We consider Orlicz spaces of differential forms on a Riemannian manifold. A Riesz-type theorem about the functionals on Orlicz spaces of forms is proved and other duality theorems are obtained therefrom. We also extend the results on the Hölder-Poincaré duality for reduced $L_{q,p}$-cohomology by Gol’dshtein and Troyanov to $L_{\Phi_I,\Phi_{II}}$-cohomology, where $\Phi_I$ and $\Phi_{II}$ are $N$-functions of class $\Delta_2 \cap \nabla_2$.

Key words: Riemannian manifold, differential form, exterior differential, Orlicz space, Orlicz cohomology

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Introduction. This article is devoted to the study of the dual spaces of Orlicz spaces of differential forms on an oriented Riemannian manifold $X$.

$L_p$-theory of differential forms on Riemannian manifolds has been the subject of many papers and several books since the beginning of the 1980s. In 1976, Atiyah defined $L_2$-cohomology for a Riemannian manifold and initiated various applications of $L_2$-methods to the study of noncompact manifolds and quotient spaces of Riemannian manifolds by discrete groups of isometries. The $L_2$-cohomology of such manifolds was studied by Gromov, Cheeger–Gromov and others (see, for example, [2] [3] [12]). In the 1980’s, Goldshtein, Kuz’minov, and Shvedov defined the $L_p$-de Rham complex on a Riemannian manifold $M$ for arbitrary $p \in [1, \infty]$ and began to investigate its cohomology, which they called the $L_p$-cohomology of $M$; they obtained many results concerning the density of smooth forms in $L_p$ (see, for example, [5]); the nontriviality and the Hausdorff property of $L_p$-cohomology on important classes of manifolds (see, for instance, [7] [8] [17]),
duality for $L_p$-related spaces of differential forms and the induced duality for $L_p$-cohomology in [6]; compactly-supported approximation of $L_p$-forms (see, for example, [16]). In studying the asymptotic invariants of infinite groups and manifolds with pinched negative curvature, Gromov and Pansu also considered $L_p$-differential forms and $l_p$-simplicial cochains (see [12] [18] [19]). Gol’dshtein and Troyanov obtained deep results about the $L_{qp}$-cohomology of Riemannian manifolds for $q \neq p$ in [9] [10] [11].

Like Orlicz function spaces, the Orlicz spaces $L^\Phi$ of differential forms are a natural nonlinear generalization of the spaces $L^p$. Orlicz spaces of differential forms on domains in $\mathbb{R}^n$ were first considered by Iwaniec and Martin in [13] and then by Agarwal, Ding, and Nolder in [1] (see also [4] [14]). In [13], Iwaniec and Martin established a Riesz-type theorem for an Orlicz space of differential forms on a domain in $\mathbb{R}^n$. Orlicz spaces of differential forms on a Riemannian manifold were apparently first examined by Panenko and the author in [15], where de Rham regularization operators were introduced and studied for Orlicz spaces of differential forms.

We prove a Riesz-type theorem for Orlicz spaces of differential forms on a Riemannian manifold and then, using it, describe the dual spaces of Orlicz–Sobolev-type spaces of differential forms, thus generalizing the results of Gol’dshtein, Kuz’minov, and Shvedov obtained in [6] for $L^p$-related spaces. The so-obtained results are applied for establishing the Hölder–Poincaré duality for the reduced Orlicz cohomology of $X$, which extends the Hölder–Poincaré duality for $L_{q,p}$-cohomology proved by Gol’dshtein and Troyanov in [11].

The structure of the article is as follows: In Section 1, we recall the main notions and necessary properties of Orlicz function spaces. In Section 2, we give the definition of Orlicz spaces of differential forms on a Riemannian manifold. The Riesz-type theorem for Orlicz spaces of differential forms (Theorem 3.1) is the contents of Section 3. Then, in Section 4, we examine the structure of the dual spaces to some $L^\Phi$-related spaces of differential forms. Finally, in Section 5, we establish a theorem on the Poincaré duality for the $L_{\Phi_1,\Phi_{II}}$-cohomology of an oriented Riemannian manifold (Theorem 5.8).

1. $N$-functions and Orlicz function spaces.

**Definition 1.1.**

A function $\Phi : \mathbb{R} \to \mathbb{R}$ is called an $N$-function if

1. $\Phi$ is even and convex;
(ii) $\Phi(x) = 0 \iff x = 0$;

(iii) $\lim_{x \to 0} \frac{\Phi(x)}{x} = 0$; $\lim_{x \to \infty} \frac{\Phi(x)}{x} = \infty$.

An $N$-function $\Phi$ has left and right derivatives (which can differ only on an at most countable set, see, for instance, [20, Theorem 1, p. 7]). The left derivative $\phi$ of $\Phi$ is left continuous, nondecreasing on $(0, \infty)$, and such that $0 < \phi(t) < \infty$ for $t > 0$, $\phi(0) = 0$, $\lim_{t \to \infty} \phi(t) = \infty$. The function

$$\psi(s) = \inf\{t > 0 : \phi(t) > s\}, \quad s > 0,$$

is called the left inverse of $\phi$.

The functions $\Phi, \Psi$ given by

$$\Phi(x) = \int_0^{|x|} \varphi(t) dt, \quad \Psi(x) = \int_0^{|x|} \psi(t) dt$$

are called complementary $N$-functions.

The $N$-function $\Psi$ complementary to an $N$-function $\Phi$ can also be expressed as

$$\Psi(y) = \sup\{x | y| - \Phi(x) : x \geq 0\}, \quad y \in \mathbb{R}.$$  

$N$-functions are classified in accordance with their growth rates as follows:

**Definition 1.2.** An $N$-function $\Phi$ is said to satisfy the $\Delta_2$-condition for large $x$ (for small $x$, for all $x$), which is written as $\Phi \in \Delta_2(\infty)$ ($\Phi \in \Delta_2(0)$, or $\Phi \in \Delta_2$), if there exist constants $x_0 > 0$, $K > 2$ such that $\Phi(2x) \leq K\Phi(x)$ for $x \geq x_0$ (for $0 \leq x \leq x_0$, or for all $x \geq 0$); and it satisfies the $\nabla_2$-condition for large $x$ (for small $x$, for all $x$), which is denoted symbolically as $\Phi \in \nabla_2(\infty)$ ($\Phi \in \nabla_2(0)$, or $\Phi \in \nabla_2$) if there are constants $x_0 > 0$ and $c > 1$ such that $\Phi(x) \leq \frac{1}{2c}\Phi(cx)$ for $x \geq x_0$ (for $0 \leq x \leq x_0$, or for all $x \geq 0$).

Henceforth, let $\Phi$ be an $N$-function and let $(\Omega, \Sigma, \mu)$ be a measure space.

**Definition 1.3.** The set $\tilde{L}^\Phi = \tilde{L}^\Phi(\Omega) = \tilde{L}^\Phi(\Omega, \Sigma, \mu)$ is defined to be the set of measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\rho_\Phi(f) := \int_\Omega \Phi(f) d\mu < \infty.$$
Definition 1.4. The linear space

\[ L^\Phi = L^\Phi(\Omega) = L^\Phi(\Omega, \Sigma, \mu) = \{ f : \Omega \to \mathbb{R} \text{ measurable} : \rho_\Phi(af) < \infty \text{ for some } a > 0 \} \]

is called an Orlicz space on \((\Omega, \Sigma, \mu)\).

The corresponding Morse–Transue space is the space

\[ M^\Phi = M^\Phi(\Omega) = M^\Phi(\Omega, \Sigma, \mu) = \{ f : \Omega \to \mathbb{R} \text{ measurable} : \rho_\Phi(af) < \infty \text{ for all } a > 0 \}. \]

For an Orlicz space \(L^\Phi = L^\Phi(\Omega, \Sigma, \mu)\), the \(N\)-function \(\Phi\) is called \(\Delta_2\)-regular if \(\Phi \in \Delta_2(\infty)\) when \(\mu(\Omega) < \infty\) or \(\Phi \in \Delta_2\) when \(\mu(\Omega) = \infty\) or \(\Phi \in \Delta_2(0)\) for \(\mu\) the counting measure on countable \(\Omega\).

Let \(\Psi\) be the complementary \(N\)-function to \(\Phi\).

Below we as usual identify two functions equal outside a set of measure zero.

