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APPROXIMATIVE PROPERTIES OF FOURIER – MEIXNER SUMS

Abstract. We consider the problem of approximation of discrete functions $f = f(x)$ defined on the set $\Omega_\delta = \{0,\, \delta,\, 2\delta,\, \ldots\}$, where $\delta = \frac{1}{N}$, $N > 0$, using the Fourier sums in the modified Meixner polynomials $M_{n,N}^\alpha(x) = M_n^\alpha(Nx)$ $(n = 0, 1, \ldots)$, which for $\alpha > -1$ constitute an orthogonal system on the grid Ω_δ with the weight function $w(x) = e^{-x}\dfrac{\Gamma(Nx+\alpha+1)}{\Gamma(Nx+1)}$. We study the approximative properties of partial sums of Fourier series in polynomials $M_{n,N}^\alpha(x)$, with particular attention paid to estimating their Lebesgue function.

Key words: *Meixner polynomials, Fourier series, Lebesgue function*

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1. Introduction. Suppose $\Omega_\delta = \{0, \delta, 2\delta, \ldots\}$, where $\delta = \frac{1}{N}$, $N > 0$. Following [5] we denote by $M_{n,N}^\alpha(x) = M_n^\alpha(Nx)$ $(n = 0, 1, \ldots)$ the modified Meixner polynomials that constitute for $\alpha > -1$ an orthogonal system on discrete set Ω_δ with the weight function $w(x) = e^{-x}\dfrac{\Gamma(Nx+\alpha+1)}{\Gamma(Nx+1)}$, that is,

$$\sum_{x\in\Omega_\delta} M_{n,N}^\alpha(x) M_{k,N}^\alpha(x) w(x) = (1-e^{-\delta})^{-\alpha-1} h_{n,N}^\alpha \delta_{nk},\ \ \alpha > -1.$$

Here $h_{n,N}^\alpha = \binom{n+\alpha}{n} e^{n\delta}\Gamma(\alpha+1)$, and the corresponding orthonormal polynomials with the weight function $\rho_N(x) = (1-e^{-\delta})^{\alpha+1} w(x)$ are denoted by $m_{n,N}^\alpha(x) = (h_{n,N}^\alpha)^{-1/2} M_{n,N}^\alpha(x)$ $(n = 0, 1, \ldots)$. These polynomials find applications in various problems of mathematical statistics, quantum physics, mathematical physics, computational mathematics and signal processing by spectral methods. In turn, these applications lead to the study of the approximative properties of Fourier – Meixner sums $S^{\alpha}_{n,N}(f,x)$, where f is the function given on the grid Ω_δ. We present the main result of this paper in Theorem 1. It holds an upper estimate for the Lebesgue function $\lambda^{\alpha}_{n,N}(x)$ of the partial sums $S^{\alpha}_{n,N}(f,x)$ for $x \in [0, \frac{\theta_n}{2}]$, where $\theta_n = 4n + 2\alpha + 2$. In the case $\alpha = -\frac{1}{2}$ this problem was solved in [2].

2. Some properties of Meixner polynomials. To study the approximative properties of the partial sums of Fourier – Meixner series we need several properties of Meixner polynomials $M^{\alpha}_{n,N}(x)$. For $q \neq 0$ and an arbitrary $\alpha \in \mathbb{R}$, the classical Meixner polynomials [1, 4, 5] can be defined by

$$M^{\alpha}_n(x) = M^{\alpha}_n(x,q) = \binom{n+\alpha}{n} \sum_{k=0}^{n} \frac{n^{[k]} x^{[k]}}{(\alpha+1)_k k!} \left(1 - \frac{1}{q}\right)^k,$$

where $x^{[k]} = x(x-1)\dots(x-k+1)$, $(a)_k = a(a+1)\dots(a+k-1)$. It is well known [1, 4, 5] that for $\alpha > -1$ and $0 < q < 1$ Meixner polynomials $M^{\alpha}_n(x)$ constitute an orthogonal system on the grid $\{0, 1, \dots\}$ with the weight function $\rho(x) = \rho(x,\alpha,q) = q^x \dfrac{\Gamma(x+\alpha+1)}{\Gamma(x+1)}$, that is,

$$\sum_{x=0}^{\infty} M^{\alpha}_n(x) M^{\alpha}_k(x) \rho(x) = (1-q)^{-\alpha-1} h^{\alpha}_n(q) \delta_{nk}, \quad 0 < q < 1, \alpha > -1,$$

where $h^{\alpha}_n(q) = \binom{n+\alpha}{n} q^{-n} \Gamma(\alpha+1)$.

