha_0$ by definition of specified order of a.l.i.f. $\mathcal{L}$, and inequality (4) follows immediately from the last representation of $P_f(z)$.

The estimation of pre-Schwarzian norm of f follows easily from (4).

To make sure that the inequality (4) is sharp consider for example the a.l.i.f. $\mathcal{C}_H$ of all univalent sense preserving close-to-convex harmonic functions. The well-known (cf., [16]) harmonic analogue of the Koebe function

$$K(z) = H_K(z) + \overline{G_K(z)} = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \overline{\left(\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}\right)} \qquad (5)$$

belongs to $\mathcal{C}_H^0$. It is easy to compute that $H_K''(0)/2 = 5/2$ is equal to specified order of $\mathcal{C}_H$ and (4) turns into equality for function K and any $z = r \in [0, 1)$.

To prove the sharpness of the inequality (4) in the class $\mathcal{K}_H$ of convex univalent sense preserving harmonic functions it is enough to note that in this case $\alpha_0 = 3/2$ (cf., [16]) and for the function

$$L(z) = \mathrm{Re}\left(\frac{z}{1-z}\right) + i\mathrm{Im}\left(\frac{z}{(1-z)^2}\right) \in \mathcal{K}_H^0 \qquad (6)$$

its pre-Schwarzian equals 3 at origin, i.e. coincides with $2\alpha_0$. The proof is complete. □

Note that for a.l.i.f. $\mathcal{K}_H$ of univalent convex harmonic mappings our estimation of the pre-Schwarzian norm ($||P_f|| \leq 5$) coincides with the estimation obtained recently by Hernández and Martín [6].

Now we are going to prove the estimation of the Schwarzian derivative of harmonic functions. As in the analytic case the key role in studying of $||S_f||$ plays the estimation of the *Fekete-Szegö functional* $|a_3 - a_2^2|$. We will obtain its upper bound via the logarithmic coefficients of harmonic function. Let $f = h + \overline{g} \in \mathcal{L}^0$. Define locally univalent analytic function $F_\varepsilon(z) = h(z) + \varepsilon g(z)$ in $\mathbb{D}$, where ε is arbitrary such that $|\varepsilon| \leq 1$. Let

$$\Phi_\varepsilon(z) = \log F_\varepsilon'(z) = \sum_{k=1}^{\infty} c_k(\varepsilon) z^k.$$

We will designate $c_k(\varepsilon)$ as *logarithmic coefficients* of f.

Theorem 2. *Let a locally univalent harmonic mapping $f \in \mathcal{L}^0$ and let $|\varepsilon| \le 1$. Then*

$$|c_k(\varepsilon)| \le \min_{r\in(0,1)} C_k(\varepsilon, r) < C_k\left(\varepsilon, \sqrt{\frac{k-1}{k+1}}\right), \tag{7}$$

where

$$C_k(\varepsilon, r) = \frac{2}{kr^{k-2}(1-r^2)}\left\{\frac{\alpha_0+\beta_0\frac{r+|\varepsilon|}{1+r|\varepsilon|}}{r} - \frac{r}{\alpha_0+\beta_0\frac{r+|\varepsilon|}{1+r|\varepsilon|}}\right\}. \tag{8}$$

Proof. For a function $f \in \mathcal{L}^0$ and arbitrary ε, $|\varepsilon| \le 1$, we define analytic function Φ_ε as above. Then $\Phi'_\varepsilon(z) = \frac{h''(z)+\varepsilon g''(z)}{h'(z)+\varepsilon g'(z)}$. The sharp estimation of the modulus of this ratio in the a.l.i.f. $\mathcal{L}$ was obtained in [11] (the proof is available in [15]) and yields the inequality

$$\left|\Phi'_\varepsilon(z) - \frac{2\overline{z}}{1-|z|^2}\right| \le \frac{2}{1-|z|^2}\left(\alpha_0+\beta_0\frac{|z|+|\varepsilon|}{1+|z||\varepsilon|}\right).$$