If \(f \in L^\Phi\) then the functional \(\| \cdot \|_\Phi\) (called the Orlicz norm) defined by

\[ \|f\|_\Phi = \|f\|_{L^\Phi(\Omega)} = \sup \left\{ \left| \int_\Omega fg \, d\mu \right| : \rho_\Psi(g) \leq 1 \right\} \]

is a seminorm. It becomes a norm if \(\mu\) satisfies the finite subset property (see [20, p. 59]): if \(A \in \Sigma\) and \(\mu(A) > 0\) then there exists \(B \in \Sigma, B \subset A\), such that \(0 < \mu(B) < \infty\).

The equivalent gauge (or Luxemburg) norm of a function \(f \in L^\Phi\) is defined by the formula

\[ \|f\|_{\Phi} = \|f\|_{L^\Phi(\Omega)} = \inf \left\{ k > 0 : \rho_\Phi\left( \frac{f}{k} \right) \leq 1 \right\}. \]

This is a norm without any constraint on the measure \(\mu\) (see [20, p. 54, Theorem 3]).

We will need the following familiar assertion (see [20, item (ii), p. 57]):

Lemma 1.5. Let

\[ 0 \leq f_1 \leq f_2 \leq \cdots \leq f_m \leq \cdots \]
be an increasing sequence of nonnegative measurable functions in the Orlicz space $L^\Phi(\Omega)$ ($\Omega, \Sigma, \mu$ is a measure space) and let $f_m \to f$ a.e. Then $\lim_{m \to \infty} \|f_m\|_{(\Phi)} \leq \|f\|_{(\Phi)} \leq \infty$.

2. Orlicz spaces of differential forms. Let $X$ be a Riemannian manifold of dimension $n$. Given $x \in X$, denote by $(\omega(x), \theta(x))$ the scalar product of exterior $k$-forms $\omega(x)$ and $\theta(x)$ on $T_xX$. This gives a function $x \mapsto (\omega(x), \theta(x))$ on $X$.

Let $\Phi : \mathbb{R} \to \mathbb{R}$ and $\Psi : \mathbb{R} \to \mathbb{R}$ be two complementary $N$-functions. Denote by $\tilde{L}^\Phi(X, \Lambda^k)$ the class of all measurable $k$-forms $\omega$ such that

$$\rho_\Phi(\omega) := \int_X \Phi(|\omega(x)|)d\mu_X < \infty.$$ 

Here $d\mu_X$ stands for the volume element of the Riemannian manifold $X$. We will identify $k$-forms differing on a set of measure zero.

Given a (not necessarily orientable) Riemannian manifold $X$, introduce the space $L^\Phi(X, \Lambda^k)$ as the class of all measurable $k$-forms $\omega$ satisfying the condition

$$\rho_\Phi(\alpha \omega) < \infty \text{ for some } \alpha > 0.$$ 

The corresponding Morse–Transue space $M^\Phi(X, \Lambda^k)$ is defined as the class of all measurable $k$-forms $\omega$ such that

$$\rho_\Phi(\alpha \omega) < \infty \text{ for all } \alpha > 0.$$ 

Obviously, $\tilde{L}^\Phi(X, \Lambda^k) \subset L^\Phi(X, \Lambda^k)$.

As in the case of Orlicz function spaces, the space $L^\Phi(X, \Lambda^k)$ is endowed with two equivalent norms: the gauge norm

$$\|\omega\|_{(\Phi)} = \inf \left\{ K > 0 : \rho_\Phi\left(\frac{\omega}{K}\right) \leq 1 \right\}$$

and the Orlicz norm

$$\|\omega\|_\Phi = \sup \left\{ \left| \int_X (\omega(x), \theta(x))d\mu_X \right| : \theta \in \tilde{L}^\Psi(X, \Lambda^k), \rho_\Psi(\theta) \leq 1 \right\}.$$ 

As in the case of function spaces, it can be proved that $L^\Phi(X, \Lambda^k)$ endowed with one of these norms is a Banach space.

Obviously, the gauge norm of a $k$-form $\omega$ is nothing but the gauge norm of its modulus function $|\omega|$. The same holds for the Orlicz norm
Moreover, similarly to the case of Orlicz function spaces \([20, \text{Proposition 10, p. 81}]\), we have

**Lemma 2.1.** The Orlicz and gauge norms of a \(k\)-form \(\omega \in L^\Phi(X, \Lambda^k)\) can be calculated by the formulas

\[
\|\omega\|_\Phi = S_\omega := \sup_{\theta \in M^\Psi(X, \Lambda^k), \|\theta\|_\Psi \leq 1} \left| \int_X (\omega(x), \theta(x))d\mu_X \right|
\]

and

\[
\|\omega\|_{\Phi} = T_\omega := \sup_{\theta \in M^\Psi(X, \Lambda^k), \|\theta\|_\Psi \leq 1} \left| \int_X (\omega(x), \theta(x))d\mu_X \right|.
\]

**Proof.** For \(\theta \in M^\Psi(X, \Lambda^k)\) with \(\|\theta\|_\Psi \leq 1\) we have

\[
\left| \int_X (\omega(x), \theta(x))d\mu_X \right| \leq \int_X |\omega(x)||\theta(x)|d\mu_X \leq \\
\leq \sup_{g \in M^\Psi(X), \|g\|_\Psi \leq 1} \left| \int_X |\omega(x)||g(x)|d\mu_X \right| = \|\omega\|_\Phi.
\]

The last equality here holds by \([20, \text{Proposition 10, p. 81}]\).

Thus,

\[
S_\omega = \sup_{\theta \in M^\Psi(X, \Lambda^k), \|\theta\|_\Psi \leq 1} \left| \int_X (\omega(x), \theta(x))d\mu_X \right| \leq \|\omega\|_\Phi.
\]

On the other hand, let \((g_m)_{m \in \mathbb{N}}\) be a sequence of functions in \(M^\Psi(X)\) with \(\|g_m\|_\Psi \leq 1\) such that

\[
\left| \int_X |\omega(x)||g_m(x)|d\mu_X \right| \to \|\omega\|_\Phi \text{ as } m \to \infty.
\]

Since

\[
\left| \int_X |\omega(x)||g_m(x)|d\mu_X \right| \leq \int_X |\omega(x)||g_m(x)|d\mu_X \leq \|\omega\|_\Phi,
\]

...
we also have
\[
\int_X |\omega(x)||g_m(x)| d\mu_x \to \|\omega\|_\Phi \text{ as } m \to \infty.
\]

Consider the sequence \((\theta_m)_{m \in \mathbb{N}}\) of \(k\)-forms \(\theta_m\) defined by
\[
\theta_m(x) = \begin{cases} 
|g_m(x)| \frac{\omega(x)}{\omega(x)} & \text{if } \omega(x) \neq 0, \\
0 & \text{otherwise}.
\end{cases}
\]
Then \(\|\theta_m\|_{(\Psi)} = \|g_m\| \leq 1\) and
\[
\left| \int_X (\omega(x), \theta_m(x)) d\mu_X \right| = \int_X |\omega(x)||g_m(x)| d\mu_X \to \|\omega\|_\Phi
\]
as \(m \to \infty\). Therefore,
\[
\|\omega\|_\Phi \leq \sup_{\theta \in M^\Psi_{(\Psi, \Lambda^k), \|\theta\|_{(\Psi)} \leq 1}} \left| \int_X (\omega(x), \theta(x)) d\mu_X \right| = \|\omega\|_\Phi.
\]
Thus, we get the desired equality for the Orlicz norm.

For the gauge norm, the equality \(\|\omega\|_\Phi = \|\|\omega\|\|_{(\Phi)}\) is obvious, and one must only prove that
\[
T_\omega = \|\omega\|_{(\Phi)},
\]
which is done in the same manner as for the Orlicz norm with the use of [20, Proposition 10, p. 81]. □

Below, when this does not lead to confusion, we use the abbreviations
\[
L^\Phi = (L^\Phi, \| \cdot \|_\Phi), \quad L^{(\Phi)} = (L^\Phi, \| \cdot \|_{(\Phi)});
\]
\[
M^\Phi = (M^\Phi, \| \cdot \|_\Phi), \quad M^{(\Phi)} = (M^\Phi, \| \cdot \|_{(\Phi)}).
\]

3. The Riesz theorem. Let \(X\) be an oriented \(n\)-dimensional Riemannian manifold.

For a \(k\)-form \(\omega\) on \(X\), let \(*\omega\) be the Hodge dual of \(\omega\) (an \((n-k)\)-form).
The bilinear function
\[
\langle \omega, \theta \rangle = \int_X \omega \wedge \theta
\]  
(1)
defines a pairing between \( L^\Phi(X, \Lambda^k) \) and \( L^{(\Psi)}(X, \Lambda^k) \) (and between \( L^{(\Phi)}(X, \Lambda^k) \) and \( L^{\Psi}(X, \Lambda^k) \)). The integral on the right-hand side of (1) exists because
\[
\omega \wedge \theta = (-1)^{kn-k} \omega \ast \theta \, d\mu_X,
\]


\[(\omega \ast \theta)_X \leq |\omega|_X \ast |\theta|_X = |\omega|_X |\theta|_X.
\]