Suppose $N>0$, $\delta = 1/N$, $q = e^{-\delta}$, $\Omega_\delta = \{0, \delta, 2\delta, \dots\}$. If $\alpha > -1$, then the polynomials $M^{\alpha}_{n,N}(x) = M^{\alpha}_n(Nx, e^{-\delta})$ and $m^{\alpha}_{n,N}(x) = m^{\alpha}_n(Nx, e^{-\delta}) = \left\{h^{\alpha}_n(e^{-\delta})\right\}^{-1/2} M^{\alpha}_{n,N}(x)$ constitute orthogonal and orthonormal on Ω_δ systems, respectively, with the weight function $\rho_N(x) = (1-e^{-\delta})^{\alpha+1} w(x)$.

Further, the following Christoffel – Darboux formula

$$\mathcal{K}^{\alpha}_{n,N}(t,x) = \sum_{k=0}^{n} m^{\alpha}_{k,N}(t) m^{\alpha}_{k,N}(x) = \frac{\delta\sqrt{(n+1)(n+\alpha+1)}}{(e^{\delta/2} - e^{-\delta/2})(x-t)} \times$$
$$\times \left[m^{\alpha}_{n+1,N}(t) m^{\alpha}_{n,N}(x) - m^{\alpha}_{n,N}(t) m^{\alpha}_{n+1,N}(x)\right] \qquad (1)$$

plays an important role for the estimate of the Lebesgue function. The formula (1) can be written [3] as

$$\mathcal{K}_{n,N}^{\alpha}(t,x)=\frac{\alpha_n}{(\alpha_n+\alpha_{n-1})}m_{n,N}^{\alpha}(t)m_{n,N}^{\alpha}(x)+\frac{\alpha_n\alpha_{n-1}}{(\alpha_n+\alpha_{n-1})}\frac{\delta}{(e^{\frac{\delta}{2}}-e^{-\frac{\delta}{2}})}\frac{1}{(x-t)}\times$$
$$\times\left[m_{n,N}^{\alpha}(x)\left(m_{n+1,N}^{\alpha}(t)-m_{n-1,N}^{\alpha}(t)\right)-\right.$$
$$\left.-\ m_{n,N}^{\alpha}(t)\left(m_{n+1,N}^{\alpha}(x)-m_{n-1,N}^{\alpha}(x)\right)\right], \tag{2}$$

where $\alpha_n=\sqrt{(n+1)(n+\alpha+1)}$, $m_{-1,N}^{\alpha}(x)=0$. For $0<\delta\le 1$, $N=\frac{1}{\delta}$, $\lambda>0$, $1\le n\le\lambda N$, $\alpha>-1$, $0\le x<\infty$, $\theta_n=4n+2\alpha+2$ the following estimates [5, 6] hold:

$$e^{-\frac{x}{2}}\left|m_{n,N}^{\alpha}(x)\right|\le c(\alpha,\lambda)\theta_n^{-\frac{\alpha}{2}}A_n^{\alpha}(x), \tag{3}$$

$$A_n^{\alpha}(x)=\begin{cases}\theta_n^{\alpha}, & 0\le x\le\frac{1}{\theta_n},\\ \theta_n^{\frac{\alpha}{2}-\frac{1}{4}}x^{-\frac{\alpha}{2}-\frac{1}{4}}, & \frac{1}{\theta_n}<x\le\frac{\theta_n}{2},\\ \left[\theta_n(\theta_n^{\frac{1}{3}}+|x-\theta_n|)\right]^{-\frac{1}{4}}, & \frac{\theta_n}{2}<x\le\frac{3\theta_n}{2},\\ e^{-\frac{x}{4}}, & \frac{3\theta_n}{2}<x<\infty,\end{cases} \tag{4}$$

$$e^{-\frac{x}{2}}\left|m_{n+1,N}^{\alpha}(x)-m_{n-1,N}^{\alpha}(x)\right|\le$$
$$\le c(\alpha,\lambda)\begin{cases}\theta_n^{\frac{\alpha}{2}-1}, & 0\le x\le\frac{1}{\theta_n},\\ \theta_n^{-\frac{3}{4}}x^{-\frac{\alpha}{2}+\frac{1}{4}}, & \frac{1}{\theta_n}<x\le\frac{\theta_n}{2},\\ x^{-\frac{\alpha}{2}}\theta_n^{-\frac{3}{4}}\left[\theta_n^{\frac{1}{3}}+|x-\theta_n|\right]^{\frac{1}{4}}, & \frac{\theta_n}{2}<x\le\frac{3\theta_n}{2},\\ e^{-\frac{x}{4}}, & \frac{3\theta_n}{2}<x<\infty,\end{cases} \tag{5}$$

where hereinafter c, $c(\alpha)$, $c(\alpha,\lambda)$ are positive numbers depending only on the indicated parameters.