Then for any fixed $r \in (0,1)$ and any ζ, $|\zeta| \le 1$, in view of the maximum principle for analytic functions we have

$$\left|\frac{1-r^2}{2}\zeta\Phi'_\varepsilon(r\zeta) - r\right| \le \alpha_0+\beta_0\frac{r+|\varepsilon|}{1+r|\varepsilon|}.$$

Consider the analytic function in $\mathbb{D}$, defined by

$$\Psi(\zeta) = \left(\frac{1-r^2}{2}\zeta\Phi'_\varepsilon(r\zeta) - r\right)\left(\alpha_0+\beta_0\frac{r+|\varepsilon|}{1+r|\varepsilon|}\right)^{-1} = \sum_{k=0}^{\infty} d_k\zeta^k,$$

where

$$d_k = kc_k(\varepsilon)r^{k-1}\frac{1-r^2}{2}\left(\alpha_0+\beta_0\frac{r+|\varepsilon|}{1+r|\varepsilon|}\right)^{-1}$$

for $k \ge 1$ in terms of logarithmic coefficients of f and

$$d_0 = -r\left(\alpha_0+\beta_0\frac{r+|\varepsilon|}{1+r|\varepsilon|}\right)^{-1}.$$

Function Ψ meets the inequality $|\Psi(\zeta)| \le 1$ and hence as a consequence of the well-known Schwarz lemma (cf., [18], pp. 319–320) we obtain

$$|d_k| \le 1 - |d_0|^2,\ k \ge 1.$$

From there taking expressions for d_k into account we have

$$|c_k(\varepsilon)| \le C_k(\varepsilon, r)$$

for any $r \in (0,1)$, where C_k are given by (8). To finish the proof we have to note that the function $r^{1-k}(1-r^2)^{-1}$ takes its minimal value when $r = \sqrt{(k-1)/(k+1)}$ and therefore,

$$|c_k(\varepsilon)| \le C_k\left(\varepsilon, \sqrt{\frac{k-1}{k+1}}\right),$$

which is the required inequality. □

As a consequence we obtain the estimation of Fekete-Szegö functional for family $\mathcal{L}^0$.

Corollary. *Let a locally univalent harmonic mapping $f \in \mathcal{L}^0$. Then*

$$\begin{aligned} \left|a_3 - a_2^2\right| &\le \frac{1}{3}\left(\alpha_0^2 + \min_{r\in(0,1)} C_2(0,r)\right) < \\ &< \frac{\alpha_0^2}{3} + \frac{1}{2}\left(\sqrt{3}\alpha_0 + \beta_0 - \frac{1}{\sqrt{3}\alpha_0 + \beta_0}\right). \end{aligned} \tag{9}$$

Proof. Let $f = h + \overline{g} \in \mathcal{L}^0$ and $h(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. Applying inequality (7) with $\varepsilon = 0$ and $k = 2$, we obtain

$$|c_2(0)| \le \frac{3}{2}\left(\sqrt{3}\alpha_0 + \beta_0 - \frac{1}{\sqrt{3}\alpha_0 + \beta_0}\right).$$

On the other hand it is easy to verify that in the current case the logarithmic coefficient $c_2(0) = 3a_3 - 2a_2^2$. Therefore

$$|a_3 - a_2^2| \le \frac{1}{3}|3a_3 - 2a_2^2| + \frac{1}{3}|a_2|^2 \le \frac{1}{3}|a_2|^2 + \frac{1}{2}\left(\sqrt{3}\alpha_0 + \beta_0 - \frac{1}{\sqrt{3}\alpha_0 + \beta_0}\right).$$

We obtain the desired estimation (9) from the last inequality because $|a_2| \le \alpha_0$ for any function $f \in \mathcal{L}^0$. □

As an immediate consequence of the Corollary 1 we obtain an estimation of the Schwarzian norm in a.l.i.f. $\mathcal{L}$.