Hence, we obtain two versions of the H"older inequality:

\[
|\langle \omega, \theta \rangle| \leq \|\omega\|_\Phi \|\theta\|_{(\Psi)}
\]  
(2)
and

\[
|\langle \omega, \theta \rangle| \leq \|\omega\|_{(\Phi)} \|\theta\|_\Psi.
\]  
(3)

Assign to each form \( \theta \in L^{(\Psi)}(X, \Lambda^{n-k}) \) the functional

\[
F_\theta(\omega) = \int_X \omega \wedge \theta.
\]  
(4)

By (2) and (3), we have

\[
|F_\theta(\omega)| \leq \|\omega\|_\Phi \|\theta\|_{(\Psi)}; \quad |F_\theta(\omega)| \leq \|\omega\|_{(\Phi)} \|\theta\|_\Psi.
\]  
(5)

**Theorem 3.1.** If \( \Phi \) is an \( N \)-function then the correspondence \( \theta \mapsto F_\theta \) yields isometric isomorphisms

\[
L^{(\Psi)}(X, \Lambda^{n-k}) \xrightarrow{\sim} (M^{\Phi}(X, \Lambda^k))'; \quad L^{\Psi}(X, \Lambda^{n-k}) \xrightarrow{\sim} (M^{(\Phi)}(X, \Lambda^k))'.
\]

**Proof.** Let us prove the first isomorphism.

By (5), \( \|F_\theta\| \leq \|\theta\|_{(\Psi)} \). Show that an arbitrary continuous functional \( F \in (M^{\Phi}(X, \Lambda^k))' \) is representable uniquely in the form (4). Let \( h : V \to \mathbb{R}^n, V \subset X \) be a local chart of \( X \) and let \( U \) be an open set with compact closure \( \text{cl}_X U \subset V \); then \( U \) is endowed with two metrics: the metric \( \rho \) of the Riemannian manifold \( X \) and the metric \( \bar{\rho} \) induced by \( h \) from the standard metric on \( \mathbb{R}^n \). It is not hard to see that the \( L^{\Phi} \)-spaces (\( M^{\Phi} \)-spaces) of \( k \)-forms on \( U \) \( L^{\Phi}(U, \Lambda^k, \rho) \) and \( L^{(\Phi)}(U, \Lambda^k, \rho) \)

(M^\Phi(U, \Lambda^k, \rho) and M^{(\Phi)}(U, \Lambda^k, \rho)) corresponding to these metrics coincide and have equivalent norms. Making use of the Riesz theorem on the general form of a linear functional on the function space M^\Phi, we, involving the coordinate representation of differential forms, conclude that every functional f \in (M^\Phi(U, \Lambda^k, \bar{\rho}))' is uniquely representable in the form

\[ f(\alpha) = \int_X \alpha \wedge \theta_f, \quad \theta_f \in L^{(\Psi)}(U, \Lambda^{n-k}, \bar{\rho}). \]

By the equivalence of the norms in M^\Phi(U, \Lambda^k, \rho) and M^\Phi(U, \Lambda^k, \bar{\rho}), the same holds for functionals in M^\Phi(U, \Lambda^k, \rho). Therefore, for F \in (M^\Phi(X, \Lambda^k))' and an open set U with compact closure, there is a unique form \theta_U \in L^{(\Psi)}(U, \Lambda^{n-k}) such that

\[ F(\omega) = \int_U \omega \wedge \theta_U \quad \text{for every } \omega \in M^\Phi(U, \Lambda^k). \]

Given two sets U_1 and U_2 as above, the forms \theta_{U_1} and \theta_{U_2} coincide on U_1 \cap U_2 by the uniqueness of \theta_{U_1 \cap U_2}. Thus, all forms \theta_U defined for different U agree with each other and thus define an (n - k)-form \theta on X. The form \theta belongs to L^{(\Psi)}(X, \Lambda^{n-k}) locally, satisfies the condition

\[ F(\omega) = \int_X \omega \wedge \theta \quad \text{for all } \omega \in M^\Phi(X, \Lambda^k) \]

and is defined by this condition uniquely.

Consider a compact set Y \subset X. Let g \in M^\Phi(X) be a function with compact support contained in Y having \|g\|_\Phi \leq 1. Let \beta_g be the k-form on X defined by the formula

\[ \beta_g(x) = \begin{cases} (-1)^{k(n-k)} \frac{g(x)}{|\theta(x)|} (\ast \theta(x)) & \text{if } x \in Y \text{ and } \theta(x) \neq 0; \\ 0 & \text{otherwise}. \end{cases} \]

We have

\[ F(\beta_g) = \int_Y \beta_g \wedge \theta = (-1)^{k(n-k)} \int_Y \frac{g(x)}{|\theta(x)|} (\ast \theta(x)) \wedge \theta(x) = \int_Y g(x) |\theta(x)| d\mu_X. \]
Since $\|g\|_\Phi \leq 1$, this gives

$$\left| \int_Y g(x)|\theta(x)|d\mu_X \right| = |F(\beta_g)| \leq \|F\|.$$ 

Hence, using Lemma 2.1, we obtain

$$\|\theta|_Y\|_{(\Psi)} = \|\theta|_Y\|_{(\Psi)} = \sup_{g \in M^\Phi(Y); \|g\|_\Phi \leq 1} \left| \int_Y g(x)|\theta(x)|d\mu_X \right| \leq \|F\|.$$ 

Let $Y_1 \subset Y_2 \subset \cdots \subset Y_m \subset \cdots \subset X$ be an exhaustion of $X$ by compact sets and let $\theta_m$ be the restriction of $\theta$ to $Y_m$. Put $f_m = |\theta_m|$. Then the sequence $\{f_m\}_{m \in \mathbb{N}}$ satisfies the conditions of Lemma 1.5. Since $\|f_m\|_{(\Psi)} \leq \|F\|$, the function $\lim_{m \to \infty} f_m = |\theta|$ lies in $L^{(\Psi)}(X)$, and so $\theta \in L^{(\Psi)}(X, \Lambda^{n-k})$ and

$$\|\theta\|_{(\Psi)} = \lim_{m \to \infty} \|\theta_m\|_{(\Psi)} \leq \|F\|. \quad (6)$$

The functionals $F$ and $F_\theta$ coincide on the set of forms in $M^\Phi(X, \Lambda^k)$ having compact support, which is, as in the case of Orlicz function spaces, dense in $M^\Phi(X, \Lambda^k)$. Thus,

$$F(\omega) = \omega \wedge \theta$$

for all $\omega \in M^\Phi(X, \Lambda^k)$. Combining (2) and (6), we infer that $\|F_\theta\| = \|\theta\|_{(\Psi)}$.

Let us now establish the second isomorphism

$$L^{\Psi}(X, \Lambda^{n-k}) \cong (M^{(\Phi)}(X, \Lambda^k))'.$$

Let $F \in (M^{(\Phi)}(X, \Lambda^k))'$. Then, as above, we see that there exists a unique $(n-k)$-form $\theta$ belonging to $L^{\Psi}$ locally that satisfies the condition

$$F(\omega) = \int_X \omega \wedge \theta \quad \text{for all } \omega \in M^{(\Phi)}(X, \Lambda^k) \text{ with compact support}.$$ 

Using Lemma 2.1, we verify in the same manner as for $\| \cdot \|_{\Psi}$ that, given any compact set $Y \subset X$,

$$\|\theta|_Y\|_\Psi \leq \|F\|.$$
Because of the inequalities
\[
\|\cdot\|_{(\Psi)} \leq \|\cdot\|_{\Psi} \leq 2\|\cdot\|_{(\Psi)},
\]
we have
\[
\|\theta|_{Y}\|_{(\Psi)} \leq \|F\|.
\]
Taking an exhaustion \(Y_1 \subset Y_2 \subset \cdots \subset Y_m \subset \cdots \subset X\) of \(X\) by compact sets, we as above conclude that \(\theta \in L^{\Psi}\).