3. Formulation of the main result. We denote by $C(\Omega_\delta)$ the space of discrete functions $f:\Omega_\delta\to\mathbb{R}$, such that

$$\lim_{x\to\infty}|f(x)|e^{-x/2}=0. \tag{6}$$

The norm in this space we define as follows:

$$\|f\|_{C(\Omega_\delta)}=\sup_{x\in\Omega_\delta}e^{-x/2}|f(x)|.$$

The following lemma holds.

Lemma 1. *Suppose that $\alpha > -1$, $p > 1$ and $l^p_{\rho_N}$ is the space of functions defined on Ω_δ with*

$$\|f\|_{l^p_{\rho_N}} = \left(\sum_{x\in\Omega_\delta} |f(x)|^p \rho_N(x)\right)^{1/p} < \infty. \tag{7}$$

Then $C(\Omega_\delta) \subset l^p_{\rho_N}$ for $1 < p < 2$.

Proof. The proof of the lemma follows immediately from (6) and (7). □

It follows from lemma 1 that for an arbitrary function $f \in C(\Omega_\delta)$ we can define Fourier – Meixner coefficients

$$f_k^\alpha = \sum_{t\in\Omega_\delta} f(t) m_{k,N}^\alpha(t) \rho_N(t) \tag{8}$$

and Fourier – Meixner series

$$f(x) \sim \sum_{k=0}^{\infty} f_k^\alpha m_{k,N}^\alpha(x). \tag{9}$$

We denote by $S_{n,N}^\alpha(f,x)$ the partial sum of the series (9):

$$S_{n,N}^\alpha(f,x) = \sum_{k=0}^{n} f_k^\alpha m_{k,N}^\alpha(x),$$

which in view of (1) and (8) can be represented as

$$S_{n,N}^\alpha(f,x) = \sum_{t\in\Omega_\delta} f(t) \mathcal{K}_{n,N}^\alpha(t,x) e^{-t} \frac{\Gamma(Nt+\alpha+1)}{\Gamma(Nt+1)} (1-e^{-\delta})^{\alpha+1}. \tag{10}$$

Let $E_n(f,\delta)$ be the best approximation of function $f \in C(\Omega_\delta)$ in the metric of the space $C(\Omega_\delta)$ by algebraic polynomials of degree n, that is,

$$E_n(f,\delta) = \inf_{p_n\in H^n} \|f - p_n\|_{C(\Omega_\delta)},$$

where H^n is the subspace of algebraic polynomials $p_n(x)$ of degree less than or equal to n. Further, let $p_n(f) = p_n(f,x)$ be a polynomial of the best approximation f in $C(\Omega_\delta)$ for which $E_n(f,\delta) = \|f - p_n(f)\|_{C(\Omega_\delta)}$. Then, since $S^\alpha_{n,N}(p_n) = p_n$ for $p_n \in H^n$, we get

$$\left|f(x) - S^\alpha_{n,N}(f,x)\right| = \left|f(x) - p_n(f,x) + p_n(f,x) - S^\alpha_{n,N}(f,x)\right| \le$$
$$\le |f(x) - p_n(f,x)| + \left|S^\alpha_{n,N}(p_n - f, x)\right|.$$

By the last inequality and (10) we have

$$e^{-\frac{x}{2}}\left|f(x)-S^\alpha_{n,N}(f,x)\right| \le e^{-\frac{x}{2}}\left|f(x)-p_n(f,x)\right|+e^{-\frac{x}{2}}\left|S^\alpha_{n,N}(p_n-f,x)\right| \le$$
$$\le E_n(f,\delta)(1+\lambda^\alpha_{n,N}(x)), \tag{11}$$

where

$$\lambda^\alpha_{n,N}(x) = \sum_{t\in\Omega_\delta} e^{-\frac{t+x}{2}} \frac{\Gamma(Nt+\alpha+1)}{\Gamma(Nt+1)}(1-e^{-\delta})^{\alpha+1}\left|\mathcal{K}^\alpha_{n,N}(t,x)\right|. \tag{12}$$

The inequality (11) needs estimating on $[0,\infty)$ the Lebesgue function $\lambda^\alpha_{n,N}(x)$ defined in (12). In this article we analyse this problem only for the cases $x \in G_1 = [0, \frac{3}{\theta_n}]$ and $x \in G_2 = [\frac{3}{\theta_n}, \frac{\theta_n}{2}]$. The case when $x \in (\frac{\theta_n}{2}, \infty)$ will be discussed in another paper. We note that this problem without proof of the main result was presented in [3]. In this subsection we give, with full proof, the result announced in paper [3]. Namely, the following theorem holds.