Theorem 3. *Let a locally univalent harmonic mapping $f \in \mathcal{L}$. Then*

$$||S_f|| \leq 2\left(\alpha_0^2 + \min_{r\in(0,1)} C_2(0,r)\right) < 2\alpha_0^2 + 3\left(\sqrt{3}\alpha_0 + \beta_0 - \frac{1}{\sqrt{3}\alpha_0 + \beta_0}\right).$$

Proof. Let $f = h+\overline{g} \in \mathcal{L}$ and $z \in \mathbb{D}$ are fixed. As in the proof of Theorem 1 we will use the property of linear invariance of $\mathcal{L}$. For the Möbius function $\varphi(\zeta) = (\zeta + z)/(1 + \overline{z}\zeta)$, we consider $F(\zeta) = H(\zeta) + \overline{G(\zeta)} = L_\varphi[f](z) \in \mathcal{L}$. Then in view of the chain rule (2) for the Schwarzian derivative and the well known fact that $S\varphi \equiv 0$ we have

$$S_f(z) = \frac{S_F(0)}{(1-|z|^2)^2}.$$

Using the affine invariance of $\mathcal{L}$ for $\varepsilon = -\overline{G'(0)}$ we define $F_0(\zeta) = H_0(\zeta) + \overline{G_0(\zeta)} = A_\varepsilon[F](\zeta) \in \mathcal{L}^0$ such that $G_0'(0) = 0$. Therefore from (3) we conclude that

$$S_F(0) = S_{F_0}(0) = SH_0(0) = \frac{H_0'''(0)}{H_0'(0)} - \frac{3}{2}\left(\frac{H_0''(0)}{H_0'(0)}\right)^2 = 6(A_3 - A_2^2),$$

where A_k are the Taylor coefficients of function H_0. Finally applying the inequality (9) we are able to estimate the Schwarzian of f:

$$|S_f(z)|(1-|z|^2)^2 = |S_{F_0}(0)| = 6|A_3 - A_2^2| \leq 2\left(\alpha_0^2 + \min_{r\in(0,1)} C_2(0,r)\right).$$

The estimation of the Schwarzian norm of f follows easily from the last expression by definition. □

Theorem 3 allows us to obtain estimations of the Schwarzian norms in a.l.i.f.s whose orders are known. For example if f belongs to the family $\mathcal{C}_H$ of close-to-convex univalent harmonic mappings (remind that $\alpha_0 = 5/2$ in this case) then $||S_f|| < 26.368$.

Note that for the class $\mathcal{K}_H$ of convex harmonic mappings estimation of the Schwarzian norm $||S_f|| \leq 6$ was obtained by Hernández and Martín [6].

3. Estimation of order of a.l.i.f. As a converse problem let us consider the question of estimating the orders α or α_0 of a.l.i.f. $\mathcal{L}$ if the estimations of Schwarzian or pre-Schwarzian norms are known for the family $\mathcal{L}$.

Theorem 4. *Let $\mathcal{L}$ be a.l.i.f. of harmonic functions and there exists a constant C such that*

$$\sup_{f\in\mathcal{L}} |P_f(z)(1-|z|^2)| \leq C+2|z|.$$

Then $\alpha_0 \leq \frac{C}{2}$, $\alpha \leq \frac{C+1}{2}$.

Estimations are sharp for example in the a.l.i.f. $\mathcal{K}_H$ of convex harmonic mappings.

Particularly, if $\sup_{f\in\mathcal{L}} ||P_f|| \leq D$, then $\alpha_0 \leq \frac{D}{2}$, $\alpha \leq \frac{D+1}{2}$.