Now, the functionals \(F\) and \(F_{\theta}\) coincide on the dense set of forms with compact support in \(M^{(\Phi)}(X, \Lambda^k)\) and hence on \(M^{(\Phi)}(X, \Lambda^k)\). By Lemma 2.1,
\[
\|F\| = \|F_{\theta}\| = \sup_{\|\theta\|_{(\Phi)} \leq 1} \left|\int_X \omega \wedge \theta\right| = \|\theta\|_{\Phi}.
\]

The theorem is completely proved. □

4. The dual spaces to \(L^{\Phi}\)-related spaces of differential forms. Throughout this section, \(X\) is an oriented smooth Riemannian manifold of dimension \(n\) and \((\Phi_1, \Psi_1)\) and \((\Phi_2, \Psi_2)\) are pairs of conjugate \(N\)-functions.

Introduce some spaces of differential forms. For \(A \in \{L, M\}\) and \(\langle \Phi_i \rangle \in \{\Phi_i, (\Phi_i)\}\), denote by \(A^{k}_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X)\) the space \(A^{\Phi_1}(X, \Lambda^k) \oplus A^{\Phi_2}(X, \Lambda^{k+1})\) with the norm
\[
\|(\alpha, \beta)\|_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle} = \|\alpha\|_{\langle \Phi_1 \rangle} + \|\beta\|_{\langle \Phi_2 \rangle}.
\]

Given \((\alpha, \beta) \in M^{k}_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X)\) and \((\omega, \theta) \in L^{n-k-1}_{\langle \Psi_2 \rangle, \langle \Psi_1 \rangle}(X)\), where
\[
\langle \Psi_i \rangle = \begin{cases} 
(\Psi_i) & \text{if } \langle \Phi_i \rangle = \Phi_i, \\
\Psi_i & \text{if } \langle \Phi_i \rangle = (\Phi_i),
\end{cases}
\]
we put
\[
\langle (\alpha, \beta), (\omega, \theta) \rangle = (-1)^k \langle \alpha, \theta \rangle + \langle \beta, \omega \rangle. \tag{7}
\]

Theorem 3.1 implies that the pairing (7) defines an isometric isomorphism
\[
(M^{k}_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X))' \cong L^{n-k-1}_{\langle \Psi_2 \rangle, \langle \Psi_1 \rangle}(X).
\]

Moreover,
\[
\|\langle (\alpha, \beta), (\omega, \theta) \rangle\| \leq \|\langle (\alpha, \beta)\|_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle} \cdot \|(\omega, \theta)\|_{\langle \Psi_2 \rangle, \langle \Psi_1 \rangle}.\]
A differential \((k + 1)\)-form \(\theta \in L^1_{\text{loc}}(X, \Lambda^{k+1})\) on \(X\) is called the weak exterior differential (or derivative) of a \(k\)-form \(\omega \in L^1_{\text{loc}}(X, \Lambda^k)\) (which is written as \(d\omega = \theta\)) if,

\[
\int_X \theta \wedge u = (-1)^{k+1} \int_X \omega \wedge du
\]

for any \(u \in \mathcal{D}^{n-k-1}(X)\), where \(\mathcal{D}^l(X)\) is the set of smooth \(l\)-forms on \(X\) with compact support included in \(\text{Int} X\).

Let \(\Phi_1\) and \(\Phi_2\) be \(N\)-functions. For \(0 \leq k \leq n\), put

\[
\Omega^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X) = \left\{ \omega \in L^{\langle \Phi_1 \rangle}(X, \Lambda^k) : d\omega \in L^{\langle \Phi_2 \rangle}(X, \Lambda^{k+1}) \right\}.
\]

This is a Banach space with the norm

\[
\|\omega\|_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle} = \|\omega\|_{\langle \Phi_1 \rangle} + \|d\omega\|_{\langle \Phi_2 \rangle}.
\]

From now on we assume that \(\Phi_1, \Phi_2 \in \Delta_2 \cap \nabla_2\), and hence also \(\Psi_1, \Psi_2 \in \Delta_2 \cap \nabla_2\).

If \(\Phi \in \Delta_2 \cap \nabla_2\) then, as is well known, the spaces \(L^\Phi\) and \(M^\Phi\) coincide and hence, by Theorem 3.1, the space \(L^\Phi\) is reflexive. Thus, there is no need in the spaces \(M^\Phi\). We will often assume that the space \(\Omega^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X)\) is embedded in \(L^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X)\) by identifying a form \(\alpha \in \Omega^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X)\) with the pair \((\alpha, d\alpha) \in L^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X)\).

Given a subspace \(H \subset L^k_{\langle \Phi_1, \Phi_2 \rangle}\), denote by \(H^\perp\) the annihilator of \(H\) in \(L^{n-k-1}_{\langle \Psi_2, \Psi_1 \rangle}(X)\) with respect to the pairing \([7]\). Since this pairing satisfies

\[
\langle(\alpha, \beta), (\omega, \theta)\rangle = (-1)^{k(n-k-1)}\langle(\omega, \theta), (\alpha, \beta)\rangle,
\]

there is no difference between the pairings between \(L^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X)\) and \(L^{n-k-1}_{\langle \Psi_2, \Psi_1 \rangle}(X)\) and between \(L^{n-k-1}_{\langle \Psi_2, \Psi_1 \rangle}(X)\) and \(L^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X)\).

The definition of \(\Omega^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X)\) implies that

\[
\Omega^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X) = (\mathcal{D}^{n-k-1}(X))^\perp.
\]

Put \(\Omega^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle, 0}(X) = (\Omega^{n-k-1}_{\langle \Psi_2, \Psi_1 \rangle}(X))^\perp\). Since \(\mathcal{D}^{n-k-1}(X) \subset \Omega^{n-k-1}_{\langle \Psi_2, \Psi_1 \rangle}(X)\), we have \(\Omega^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle, 0}(X) \subset \Omega^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X)\).
Observe that if \( \Omega^k_{\langle \Phi_1, \Phi_2 \rangle, 0}(X) = \Omega^k_{\langle \Phi_1, \Phi_2 \rangle}(X) \) then
\[
\Omega^{n-k-1}_{\langle \Psi_2, \langle \Phi_1 \rangle \rangle, 0}(X) = \Omega^{n-k-1}_{\langle \Psi_2, \langle \Phi_1 \rangle \rangle}(X).
\]

**Lemma 4.1.** The following hold for \( \Phi_1, \Phi_2 \in \Delta_2 \cap \nabla_2 \):

1. Smooth forms constitute a dense set in \( \Omega^k_{\langle \Phi_1, \Phi_2 \rangle}(X) \).
2. Smooth forms with compact support constitute a dense set in \( \Omega^k_{\langle \Phi_1, \Phi_2 \rangle, 0}(X) \).

**Proof.** Item (1) stems from the only theorem of [15] about the properties of the de Rham regularization operators in Orlicz spaces of differential forms. Prove (2). Denote the closure of \( D^k(X) \) in \( L_{\langle \Phi_1, \Phi_2 \rangle}(X) \) by \( D^k_{\langle \Phi_1, \Phi_2 \rangle} \).

Then, by [21, Theorem 4.7],
\[
\overline{D^k_{\langle \Phi_1, \Phi_2 \rangle}} = \left( \overline{(D^k)_{\langle \Phi_1, \Phi_2 \rangle}} \right) = (\Omega^{k}_{\langle \Psi_2, \langle \Phi_1 \rangle \rangle}(X)) = \Omega^k_{\langle \Phi_1, \Phi_2 \rangle, 0}(X).
\]

□

**Lemma 4.2.** If \( \Phi_1, \Phi_2 \in \Delta_2 \cap \nabla_2 \) and a form \( \omega \in \Omega^k_{\langle \Phi_1, \Phi_2 \rangle}(X) \) has compact support then \( \omega \in \Omega^k_{\langle \Phi_1, \Phi_2 \rangle, 0}(X) \).