Theorem 1. *Suppose that $\alpha > -1$, $\theta_n = 4n+2\alpha+2$, $\lambda > 0$, $0 < \delta \le 1$, $1 \le n \le \lambda N$. We have the following:*
1) if $x \in G_1$, then

$$\lambda^\alpha_{n,N}(x) \le c(\alpha,\lambda)\begin{cases} 1, & \alpha \in (-1, -\frac{1}{2}), \\ \log(n+1), & \alpha = -\frac{1}{2}, \\ n^{\alpha+\frac{1}{2}}, & \alpha > -\frac{1}{2}; \end{cases} \tag{13}$$

2) if $x \in G_2$, then

$$\lambda^\alpha_{n,N}(x) \le c(\alpha,\lambda)\begin{cases} \log(nx+1), & \alpha \in (-1, -\frac{1}{2}), \\ \log(n+1), & \alpha = -\frac{1}{2}, \\ \log(n+1) + \left(\frac{n}{x}\right)^{\frac{\alpha}{2}+\frac{1}{4}}, & \alpha > -\frac{1}{2}. \end{cases} \tag{14}$$

4. Proof of Theorem 1. Suppose that $x \in G_1$, then

$$\lambda^\alpha_{n,N}(x) = I_1 + I_2, \tag{15}$$

where

$$I_1 \le c(\alpha)\delta \sum_{\substack{t\in\Omega_\delta,\\ 0\le t\le \frac{4}{\theta_n}}} e^{-\frac{t+x}{2}}(t+\delta)^\alpha \left|\sum_{k=0}^{n} m_{k,N}^\alpha(t)m_{k,N}^\alpha(x)\right|,$$

$$I_2 = \sum_{\substack{t\in\Omega_\delta,\\ \frac{4}{\theta_n}<t<\infty}} e^{-\frac{t+x}{2}}\frac{\Gamma(Nt+\alpha+1)}{\Gamma(Nt+1)}(1-e^{-\delta})^{\alpha+1}\left|\mathcal{K}_{n,N}^\alpha(t,x)\right|.$$

Let us estimate I_1. From (3) and (4) we obtain

$$I_1 \le c(\alpha)\delta \sum_{\substack{t\in\Omega_\delta,\\ 0\le t\le \frac{4}{\theta_n}}} (t+\delta)^\alpha \sum_{k=0}^{n} |e^{-\frac{x}{2}}m_{k,N}^\alpha(x)||e^{-\frac{t}{2}}m_{k,N}^\alpha(t)| \le$$

$$\le c(\alpha,\lambda)\delta \sum_{\substack{t\in\Omega_\delta,\\ 0\le t\le \frac{4}{\theta_n}}} (t+\delta)^\alpha \sum_{k=0}^{n}\theta_k^\alpha \le c(\alpha,\lambda)\delta \sum_{\substack{t\in\Omega_\delta,\\ 0\le t\le \frac{4}{\theta_n}}} (t+\delta)^\alpha\theta_n^{\alpha+1} \le$$

$$\le c(\alpha,\lambda)\theta_n^{\alpha+1}\left[\delta^{\alpha+1} + \int_0^{\frac{4}{\theta_n}+\delta}(t+\delta)^\alpha dt\right] \le c(\alpha,\lambda). \tag{16}$$

Now we proceed to estimating I_2. Taking (2) into account, we can write

$$I_2 \le I_{21} + I_{22} + I_{23}, \tag{17}$$

where

$$I_{21} = \frac{\alpha_n}{\alpha_n+\alpha_{n-1}}e^{-\frac{x}{2}}|m_{n,N}^\alpha(x)|\times$$

$$\times \sum_{\substack{t\in\Omega_\delta,\\ \frac{4}{\theta_n}<t<\infty}} \frac{e^{-\frac{t}{2}}\Gamma(Nt+\alpha+1)}{\Gamma(Nt+1)}(1-e^{-\delta})^{\alpha+1}\left|m_{n,N}^\alpha(t)\right|,$$

$$I_{22} = \frac{\alpha_n\alpha_{n-1}}{\alpha_n+\alpha_{n-1}}\frac{\delta}{e^{\delta/2}-e^{-\delta/2}}e^{-\frac{x}{2}}\left|m_{n+1,N}^\alpha(x)-m_{n-1,N}^\alpha(x)\right|\times$$

$$\times \sum_{\substack{t\in\Omega_\delta,\\ \frac{4}{\theta_n}<t<\infty}} \frac{e^{-\frac{t}{2}}\Gamma(Nt+\alpha+1)}{\Gamma(Nt+1)(t-x)}(1-e^{-\delta})^{\alpha+1}\left|m_{n,N}^\alpha(t)\right|,$$

$$I_{23} = \frac{\alpha_n \alpha_{n-1}}{\alpha_n + \alpha_{n-1}} \frac{\delta}{e^{\delta/2} - e^{-\delta/2}} e^{-\frac{x}{2}} \left| m_{n,N}^{\alpha}(x) \right| \times$$
$$\times \sum_{\substack{t \in \Omega_\delta, \\ \frac{4}{\theta_n} < t < \infty}} \frac{e^{-\frac{t}{2}} \Gamma(Nt+\alpha+1)}{\Gamma(Nt+1)(t-x)} (1-e^{-\delta})^{\alpha+1} \left| m_{n+1,N}^{\alpha}(t) - m_{n-1,N}^{\alpha}(t) \right| .$$