Proof. If there exists a finite C such that $|P_f(z)(1-|z|^2)| \leq C+2|z|$ for any $f\in\mathcal{L}$ and all $z\in\mathbb{D}$ then, particularly, $|P_f(0)|\leq C$. In view of definition of P_f this means that

$$\left|2\,a_2 - \overline{b}_1 \frac{\omega'(0)}{1-|\omega(0)|^2}\right| \leq C$$

for any $f\in\mathcal{L}$ with dilatation ω and $b_1 = \overline{f}_z(0)$. Therefore,

$$|a_2| \leq \frac{1}{2}\left(C+|b_1|\frac{|\omega'(0)|}{1-|\omega(0)|^2}\right). \tag{10}$$

Obviously $|b_1| < 1$. Applying invariant form of the Schwarz lemma (cf. [18], pp. 319–320) to the dilatation ω we deduce that $|\omega'(0)|/(1-|\omega(0)|^2)\leq 1$ and hence $|a_2|\leq (C+1)/2$ for any $f\in\mathcal{L}$. This inequality gives us the desired estimation of order α of $\mathcal{L}$.

In the case when $f\in\mathcal{L}^0$ we have $b_1=0$ and estimation of the specified order α_0 easily follows from (10).

The sharpness of the result of the theorem in class $\mathcal{K}_H$ of convex harmonic mappings can be verified directly. Indeed, from Theorem 1 it follows that the best possible value of constant C is equal to $2\alpha_0 = 3$ in a.l.i.f. $\mathcal{K}_H$. On the other hand, for function L given by (6) the second coefficient $a_2 = 3/2$ while $b_1 = 0$. So (10) turns into equality. Therefore the order and specified order of a.l.i.f. $\mathcal{K}_H$ should not be greater than 2 and 3/2, respectively, and that represent the facts. □

To obtain the similar result in terms of the Schwarzian norm we first have to estimate the Fekete-Szegö functional $|a_3 - a_2^2|$ by supremum of $||S_f||$ over a.l.i.f. $\mathcal{L}$.

Theorem 5. *Let $\mathcal{L}$ be a.l.i.f. of harmonic functions and there exists a constant C such that*

$$\sup_{f\in\mathcal{L}} ||S_f|| \leq C.$$

Then $|a_3 - a_2^2| \leq C/6$ for any $f \in \mathcal{L}^0$ and

$$|a_3 - a_2^2| \leq \frac{C + 2\alpha_0 + 3/2}{6}$$

for any $f \in \mathcal{L}$.

Proof. First, if the condition of Theorem 5 is satisfied and $f \in \mathcal{L}^0$, then $S_f(0) = 6(a_3 - a_2^2)$ and statement of theorem is clear. Now, let $f \in \mathcal{L}$. Then there exists a function $f_0 = h_0 + \overline{g_0} \in \mathcal{L}^0$ such that

$$f(z) = f_0(z) + \overline{b_1 f_0(z)}.$$

So $a_k = a_k^0 + \overline{b_1} b_k^0$, where $|b_1| < 1$, a_k are Taylor coefficients of the analytic part of f while a_k^0, b_k^0 are the coefficients of analytic and co-analytic parts of f_0, respectively.

Then the Fekete-Szegö functional of f takes the form

$$|a_3 - a_2^2| = \left| a_3^0 - \left(a_2^0\right)^2 + \overline{b_1}\left(b_3^0 - 2a_2^0 b_2^0 - \overline{b_1}\left(b_2^0\right)^2\right)\right|. \tag{11}$$

To estimate the last quantity we will evaluate b_3^0 via derivative of dilatation $\omega_0 = g_0'/h_0'$ of function f_0 at origin:

$$\omega_0'' = \frac{\left(g_0''' h_0' - g_0' h_0'''\right) h_0' - 2h_0''\left(g_0'' h_0' - g_0' h_0''\right)}{\left(h_0'\right)^3}.$$

But $g_0'(0) = 0$ for $f_0 \in \mathcal{L}^0$ and $h_0'(0) = 1$. Therefore

$$\omega_0''(0) = 6b_3^0 - 8a_2^0 b_2^0. \tag{12}$$

Note that analytic function $\omega_0(z) = 2b_2^0 z + 2^{-1}\omega''(0)z^2 + \cdots$ satisfies conditions of the Schwarz lemma, so $|\omega_0(z)| \leq |z|$ and function $u(z) = \omega_0(z)/z = 2b_2^0 + 2^{-1}\omega''(0)z + \cdots$ satisfies the invariant form of the Schwarz lemma. Hence $|u'(0)| \leq 1 - |u(0)|^2$ and