**Proof.** Suppose that \( \omega \in \Omega^k_{\langle \Phi_1, \Phi_2 \rangle}(X) \) has compact support. Assume first that \( \theta \) is a smooth \((n-k-1)\)-form. By Lemma 4.1, there exists a sequence \( \{\omega_j\} \) of smooth forms with compact support such that \( \omega_j \to \omega \) in norm as \( j \to \infty \). Then
\[
\langle (\omega, d\omega), (\theta, d\omega) \rangle = \lim_{j \to \infty} \langle (\omega_j, d\omega_j), (\theta, d\theta) \rangle =
\]
\[
= \lim_{j \to \infty} \int_X \left[ (-1)^k \omega_j \wedge d\theta + d\omega_j \wedge \theta \right] = \lim_{j \to \infty} d(\omega_j \wedge \theta) = 0. \tag{8}
\]
The last equality in (8) is due to the Stokes theorem. Now, let \( \theta \) be an arbitrary form in \( \Omega^{n-k-1}_{\langle \Psi_2, \langle \Phi_1 \rangle \rangle}(X) \). By Lemma 4.1, there is a sequence \( \{\theta_j\} \) of smooth forms converging to \( \theta \) in norm as \( j \to \infty \). Then
\[
\langle (\omega, d\omega), (\theta, d\omega) \rangle = \lim_{j \to \infty} \langle (\omega, d\omega), (\theta_j, d\theta_j) \rangle = 0.
\]

Thus, \( \theta \in \Omega^k_{\langle \Phi_1, \Phi_2 \rangle, 0}(X) \). □

Each pair of forms \( (\omega, \theta) \in L^{n-k}_{\langle \Psi_2, \langle \Phi_1 \rangle \rangle}(X) \) defines by (7) a continuous linear functional on \( L^k_{\langle \Phi_1, \Phi_2 \rangle}(X) \) and hence on \( \Omega^k_{\langle \Phi_1, \Phi_2 \rangle}(X) \) and
Theorem 4.3. If \( \Phi_1, \Phi_2 \in \Delta_2 \cap \nabla_2 \) and \( \Psi_1, \Psi_2 \) are the corresponding complementary functions then any continuous linear functional on \( \Omega_{(\Phi_1),(\Phi_2),0}^k(X) \) (on \( \Omega_{(\Phi_1),(\Phi_2),0}^k(X) \)) can be represented in the form (9). A pair of forms \((\omega, \theta)\) defines the zero functional on \( \Omega_{(\Phi_1),(\Phi_2),0}^k(X) \) (on \( \Omega_{(\Phi_1),(\Phi_2),0}^k(X) \)) if and only if \( \omega \in \Omega_{(\Psi_2),(\Psi_1),0}^{n-k-1}(X) \) and \( \theta = d\omega \). The norm of the functional (9) on \( \Omega_{(\Phi_1),(\Phi_2),0}^k(X) \) (on \( \Omega_{(\Phi_1),(\Phi_2),0}^k(X) \)) has the form

\[
\|F\| = \inf \left\{ \|\theta + d\beta\|_{(\Psi_1)} + \|\omega + \beta\|_{(\Psi_2)} : \beta \in \Omega_{(\Psi_2),(\Psi_1),0}^{n-k-1}(X) \right\}
\]

\[
\|F\| = \inf \left\{ \|\theta + d\beta\|_{(\Psi_1)} + \|\omega + \beta\|_{(\Psi_2)} : \beta \in \Omega_{(\Psi_2),(\Psi_1),0}^{n-k-1}(X) \right\}.
\]

Proof. In accordance with [21, Theorem 4.9], if \( H \) is a closed subspace in a Banach space \( Y \) then \( Y'/H^\perp = H' \), where the isomorphism is induced by the canonical pairing between \( Y \) and \( Y' \). Therefore,

\[
\left( \Omega_{(\Phi_1),(\Phi_2),0}^k(X) \right)' = L_{n-k-1}^{(\Psi_2),(\Psi_1)}(X) \big/ \left( \Omega_{(\Phi_1),(\Phi_2),0}^k(X) \right) = L_{n-k-1}^{(\Psi_2),(\Psi_1)}(X) \big/ \Omega_{(\Psi_2),(\Psi_1),0}^{n-k-1}(X);
\]

\[
\left( \Omega_{(\Phi_1),(\Phi_2),0}^k(X) \right)' = L_{n-k-1}^{(\Psi_2),(\Psi_1)}(X) \big/ \left( \Omega_{(\Phi_1),(\Phi_2),0}^k(X) \right) = L_{n-k-1}^{(\Psi_2),(\Psi_1)}(X) \big/ \Omega_{(\Psi_2),(\Psi_1),0}^{n-k-1}(X).
\]

\( \square \)

Theorem 4.4. If \( \Phi_1, \Phi_2 \in \Delta_2 \cap \nabla_2 \) and \( \Psi_1, \Psi_2 \) are their complementary \( N \)-functions then the dual of the space \( \Omega_{(\Phi_1),(\Phi_2),0}^k(X) \) is isomorphic to the completion of \( D_{n-k}^k(X) \) with respect to the norm

\[
\|\omega\| = \inf \left\{ \|\omega + d\theta\|_{(\Psi_1)} + \|\theta\|_{(\Psi_2)} : \theta \in D_{n-k}^k(X) \right\}.
\]

(10)
This isomorphism is given by the action
\[ \langle \alpha, \omega \rangle = (-1)^k \int_X \alpha \wedge \omega. \] (11)

**Proof.** Consider the embedding \( j : L^{\Psi_1}(X, \Lambda^{n-k}) \to L^{n-k-1}_{\Psi_2,\Psi_1}(X) \) defined by \( j(\omega) = (0, \omega) \). Let
\[
\pi : L^{n-k-1}_{\Psi_2,\Psi_1}(X) \to L^{n-k-1}_{\Psi_2,\Psi_1}(X) / \Omega^{n-k-1}_{\Psi_2,\Psi_1,0}(X)
\]
be the canonical projection. It is not hard to see that \( \pi \circ j \) is a monomorphism. Since the set \( S = \{ (\omega, \theta) : \omega \in \mathcal{D}^{n-k-1}(X), \theta \in \mathcal{D}^{n-k}(X) \} \) is dense in \( L^{n-k-1}_{\Psi_2,\Psi_1}(X) \), \( \pi(S) \) is dense in \( L^{n-k-1}_{\Psi_2,\Psi_1}(X) / \Omega^{n-k-1}_{\Psi_2,\Psi_1,0}(X) \).

Let \( \omega \in \mathcal{D}^{n-k-1}(X), \theta \in \mathcal{D}^{n-k}(X) \). Since \( (\omega, d\omega) \in \Omega^{n-k-1}_{\Psi_2,\Psi_1,0}(X) \), we have \( \pi(\omega, \theta) = \pi(0, \theta - d\omega) = \pi \circ j(\theta - d\omega) \). Hence, the set \( \pi \circ j(\mathcal{D}^{n-k}) \) is dense in \( L^{n-k-1}_{\Psi_2,\Psi_1}(X) / \Omega^{n-k-1}_{\Psi_2,\Psi_1,0}(X) \). Moreover,
\[
\|\pi \circ j(\omega)\|_{L^{n-k-1}_{\Psi_2,\Psi_1}(X) / \Omega^{n-k-1}_{\Psi_2,\Psi_1,0}(X)} = \inf \left\{ \|\omega + d\theta\|_{\Psi_1} + \|\theta\|_{\Psi_2} : \theta \in \Omega^{n-k-1}_{\Psi_2,\Psi_1,0}(X) \right\}.
\]

By Lemma 4.1(2), the set \( \mathcal{D}^{n-k-1}(X) \) is dense in \( \Omega^{n-k-1}_{\Psi_2,\Psi_1,0}(X) \). Hence,
\[
\|\pi \circ j(\omega)\|_{L^{n-k-1}_{\Psi_2,\Psi_1}(X) / \Omega^{n-k-1}_{\Psi_2,\Psi_1,0}(X)} = \inf \left\{ \|\omega + d\theta\|_{\Psi_1} + \|\theta\|_{\Psi_2} : \theta \in \mathcal{D}^{n-k-1}(X) \right\}.
\]

Thus, the space \( L^{n-k-1}_{\Psi_2,\Psi_1}(X) / \Omega^{n-k-1}_{\Psi_2,\Psi_1,0}(X) \) is isomorphic to the completion of \( \mathcal{D}^{n-k}(X) \) with respect to the norm \( (10) \). Now, in view of [21, Theorem 4.9], if \( H \) is a closed subspace in a Banach space \( Y \) then \( (Y/H)' = H^\perp \), where the isomorphism is induced by the canonical pairing between \( Y \) and \( Y' \). Thus, \( \left( L^{n-k-1}_{\Psi_2,\Psi_1}(X) / \Omega^{n-k-1}_{\Psi_2,\Psi_1,0}(X) \right)' = \left( \Omega^{n-k-1}_{\Psi_2,\Psi_1,0}(X) \right)^{\perp} = \Omega^{k}_{\Psi_1}(X), \) and the first claim of the theorem is established.
Further, since

\[ \langle (\alpha, d\alpha), (0, \omega) \rangle = (-1)^k \int_X \alpha \wedge \omega, \]

the form \( \alpha \in \Omega^k_{\langle \Phi_1, \Phi_2 \rangle}(X) \) acts at the forms \( \pi \circ j(\omega), \omega \in D^{n-k}(X) \), by the formula

\[ \langle \alpha, \pi \circ j(\omega) \rangle = (-1)^k \int_X \alpha \wedge \omega. \]

The theorem is proved. \( \square \)

5. Hölder–Poincaré duality for \( L_{\Phi_I, \Phi_{II}} \)-cohomology. Let \( X \) be an oriented Riemannian manifold of dimension \( n \).