Let us estimate I_{21}. From (3) and (4) we have

$$I_{21} \leq c(\alpha, \lambda) \theta_n^{\frac{\alpha}{2}} \sum_{\substack{t \in \Omega_\delta, \\ \frac{4}{\theta_n} < t < \infty}} e^{-\frac{t}{2}} \frac{\Gamma(Nt+\alpha+1)}{\Gamma(Nt+1)} (1-e^{-\delta})^{\alpha+1} \left| m_{n,N}^{\alpha}(t) \right| . \quad (18)$$

We put

$$W = \sum_{t \in \Omega_\delta} \frac{e^{-\frac{t}{2}} \Gamma(Nt+\alpha+1)}{\Gamma(Nt+1)} (1-e^{-\delta})^{\alpha+1} \left| m_{n,N}^{\alpha}(t) \right| = W_1 + W_2, \quad (19)$$

where

$$W_1 = \sum_{\substack{t \in \Omega_\delta, \\ 0 \leq t \leq \frac{3\theta_n}{2}}} \frac{e^{-\frac{t}{2}} \Gamma(Nt+\alpha+1)}{\Gamma(Nt+1)} (1-e^{-\delta})^{\alpha+1} \left| m_{n,N}^{\alpha}(t) \right| ,$$

$$W_2 = \sum_{\substack{t \in \Omega_\delta, \\ \frac{3\theta_n}{2} < t < \infty}} \frac{e^{-\frac{t}{2}} \Gamma(Nt+\alpha+1)}{\Gamma(Nt+1)} (1-e^{-\delta})^{\alpha+1} \left| m_{n,N}^{\alpha}(t) \right| .$$

By the Cauchy–Bunyakovsky inequality

$$\begin{aligned} W_1 \quad &\leq \quad \left(\sum_{\substack{t \in \Omega_\delta, \\ 0 \leq t \leq \frac{3\theta_n}{2}}} (1-e^{-\delta})^{\alpha+1} \frac{\Gamma(Nt+\alpha+1)}{\Gamma(Nt+1)} \right)^{1/2} \times \\ &\times \quad \left(\sum_{\substack{t \in \Omega_\delta, \\ 0 \leq t \leq \frac{3\theta_n}{2}}} (1-e^{-\delta})^{\alpha+1} \frac{e^{-t} \Gamma(Nt+\alpha+1)}{\Gamma(Nt+1)} \left(m_{n,N}^{\alpha}(t) \right)^2 \right)^{1/2} \leq \\ &\leq \quad c(\alpha) \left(\delta^{\alpha+1} + \int_0^{\frac{3\theta_n}{2}+\delta} (t+\delta)^\alpha dt \right)^{1/2} \leq c(\alpha) \theta_n^{\frac{\alpha+1}{2}} , \end{aligned} \quad (20)$$

$$W_2 \le c(\alpha,\lambda)\theta_n^{-\frac{\alpha}{2}}\delta \sum_{\substack{t\in\Omega_\delta,\\ \frac{3\theta_n}{2}<t<\infty}} e^{-\frac{t}{4}}(t+\delta)^\alpha \le c(\alpha,\lambda)\theta_n^{-\frac{\alpha}{2}}e^{-n}. \tag{21}$$

From (19) – (21) we conclude that

$$W \le c(\alpha,\lambda)\theta_n^{\frac{\alpha+1}{2}}. \tag{22}$$

From (18), (19) and the last inequality we have

$$I_{21} \le c(\alpha,\lambda)\theta_n^{\alpha+\frac{1}{2}}. \tag{23}$$

Now we proceed to estimating I_{22}. From (5) and (3) we have

$$I_{22} \le c(\alpha,\lambda)n\theta_n^{\frac{\alpha}{2}-1}\theta_n^{-\frac{\alpha}{2}}\delta \sum_{\substack{t\in\Omega_\delta,\\ \frac{4}{\theta_n}<t<\infty}} \frac{t^\alpha A_n^\alpha(t)}{t-x} = I_{22}^1 + I_{22}^2 + I_{22}^3,$$

where

$$I_{22}^i = c(\alpha,\lambda)n\theta_n^{-1}\delta\sum_{t\in B_i}\frac{t^\alpha A_n^\alpha(t)}{t-x},\ i=1,2,3,$$

$B_1 = (4/\theta_n, \theta_n/2]\cap\Omega_\delta,\ B_2 = (\theta_n/2, 3\theta_n/2]\cap\Omega_\delta,\ B_3 = (3\theta_n/2,\infty)\cap\Omega_\delta.$