$$|\omega_0''(0)| \leq 2\left(1 - 4\left|b_2^0\right|^2\right). \tag{13}$$

Now we will use equality (12) together with estimation (13) in order to obtain an upper bound for $|a_3 - a_2^2|$ in (11):

$$|a_3 - a_2^2| \leq \left|a_3^0 - \left(a_2^0\right)^2\right| + |b_1| \left|\frac{1}{6}\omega_0''(0) - \frac{2}{3}a_2^0 b_2^0 - \overline{b_1}\left(b_2^0\right)^2\right| \leq$$
$$\leq \frac{1}{6}C + \frac{1}{6}|\omega''(0)| + \frac{2}{3}\left|a_2^0\right|\left|b_2^0\right| + \left|b_2^0\right|^2 \leq$$
$$\leq \frac{1}{6}C + \frac{1}{3}\left(1 - \left|b_2^0\right|^2 + 2\left|a_2^0\right|\left|b_2^0\right|\right) \leq \frac{1}{6}C + \frac{1}{3}\left(\alpha_0 + \frac{3}{4}\right)$$

because $|a_2^0| \leq \alpha_0$ for $f_0 \in \mathcal{L}^0$ and function $y(t) = 1 - t^2 + 2\alpha_0 t$ is increasing for $0 \leq t \leq \beta_0 \leq 1/2$ and $\alpha_0 \geq 1/2$. Here we have to mark out that the order α_0 of any a.l.i.f. of a harmonic sense preserving functions is not less than $1/2$. Indeed, it is known (cf., [8]) that order α of any l.i.f. (even in the analytic case) is not less than 1, and $\alpha_0 \geq \alpha - 1/2$ as was mentioned above. □

Corollary. *Let $\mathcal{L}$ be a.l.i.f. of harmonic functions and there exists a constant C such that*

$$\sup_{f\in\mathcal{L}} ||S_f|| \leq C.$$

Then $\alpha \leq \frac{1+\sqrt{8+2C}}{2}$.

Proof. First of all note that the set of analytic parts h of harmonic functions $f \in \mathcal{L}$ forms a linear invariant family $\mathcal{U}$ of analytic functions. Ch. Pommerenke [8] proved that in any l.i.f. $\mathcal{U}$ of analytic functions h the order α satisfies the inequality

$$\alpha^2 - 1 \leq 3 \sup_{h\in\mathcal{U}} |a_3 - a_2^2|.$$

From Theorem 5 we deduce that

$$\sup_{h\in\mathcal{U}} |a_3 - a_2^2| = \sup_{f\in\mathcal{L}} |a_3 - a_2^2| \leq \frac{C + 2\alpha + 3/2}{6}$$

because specified order $\alpha_0 \leq \alpha$ in any a.l.i.f. $\mathcal{L}$. Therefore

$$\alpha^2 - \alpha - \frac{7}{4} - \frac{1}{2}C \leq 0.$$

Solution of this quadratic inequality gives us the desired estimation of order α in a.l.i.f. $\mathcal{L}$. □

Note that relations (12) and (13) allow us to obtain the sharp estimation of $|b_3|$ in the family $\mathcal{L}^0$ with specified order $\alpha_0 \geq 1$. Really, if $f \in \mathcal{L}^0$ with dilatation ω then coefficients of f satisfy the conditions

$$|b_3| = \left| \frac{\omega''(0) + 8a_2b_2}{6} \right| \leq \frac{1 - 4|b_2|^2 + 4|a_2||b_2|}{3}.$$

Taking the maximum of the last ratio over all $|b_2| \leq 1/2$ and $|a_2| \leq$ $\leq \alpha_0$, $\alpha_0 \geq 1$, we obtain

$$|b_3| \leq \frac{1 - 4|b_2|^2 + 4\alpha_0|b_2|}{3} \leq \frac{2}{3}\alpha_0.$$

Inequality is sharp. Equality occurs on functions (5) and (6) with $\alpha_0 = 5/2$ and $3/2$ for subfamilies $\mathcal{C}_H^0$ and $\mathcal{K}_H^0$ respectively.