Given \( N \)-functions \( \Phi_I \) and \( \Phi_{II} \), consider the spaces

\[ Z^k_{\langle \Phi_{II} \rangle}(X) = \{ \omega \in L^k_{\langle \Phi_{II} \rangle}(X, \Lambda^k) : d\omega = 0 \}; \]

\[ B^k_{\langle \Phi_I, \Phi_{II} \rangle}(X) = \{ \omega \in L^k_{\langle \Phi_{II} \rangle}(X, \Lambda^k) : \omega = d\beta \text{ for some } \beta \in L^k_{\langle \Phi_I \rangle}(X, \Lambda^{k-1}) \}. \]

Denote by \( \overline{B}^k_{\langle \Phi_I, \Phi_{II} \rangle}(X) \) the closure of \( B^k_{\langle \Phi_I, \Phi_{II} \rangle}(X) \) in \( L^k_{\langle \Phi_{II} \rangle}(X, \Lambda^k) \). The quotient spaces

\[ H^k_{\langle \Phi_I, \Phi_{II} \rangle}(X) := Z^k_{\langle \Phi_{II} \rangle}(X)/B^k_{\langle \Phi_I, \Phi_{II} \rangle}(X) \]

and

\[ \overline{H}^k_{\langle \Phi_I, \Phi_{II} \rangle}(X) := Z^k_{\langle \Phi_{II} \rangle}(X)/\overline{B}^k_{\langle \Phi_I, \Phi_{II} \rangle}(X) \]

are called the \( k \)th \( L_{\langle \Phi_I, \Phi_{II} \rangle} \)-cohomology and the \( k \)th reduced \( L_{\langle \Phi_I, \Phi_{II} \rangle} \)-cohomology of the Riemannian manifold \( X \), the latter cohomology being a Banach space.

If \( \Phi_I = \Phi_{II} = \Phi \) then we use the notations \( \Omega^k_{\langle \Phi \rangle}(X), H^k_{\langle \Phi \rangle}(X), \) and \( \overline{H}^k_{\langle \Phi \rangle}(X) \) instead of \( \Omega^k_{\langle \Phi_I, \Phi_{II} \rangle}(X), H^k_{\langle \Phi_I, \Phi_{II} \rangle}(X), \) and \( \overline{H}^k_{\langle \Phi_I, \Phi_{II} \rangle}(X) \) respectively. Thus, the \( L_{\langle \Phi \rangle} \)-cohomology \( H^k_{\langle \Phi \rangle}(X) \) (respectively, the reduced \( L_{\langle \Phi \rangle} \)-cohomology \( \overline{H}^k_{\langle \Phi \rangle}(X) \)) is the \( k \)th cohomology (respectively, the \( k \)th reduced cohomology) of the cochain complex \( \{ \Omega^*_{\langle \Phi \rangle}(X), d \} \).
The \textit{kth interior reduced} $L_{(\Phi_I), (\Phi_{II})}$-cohomology of a Riemannian manifold $X$ is the Banach space

$$\overline{H}^k_{(\Phi_I), (\Phi_{II}), 0}(X) = \frac{Z^k_{(\Phi_I), (\Phi_{II}), 0}(X)}{d\mathcal{D}^{k-1}(X)},$$

where $d\mathcal{D}^{k-1}(X)$ is the closure of $d\mathcal{D}^k(X)$ in $L^{(\Phi_{II})}(X, \Lambda^k)$ and

$$Z^k_{(\Phi_I), (\Phi_{II}), 0}(X) = \text{Ker} \left\{ d : \Omega^k_{(\Phi_I), (\Phi_{II})} \rightarrow \Omega^{k+1}_{(\Phi_I), (\Phi_{II})} \right\} \cap \overline{d\mathcal{D}^k(X)}^{\Omega^k_{(\Phi_I), (\Phi_{II})}}.$$

Thus, a $k$-form $\theta$ belongs to $Z^k_{(\Phi_I), (\Phi_{II}), 0}(X)$ if and only if $\theta \in L^k_{(\Phi_I)}(X, \Lambda^k)$, $d\theta = 0$, and there is a sequence is a weakly closed forms $\theta_j \in \mathcal{D}^k(X)$ such that

$$\|\theta_j - \theta\|_{(\Phi_I)} \rightarrow 0 \text{ and } \|d\theta_j\|_{(\Phi_{II})} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

The quotient (semi)norm on each of the above-introduced cohomology spaces depends on the choice of the norm on $L^k_{\Phi_I}$ and $L^k_{\Phi_{II}}$ but the resulting topology does not.

From now on, we assume all $N$-functions under consideration to belong to $\Delta_2 \cap \nabla_2$.

In [11], Gol’dshtein and Troyanov realized the $k$th $L_{q,p}$-cohomology as the $k$th cohomology of some Banach complex. Here we apply this approach to $L_{(\Phi_I), (\Phi_{II})}$-cohomology.

Fix an $(n+1)$-tuple of $N$-functions $\mathcal{F} = \{\Phi_0, \Phi_1, \ldots, \Phi_n\}$ and put

$$\Omega^k_{\mathcal{F}}(X) = \Omega^k_{\Phi_k, \Phi_{k+1}}(X); \quad \Omega^k_{(\mathcal{F})}(X) = \Omega^k_{(\Phi_k), (\Phi_{k+1})}(X).$$

Use the unified notation $\Omega^k_{(\mathcal{F})}(X)$ for $\Omega^k_{\mathcal{F}}(X)$ and $\Omega^k_{(\mathcal{F})}(X)$. Since the weak exterior differential is a bounded operator $d : \Omega^k_{(\mathcal{F})}(X) \rightarrow \Omega^{k+1}_{(\mathcal{F})}(X)$, we obtain a Banach complex

$$0 \rightarrow \Omega^0_{(\mathcal{F})}(X) \rightarrow \Omega^1_{(\mathcal{F})}(X) \rightarrow \cdots \rightarrow \Omega^k_{(\mathcal{F})}(X) \rightarrow \cdots \rightarrow \Omega^n_{(\mathcal{F})}(X) \rightarrow 0.$$

The $L_{(\mathcal{F})}$-cohomology $H^k_{(\mathcal{F})}(X)$ (respectively, the \textit{reduced} $L_{(\mathcal{F})}$-cohomology $\overline{H}^k_{(\mathcal{F})}(X)$) of $X$ is the $k$th cohomology (respectively, the $k$th reduced cohomology) of the Banach complex $(\Omega^*_{(\mathcal{F})}, d)$. 
The above-defined cohomology spaces $H^k_{(\mathcal{F})}(X)$ and $\overline{H}^k_{(\mathcal{F})}(X)$ in fact depend only on $\Phi_{k-1}$ and $\Phi_k$:

$$H^k_{(\mathcal{F})}(X) = H^k_{(\Phi_{k-1}),\langle \Phi_k \rangle}(X) = Z^k_{(\Phi_k)}(X) / B^k_{(\Phi_{k-1}),\langle \Phi_k \rangle};$$

$$\overline{H}^k_{(\mathcal{F})}(X) = \overline{H}^k_{(\Phi_{k-1}),\langle \Phi_k \rangle}(X) = Z^k_{(\Phi_k)}(X) / \overline{B}^k_{(\Phi_{k-1}),\langle \Phi_k \rangle}.$$

Denote by $\Omega^k_{(\mathcal{F}),0}(X)$ the closure of $D^k(X)$ in $\Omega^k_{(\mathcal{F})}(X)$. The \textit{interior reduced $L_{(\mathcal{F})}$-cohomology} of $X$ is the reduced cohomology of the Banach complex

$$0 \to \Omega^0_{(\mathcal{F}),0}(X) \to \Omega^1_{(\mathcal{F}),0}(X) \to \cdots \to \Omega^k_{(\mathcal{F}),0}(X) \to \cdots \to \Omega^n_{(\mathcal{F}),0}(X) \to 0;$$

$$\overline{H}^k_{(\mathcal{F}),0}(X) = \overline{H}^k_{(\Phi_k),\langle \Phi_{k+1} \rangle,0}(X) = Z^k_{(\Phi_k),\langle \Phi_{k+1} \rangle,0}(X) / dD^{k-1}(X).$$

The dual of an $(n+1)$-tuple of $N$-functions $\mathcal{F} = \{\Phi_0, \Phi_1, \ldots, \Phi_n\}$ is the $(n+1)$-tuple $\mathcal{F}' = \{\Psi_0, \Psi_1, \ldots, \Psi_n\}$, where $\Psi_k$ and $\Phi_{n-k}$ are complementary $N$-functions for all $k$. Henceforth, we assume all $N$-functions to belong to the class $\Delta_2 \cap \nabla_2$.