Turning to inequality (4), we obtain

$$\begin{aligned} I_{22}^1 &\le c(\alpha,\lambda)\theta_n^{\frac{\alpha}{2}-\frac{1}{4}}\delta\sum_{t\in B_1}\frac{t^\alpha t^{-\frac{\alpha}{2}-\frac{1}{4}}}{t-x} \le c(\alpha,\lambda)\theta_n^{\frac{\alpha}{2}-\frac{1}{4}}\delta\sum_{t\in B_1} t^{\frac{\alpha}{2}-\frac{5}{4}} \le \\ &\le c(\alpha,\lambda)\begin{cases} 1, & \alpha\in(-1,-\frac{1}{2}), \\ \log(n+1), & \alpha=\frac{1}{2}, \\ \theta_n^{\alpha-\frac{1}{2}}, & \alpha>\frac{1}{2}. \end{cases} \end{aligned} \tag{24}$$

$$\begin{aligned} I_{22}^2 &\le c(\alpha,\lambda)\theta_n^{-\frac{1}{4}}\delta\sum_{t\in B_2}\frac{t^\alpha[\theta_n^{\frac{1}{3}}+|t-\theta_n|]^{-\frac{1}{4}}}{t-x} \le c(\alpha,\lambda)\theta_n^{\alpha-\frac{5}{4}} \times \\ &\times \int_{\frac{\theta_n}{2}}^{\frac{3\theta_n}{2}}[\theta_n^{\frac{1}{3}}+|t-\theta_n|]^{-\frac{1}{4}}dt \le c(\alpha,\lambda)\theta_n^{\alpha-\frac{5}{4}}\theta_n^{\frac{3}{4}} \le c(\alpha,\lambda)\theta_n^{\alpha-\frac{1}{2}}, \end{aligned} \tag{25}$$

$$I_{22}^3 \le c(\alpha,\lambda)\delta \sum_{t\in B_3} \frac{t^\alpha e^{-\frac{t}{4}}}{t-x} \le c(\alpha,\lambda) \int\limits_{\frac{3\theta_n}{2}-\delta}^{\infty} t^\alpha e^{-\frac{t}{4}} dt \le c(\alpha,\lambda) e^{-n}. \quad (26)$$

Combining estimates (24) – (26), we obtain

$$I_{22} \le c(\alpha,\lambda) \begin{cases} 1, & \alpha\in(-1,-\frac{1}{2}), \\ \log(n+1), & \alpha=\frac{1}{2}, \\ \theta_n^{\alpha-\frac{1}{2}}, & \alpha>\frac{1}{2}. \end{cases} \quad (27)$$

Proceed to estimating I_{23} for $x\in G_1$. Using (3) and (4), we can write

$$\begin{aligned} I_{23} \quad &\le \quad c(\alpha,\lambda) n \theta_n^{\frac{\alpha}{2}} \delta \sum_{\substack{t\in\Omega_\delta, \\ \frac{4}{\theta_n}<t<\infty}} \frac{t^\alpha}{t-x} e^{-\frac{t}{2}} \left| m_{n+1,N}^\alpha(t) - m_{n-1,N}^\alpha(t) \right| = \\ &= \quad I_{23}^1 + I_{23}^2 + I_{23}^3, \end{aligned} \quad (28)$$

where

$$I_{23}^i = c(\alpha,\lambda) n \theta_n^{\frac{\alpha}{2}} \delta \sum_{t\in B_i} \frac{t^\alpha}{t-x} e^{-\frac{t}{2}} \left| m_{n+1,N}^\alpha(t) - m_{n-1,N}^\alpha(t) \right|, \; i=1,2,3.$$

$$\begin{aligned} I_{23}^1 \quad &\le \quad c(\alpha,\lambda) n \theta_n^{\frac{\alpha}{2}} \delta \sum_{t\in B_1} \frac{\theta_n^{-\frac{3}{4}} t^\alpha t^{-\frac{\alpha}{2}+\frac{1}{4}}}{t-x} \le c(\alpha,\lambda) \theta_n^{\frac{\alpha}{2}+\frac{1}{4}} \delta \sum_{t\in B_1} \frac{t^{\frac{\alpha}{2}+\frac{1}{4}}}{t-x} \le \\ &\le \quad c(\alpha,\lambda) \theta_n^{\frac{\alpha}{2}+\frac{1}{4}} \delta \sum_{t\in B_1} t^{\frac{\alpha}{2}-\frac{3}{4}} \le \\ &\le \quad c(\alpha,\lambda) \begin{cases} 1, & \alpha\in(-1,-\frac{1}{2}), \\ \log(n+1), & \alpha=-\frac{1}{2}, \\ \theta_n^{\alpha+\frac{1}{2}}, & \alpha>-\frac{1}{2}, \end{cases} \end{aligned} \quad (29)$$