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References

[1] Duren P. *Univalent functions.* N. Y.: Springer-Verlag, 1983, 395 p.

[2] Krauss W. *Über den Zusammenhang einiger Charakteristiken eines einfach zusammenhängenden Bereiches mit der Kreisabbildung.* Mitt. Math. Sem. Geissen, 1932, vol. 21, pp. 1–28.

[3] Nehari Z. *The Schwarzian derivatives and schlicht functions.* Bull. Amer. Math. Soc., 1949, vol. 55, no. 6, pp. 545–551.

[4] Ahlfors L., Weill G. *A uniqueness theorem for Beltrami equations.* Proc. Amer. Math. Soc., 1962, vol. 13, pp. 1975–1978.

[5] Chuaqui M., Duren P., Osgood B. *The Schwarzian derivative for harmonic mappings.* J. Anal. Math., 2003, vol. 91, pp. 329–351. DOI: 10.1007/BF02788793.

[6] Hernández R., Martín M. J. *Pre-Schwarzian and Schwarzian Derivatives of Harmonic Mappings.* J. Geom. Anal., 2015, vol. 25 (1), pp. 64–91. DOI: 10.1007/s12220-013-9413-x.

[7] Sheil-Small T. *Constants for planar harmonic mappings.* J. Lond. Math. Soc., 1990, vol. s2-42, pp. 237–248. DOI: 10.1112/jlms/s2-42.2.237.

[8] Pommerenke Ch. *Linear-invariante Familien analytischer Functionen. I.* Math. Ann., 1964, vol. 155 (2), pp. 108–154. DOI: 10.1007/BF01344077.

[9] Schaubroeck L. E. *Subordination of planar harmonic functions.* Complex Var. Elliptic Equ., 2000, vol. 41, pp. 163–178.

[10] Sobczak-Knec M., Starkov V. V., Szynal J. *Old and new order of linear invariant family of harmonic mappings and the bound for Jacobian.* Ann. Univ. Mariae Curie-Sklodowska. Sect. A., 2011, vol. 65 (2), pp. 191–202.

[11] Graf S. Yu., Eyelangoli O. R. *Differential inequalities in linear- and affine-invariant families of harmonic mappings.* Russian Math. (Iz. VUZ), 2010, vol. 55, no. 10, pp. 60–62. DOI: 10.3103/S1066369X10100075.

[12] Starkov V. V. *Application of the linear invariance idea in the theory of harmonic mappings. New order.* Materials of the 12th Saratov conference "Modern problems of the function theory and their applications", 2004, p. 173 (in Russian).

[13] Graf S. Yu. *An exact bound for the Jacobian in linear and affine invariant families of harmonic mappings.* Tr. Petrozavodsk. Univ. Ser. Matem., 2008, vol. 14, pp. 31–38 (in Russian).

[14] Ganenkova E.G., Starkov V. V. *Regularity theorems for harmonic functions.* J. Appl. Anal., 2015, vol. 21 (1), pp. 25–36. DOI: 10.1515/jaa-2015-0003.

[15] Graf S. Yu. *To the theory of linear and affine invariant families of harmonic mappings.* Application of functional analysis in the approximation theory (Primenenie functional'nogo analiza v teorii priblizhenij), 2012, issue 33, pp. 12–40 (in Russian).

[16] Clunie J., Sheil-Small T. *Harmonic univalent functions.* Ann. Acad. Sci. Fenn. Math., 1984, vol. 9, pp. 3–25.

[17] Wang X.-T., Liang X.-O., ZhangY.-L. *Precise coefficient estimates for close-to-convex harmonic univalent mappings.* J. Math. Anal. Appl., 2001, vol. 263 (2), pp. 501–509. DOI:10.1006/jmaa.2001.7626.

[18] Golusin G. M. *Geometrical function theory.* Moscow, 1966, 628 p. (in Russian).

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