Fix an $(n+1)$-tuple of $N$-functions $\mathcal{F} = \{\Phi_0, \Phi_1, \ldots, \Phi_n\}$ and let $\mathcal{F}' = \{\Psi_0, \Psi_1, \ldots, \Psi_n\}$ be its dual $(n+1)$-tuple. For $-1 \leq k \leq n$, introduce the vector spaces

$$\mathcal{P}^k_{(\mathcal{F})}(X) = L^k_{(\Phi_k),\langle \Phi_{k+1} \rangle}(X) = L_{(\Phi_k)}(X, \Lambda^k) \oplus L_{(\Phi_{k+1})}(X, \Lambda^{k+1})$$

(here $L_{(\Phi_k)}(X, \Lambda^k) = 0$ for $k = -1, n+1$). If $(\alpha, \beta) \in \mathcal{P}_{(\mathcal{F})}(X)$ with $\alpha \in L_{(\Phi_k)}(X, \Lambda^k)$ and $\beta \in L_{(\Phi_{k+1})}(X, \Lambda^{k+1})$ then $\mathcal{P}_{(\mathcal{F})}(X)$ is endowed with the norm

$$\|(\alpha, \beta)\|_{\mathcal{P}_{(\mathcal{F})}(X)} = \|\alpha\|_{(\Phi_k)} + \|\beta\|_{(\Phi_{k+1})}.$$ 

Let $d_\mathcal{P} : \mathcal{P}^k_{(\mathcal{F})}(X) \to \mathcal{P}^{k+1}_{(\mathcal{F})}(X)$ be defined as

$$d_\mathcal{P}(\alpha, \beta) = (\beta, 0).$$

The so-obtained Banach complex $\left(\mathcal{P}^*_{(\mathcal{F})}(X), d_\mathcal{P}\right)$ has trivial cohomology.

\textbf{Lemma 5.1.} \textit{Let $\mathcal{F} = \{\Phi_0, \Phi_1, \ldots, \Phi_n\}$ be an $(n+1)$-tuple of $N$-functions and let $\mathcal{F}' = \{\Psi_0, \Psi_1, \ldots, \Psi_n\}$ be its dual $(n+1)$-tuple. Then the spaces}
\( \mathcal{P}_{(\mathcal{F})}^k(X) \) and \( \mathcal{P}_{(\mathcal{F})}^{n-k-1}(X) \) (here, as above, the bar changes the type of the norm) are dual with respect to the pairing
\[
\langle (\alpha, \beta), (\omega, \theta) \rangle = \int_X ((-1)^k \alpha \wedge \omega + \beta \wedge \theta) .
\] (12)

Lemma 5.1 easily follows from Theorem 4.3.

**Lemma 5.2.** The operators
\[
d : \mathcal{P}_{(\mathcal{F})}^{k-1}(X) \to \mathcal{P}_{(\mathcal{F})}^k(X)
\]
and
\[
d : \mathcal{P}_{(\mathcal{F})}^{n-k-1}(X) \to \mathcal{P}_{(\mathcal{F})}^{n-k}(X)
\]
are adjoint.

**Proof.** If \((\alpha, \beta) \in \mathcal{P}_{(\mathcal{F})}^{k-1}(X)\) and \((\omega, \theta) \in \mathcal{P}_{(\mathcal{F})}^{n-k-1}(X)\) then
\[
\langle d(\alpha, \beta), (\omega, \theta) \rangle = \langle (\beta, 0), (\omega, \theta) \rangle = \int_X (-1)^k \beta \wedge \theta,
\]
\[
\langle (\alpha, \beta), d(\omega, \theta) \rangle = \langle (\alpha, \beta), (\theta, 0) \rangle = \int_X \beta \wedge \theta. \quad \square
\]

Put
\[
\Sigma_{(\mathcal{F})}^k(X) = \left\{ (\omega, d\omega) \in \mathcal{P}_{(\mathcal{F})}^k(X) : \omega \in \Omega_{(\mathcal{F})}^k(X) \right\};
\]
\[
\Sigma_{(\mathcal{F}),0}^k(X) = \left\{ (\omega, d\omega) \in \mathcal{P}_{(\mathcal{F})}^k(X) : \omega \in \Omega_{(\mathcal{F}),0}^k(X) \right\}.
\]

Clearly, these spaces form Banach complexes \( \Sigma_{(\mathcal{F})}(X) \) and \( \Sigma_{(\mathcal{F}),0}(X) \) which are isomorphic to \( \Omega_{(\mathcal{F})}(X) \) and \( \Omega_{(\mathcal{F}),0}(X) \) respectively.

Introduce the following quotient complex of \( \mathcal{P}_{(\mathcal{F})}(X) \):
\[
\mathcal{A}_{(\mathcal{F})}^*(X) = \mathcal{P}_{(\mathcal{F})}^*(X) / \Sigma_{(\mathcal{F}),0}^*(X).
\]

What was said above implies:

**Proposition 5.3.** The graded vector space \( \mathcal{A}_{(\mathcal{F})}^*(X) \) possesses the following properties:

(1) \( \mathcal{A}_{(\mathcal{F})}^*(X) \) is a Banach space with respect to the norm
\[
\|(\omega, \theta)\|_\mathcal{A} = \inf \left\{ \|\omega + \rho\|_{\mathcal{P}_k} + \|\theta + d\rho\|_{\mathcal{P}_{k+1}} \right\}.
\]
(2) $A^k_{\langle F \rangle}(X)$ is dual to $\Sigma^{n-k-1}_{\langle F \rangle}(X)$ with respect to the pairing (12).

(3) The differential $d_P : P^k_{\langle F \rangle}(X) \to P^{k+1}_{\langle F \rangle}(X)$ induces a differential $d_A : A^k_{\langle F \rangle}(X) \to A^{k+1}_{\langle F \rangle}(X)$ and $(A^*_{\langle F \rangle}(X), d_A)$ is a Banach complex.

(4) The operators $d_A : A^{k-1}_{\langle F \rangle}(X) \to A^k_{\langle F \rangle}(X)$ and $d_\Sigma : \Sigma^{n-k-1}_{\langle F \rangle}(X) \to \Sigma^{n-k}_{\langle F \rangle}(X)$ are adjoint up to sign with respect to the pairing (12).

Examine the cohomology of the Banach complex $(A^*_{\langle F \rangle}(X), d_A)$.

If we put

$Z^k \left( A^*_{\langle F \rangle}(X) \right) = \text{Ker} \ d_A : A^k_{\langle F \rangle}(X) \to A^{k+1}_{\langle F \rangle}(X)$

and

$B^k \left( A^*_{\langle F \rangle}(X) \right) = \text{Im} \ d_A \left( A^{k-1}_{\langle F \rangle}(X) \right)$

and denote by $\overline{B}^k \left( A^*_{\langle F \rangle}(X) \right)$ the closure of $B^k \left( A^*_{\langle F \rangle}(X) \right)$ then the cohomology and the reduced cohomology of $A^*_{\langle F \rangle}(X)$ are the spaces

$H^k \left( A^*_{\langle F \rangle}(X) \right) = Z^k \left( A^*_{\langle F \rangle}(X) \right) / B^k \left( A^*_{\langle F \rangle}(X) \right)$;

$\overline{H}^k \left( A^*_{\langle F \rangle}(X) \right) = Z^k \left( A^*_{\langle F \rangle}(X) \right) / \overline{B}^k \left( A^*_{\langle F \rangle}(X) \right)$.

We will need the following assertion [11, Lemma 3.1]:

**Lemma 5.4.** Let $I : Y_0 \times Y_1 \to \mathbb{R}$ be a duality between two reflexive Banach spaces. Let $B_0, B_1, A_0, A_1$ be linear subspaces such that

$B_0 \subset A_0 = B_1^\perp \subset Y_0; \quad B_1 \subset A_1 = B_0^\perp \subset Y_1.$

Then the pairing $\overline{I} : (A_0/\overline{B_0}) \times (A_1/\overline{B_1}) \to \mathbb{R}$ (with the bars standing for closures) is well-defined and induces duality between $A_0/\overline{B_0}$ and $A_1/\overline{B_1}$.

**Lemma 5.5.** The pairing (12) induces a pairing between the reduced cohomologies of $A^*_{\langle F \rangle}(X)$ and $\Sigma^*_{\langle F \rangle}(X)$.