$$\begin{aligned} I_{23}^2 \quad &\le \quad c(\alpha,\lambda) n \theta_n^{\frac{\alpha}{2}} \delta \sum_{t\in B_2} \frac{\theta_n^{-\frac{3}{4}} t^\alpha t^{-\frac{\alpha}{2}} [\theta_n^{\frac{1}{3}} + |t-\theta_n|]^{\frac{1}{4}}}{t-x} \le c(\alpha,\lambda) \theta_n^{\alpha-\frac{3}{4}} \times \\ &\times \quad \int\limits_{\frac{\theta_n}{2}-\delta}^{\frac{3\theta_n}{2}+\delta} [\theta_n^{\frac{1}{3}} + |t-\theta_n|]^{\frac{1}{4}} dt \le c(\alpha,\lambda) \theta_n^{\alpha-\frac{3}{4}} \theta_n^{\frac{5}{4}} = c(\alpha,\lambda) \theta_n^{\alpha+\frac{1}{2}}, \end{aligned} \quad (30)$$

$$\begin{aligned} I_{23}^3 &\le c(\alpha,\lambda) n \theta_n^{\frac{\alpha}{2}} \delta \sum_{t\in B_3} \frac{t^\alpha e^{-\frac{t}{4}}}{t-x} \le \\ &\le c(\alpha,\lambda) \theta_n^{\frac{\alpha}{2}-1} \int\limits_{\frac{3\theta_n}{2}-\delta}^{\infty} t^{\alpha-1} e^{-\frac{t}{4}} dt \le c(\alpha,\lambda)\theta_n^{\frac{\alpha}{2}-1} e^{-n}. \end{aligned} \tag{31}$$

The inequality (28) and estimates (29) – (31) yield

$$I_{23} \le c(\alpha,\lambda) \begin{cases} 1, & \alpha\in(-1,-\frac{1}{2}), \\ \log(n+1), & \alpha=-\frac{1}{2}, \\ \theta_n^{\alpha+\frac{1}{2}}, & \alpha>-\frac{1}{2}, \end{cases} . \tag{32}$$

From inequalities (17), (23), (27) and (32) we obtain

$$I_2 \le c(\alpha,\lambda) \begin{cases} 1, & \alpha\in(-1,-\frac{1}{2}), \\ \log(n+1), & \alpha=-\frac{1}{2}, \\ n^{\alpha+\frac{1}{2}}, & \alpha>-\frac{1}{2}. \end{cases} \tag{33}$$

In turn, from (15), (16) and (33) we have

$$\lambda_{n,N}^\alpha(x) \le c(\alpha,\lambda) \begin{cases} 1, & \alpha\in(-1,-\frac{1}{2}), \\ \log(n+1), & \alpha=-\frac{1}{2}, \\ n^{\alpha+\frac{1}{2}}, & \alpha>-\frac{1}{2}. \end{cases}$$

Thus, (13) is proved.

Now let us prove (14). Assume that $x\in G_2$. We introduce the notation: $D_1=[0,x-\sqrt{x/\theta_n}]\cap\Omega_\delta$, $D_2=(x-\sqrt{x/\theta_n}, x+\sqrt{x/\theta_n}]\cap\Omega_\delta$, $D_3=(x+\sqrt{x/\theta_n},\infty)\cap\Omega_\delta$. Then

$$\lambda_{n,N}^\alpha(x)=J_1+J_2+J_3,$$

where

$$J_i=e^{-\frac{x}{2}} \sum_{t\in D_i} e^{-\frac{t}{2}} \frac{\Gamma(Nt+\alpha+1)}{\Gamma(Nt+1)} (1-e^{-\delta})^{\alpha+1} \left|\mathcal{K}_{n,N}^\alpha(t,x)\right|, i=1,2,3.$$

Let us estimate J_2. To this end, observe that the Cauchy-Bunyakovsky inequality yields

$$\left|\mathcal{K}_{n,N}^{\alpha}(t,x)\right| \le \left|\mathcal{K}_{n,N}^{\alpha}(t,t)\right|^{1/2} \left|\mathcal{K}_{n,N}^{\alpha}(x,x)\right|^{1/2}.$$

Further, if $\frac{3}{\theta_n} \le x \le \frac{\theta_n}{2}$, then $x - \sqrt{\frac{x}{\theta_n}} \ge \frac{1}{\theta_n}$, moreover, for $t \in D_2$, we have $c_1 x \le t \le c_2 x$. Then

$$J_2 \le c(\alpha) \left|e^{-x}\mathcal{K}_{n,N}^{\alpha}(x,x)\right|^{1/2} \delta \sum_{t\in D_2} t^{\alpha} \left|e^{-t}\mathcal{K}_{n,N}^{\alpha}(t,t)\right|^{1/2}.$$

Let us now estimate $|e^{-t}\mathcal{K}_{n,N}^{\alpha}(t,t)|$. Using (1), (3), (4) and (5), repeating almost verbatim the arguments of the proof of lemma 7.1 [7], we can prove the following assertion.