**Proof.** We have

$B^{k-1}(A^*_{\langle F \rangle}(X)) \subset Z^{k-1}(A^*_{\langle F \rangle}(X)) = \left( B^{n-k}(\Sigma^*_{\langle F \rangle}(X)) \right)^\perp \subset A^{k-1}_{\langle F \rangle}(X)$,
and, similarly,

\[ \text{Im } d_{\Sigma}^{n-k-1} \subseteq \text{Ker } d_{\Sigma}^{n-k} = \left( \text{Im } d_{\mathcal{A}}^{k-2} \right)^\perp \subseteq \Sigma_{(\mathcal{F})}^{n-k}(X), \]

where the equalities are due to the fact that \( d_{\Sigma} \) and \( d_{\mathcal{A}} \) are adjoint operators. It remains to apply Lemma 5.4 with \( X_0 = \mathcal{A}_{(\mathcal{F}'),0}^{k-1} \) and \( X_1 = \Sigma_{(\mathcal{F})}^{n-k}(X). \) □

**Lemma 5.6.** The reduced cohomology of the Banach complex \((\mathcal{A}_{(\mathcal{F}'),0}^{*}, d_{\mathcal{A}})\) is isomorphic to the interior cohomology of \( X \) up to a shift:

\[ \overline{H}^{k}_{(\mathcal{F}')}^{(\mathcal{F})}(X) \cong \overline{H}^{k-1}_{(\mathcal{F}')}^{(\mathcal{F})}(\mathcal{A}_{(\mathcal{F}')}^{*}(X)). \]

The isomorphism is induced by the mapping \( j : Z^{k}_{(\mathcal{F}'),0}(X) \to \mathcal{P}^{k-1}_{(\mathcal{F}')}^{(\mathcal{F})}(X), \)

\[ j(\beta) = (0, \beta). \]

**Proof.** Every element in \( \mathcal{A}_{(\mathcal{F}')}^{k-1}(X) \) is represented by an element \((\alpha, \beta) \in \mathcal{P}^{k-1}_{(\mathcal{F}')}^{(\mathcal{F})}(X)\) modulo \( \Sigma_{(\mathcal{F}'),0}^{k-1}(X); \) thus, \((\alpha, \beta) \) and \((\alpha_1, \beta_1) \) represent one element in \( \mathcal{A}_{(\mathcal{F}')}^{k-1}(X) \) if and only if \( \alpha - \alpha_1 = \omega \) and \( \beta - \beta_1 = d\omega, \) where \( \omega \in \Sigma_{(\mathcal{F}'),0}^{k-1}(X). \)

Further, \((\alpha, \beta) \in \mathcal{P}^{k-1}_{(\mathcal{F}')}^{(\mathcal{F})}(X)\) represents an element of \( Z^{k-1}_{(\mathcal{F}')}^{(\mathcal{F})}(\mathcal{A}_{(\mathcal{F}')}^{*}(X)) \) whenever \( d_{\mathcal{P}}(\alpha, \beta) = (\beta, 0) \in \Sigma_{(\mathcal{F}'),0}^{k}(X), \) that is, \( \beta \in Z^{k}_{(\mathcal{F}'),0}(X). \) Thus,

\[ Z^{k-1}_{(\mathcal{F}')}^{(\mathcal{F})}(\mathcal{A}_{(\mathcal{F}')}^{*}(X)) = \left\{ (\alpha, \beta) \in \mathcal{P}^{k-1}_{(\mathcal{F}')}^{(\mathcal{F})}(X) : \beta \in Z^{k}_{(\mathcal{F}'),0}(X) \right\} \bigg/ \Sigma_{(\mathcal{F}'),0}^{k-1}(X). \]

Similarly, \((\alpha, \beta) \) represents an element in \( B^{k-1}_{(\mathcal{F}')}^{(\mathcal{F})}(\mathcal{A}_{(\mathcal{F}')}^{*}(X)) \) if there is \((\gamma, \delta) \in \mathcal{P}^{k-2}_{(\mathcal{F}')}^{(\mathcal{F})}(X) \) with \((\alpha, \beta) = d_{\mathcal{A}}(\gamma, \delta) = (\delta, 0) \) modulo \( \Sigma_{(\mathcal{F}'),0}^{k-1}(X), \) which means that \( \beta = d\omega \in B^{k}_{(\mathcal{F}'),0}(X). \) Thus,

\[ B^{k-1}_{(\mathcal{F}')}^{(\mathcal{F})}(\mathcal{A}_{(\mathcal{F}')}^{*}(X)) = \left\{ (\alpha, \beta) \in \mathcal{P}^{k-1}_{(\mathcal{F}')}^{(\mathcal{F})}(X) : \beta \in B^{k}_{(\mathcal{F}'),0}(X) \right\} \bigg/ \Sigma_{(\mathcal{F}'),0}^{k-1}(X) \]

and

\[ B^{k-1}_{(\mathcal{F}')}^{(\mathcal{F})}(\mathcal{A}_{(\mathcal{F}')}^{*}(X)) = \left\{ (\alpha, \beta) \in \mathcal{P}^{k-1}_{(\mathcal{F}')}^{(\mathcal{F})}(X) : \beta \in \overline{B}^{k}_{(\mathcal{F}'),0}(X) \right\} \bigg/ \Sigma_{(\mathcal{F}'),0}^{k-1}(X). \]
Therefore,

\[
H^{k-1} \left( A^*_{(\mathcal{F}')} (X) \right) = \left\{ (\alpha, \beta) \in \mathcal{P}^{k-1}_{(\mathcal{F}')} (X) : \beta \in Z^k_{(\mathcal{F}'),0} (X) \right\}
\]

Thus, the embedding \( j : Z^k_{(\mathcal{F}'),0} (X) \to \mathcal{P}^{k-1}_{(\mathcal{F}')} (X) \), \( j(\beta) = (0, \beta) \), induces an algebraic isomorphism \( j_* : H^k_{(\mathcal{F}'),0} (X) \cong H^{k-1} \left( A^*_{(\mathcal{F}')} (X) \right) \). We also have the relation

\[
\overline{H}^{k-1} \left( A^*_{(\mathcal{F}')} (X) \right) = \left\{ (0, \beta) \in \mathcal{P}^{k-1}_{(\mathcal{F}')} (X) : \beta \in Z^k_{(\mathcal{F}'),0} (X) \right\}
\]

The quotient on the right-hand side is endowed with the natural quotient norm and \( j \) induces an isometric isomorphism \( \bar{j}_* : \overline{H}^k_{(\mathcal{F}'),0} (X) \cong \overline{H}^{k-1} \left( A^*_{(\mathcal{F}')} (X) \right) \). □

Thus, we have

**Theorem 5.7.** Let \( X \) be a smooth \( n \)-dimensional oriented Riemannian manifold and let \( \mathcal{F} = (\Phi_0, \Phi_1, \ldots, \Phi_n) \) and \( \mathcal{F}' = (\Psi_0, \Psi_1, \ldots, \Psi_n) \) be dual sequences of \( N \)-functions with \( \Phi_i \in \Delta_2 \cap \nabla_2 \). Then the Banach spaces \( \overline{H}^k_{(\mathcal{F})} (X) \) and \( \overline{H}^{n-k}_{(\mathcal{F}'),0} (X) \) are dual with respect to the pairing \( \langle \omega, \theta \rangle = \int_X \omega \wedge \theta \) for \( \omega \in Z^k_{(\mathcal{F})} (X) \) and \( \theta \in Z^{n-k}_{(\mathcal{F}'),0} (X) \).

This gives the following duality theorem for \( L_{\Phi_I,\Phi_{II}} \)-cohomology:

**Theorem 5.8.** Let \( X \) be an oriented \( n \)-dimensional Riemannian manifold. If \( \Phi_I, \Phi_{II} \) are \( N \)-functions belonging to \( \Delta_2 \cap \nabla_2 \) and \( \Psi_I \) and \( \Psi_{II} \) are their respective complementary \( N \)-functions then \( \overline{H}_I,\Phi_{II} (X) \) is isomorphic to the dual of \( \overline{H}^{n-k}_{(\Phi_{II}),0} (X) \) and \( \overline{H}^k_{(\Phi_I),0} (X) \) is isomorphic to the dual of \( \overline{H}^{n-k}_{(\Phi_I),0} (X) \). The dualities are given by the pairing

\[
\langle [\omega], [\theta] \rangle = \int_X \omega \wedge \theta.
\]
Proof. The theorem results from Theorem 5.7 by considering any sequence of $N$-functions $(\Phi_0, \ldots, \Phi_n)$ with $\Phi_{k-1} = \Phi_I$ and $\Phi_k = \Phi_{II}$ and its dual sequence. □

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References


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