Lemma 2. *If* $\alpha > -1$, $\theta_n = 4n + 2\alpha + 2$, $\lambda > 0$, $1 \le n \le \lambda N$, $t \ge 3/\theta_n$ *then*

$$|e^{-t}\mathcal{K}_{n,N}^{\alpha}(t,t)| \le c(\alpha,\lambda) t^{-\alpha-1/2} n^{1/2}.$$

By lemma 2 we have

$$J_2 \le c(\alpha,\lambda) x^{-\frac{\alpha}{2}-\frac{1}{4}} n^{\frac{1}{4}} \delta \sum_{t\in D_2} (t+\delta)^{\alpha} t^{-\frac{\alpha}{2}-\frac{1}{4}} n^{\frac{1}{4}} = c(\alpha,\lambda) x^{-\frac{\alpha}{2}-\frac{1}{4}} n^{\frac{1}{2}} \delta \times$$
$$\times \sum_{t\in D_2} (t+\delta)^{\frac{\alpha}{2}-\frac{1}{4}} \le c(\alpha,\lambda) x^{-\frac{1}{2}} n^{\frac{1}{2}} \sum_{t\in D_2} \delta \le c(\alpha,\lambda). \tag{34}$$

Let us estimate J_1. Using (2) we can write

$$J_1 \le J_{11} + J_{12} + J_{13},$$

where

$$J_{11} = c(\alpha) e^{-\frac{x}{2}} \delta \sum_{t\in D_1} e^{-\frac{t}{2}} (t+\delta)^{\alpha} \left|m_{n,N}^{\alpha}(x) m_{n,N}^{\alpha}(t)\right|,$$

$$J_{12} = c(\alpha) n e^{-\frac{x}{2}} \left|m_{n+1,N}^{\alpha}(x) - m_{n-1,N}^{\alpha}(x)\right| \delta \sum_{t\in D_1} \frac{e^{-\frac{t}{2}}(t+\delta)^{\alpha}}{|t-x|} \left|m_{n,N}^{\alpha}(t)\right|,$$

$$J_{13} = c(\alpha) n e^{-\frac{x}{2}} \left|m_{n,N}^{\alpha}(x)\right| \delta \sum_{t\in D_1} \frac{e^{-\frac{t}{2}}(t+\delta)^{\alpha}}{|t-x|} \left|m_{n+1,N}^{\alpha}(t) - m_{n-1,N}^{\alpha}(t)\right|.$$

To estimate J_{11} we have

$$J_{11} \leq c(\alpha)(J_{11}^1 + J_{11}^2), \tag{35}$$

where

$$J_{11}^1 \leq c(\alpha,\lambda)\theta_n^{\frac{\alpha}{2}-\frac{1}{4}} x^{-\frac{\alpha}{2}-\frac{1}{4}} \delta \sum_{\substack{t\in\Omega_\delta,\\ 0\leq t\leq \frac{1}{\theta_n}}} (t+\delta)^\alpha \leq c(\alpha,\lambda)\theta_n^{\frac{\alpha}{2}-\frac{1}{4}} x^{-\frac{\alpha}{2}-\frac{1}{4}} \times$$
$$\times \left(\delta^{\alpha+1} + \int\limits_0^{\frac{1}{\theta_n}+\delta} (t+\delta)^\alpha dt\right) \leq c(\alpha,\lambda) x^{-\frac{\alpha}{2}-\frac{1}{4}} \theta_n^{\frac{\alpha}{2}-\frac{1}{4}} \theta_n^{-\alpha-1} =$$
$$= c(\alpha,\lambda)(x\theta_n)^{-\frac{\alpha}{2}-\frac{1}{4}} \theta_n^{-1} \leq c(\alpha,\lambda)\theta_n^{-\frac{1}{2}}, \tag{36}$$

$$J_{11}^2 \leq c(\alpha,\lambda) x^{-\frac{\alpha}{2}-\frac{1}{4}} \theta_n^{-\frac{1}{2}} \delta \sum_{\substack{t\in\Omega_\delta,\\ \frac{1}{\theta_n} < t \leq x - \sqrt{\frac{x}{\theta_n}}}} (t+\delta)^\alpha t^{-\frac{\alpha}{2}-\frac{1}{4}} \leq$$
$$\leq c(\alpha,\lambda) x^{\frac{1}{2}} \theta_n^{-\frac{1}{2}}. \tag{37}$$

From the inequalities (35), (36) and (37) we have

$$J_{11} \leq c(\alpha,\lambda) \left[\left(\frac{x}{\theta_n}\right)^{\frac{1}{2}} + \theta_n^{-\frac{1}{2}} \right]. \tag{38}$$

In order to estimate J_{12} we represent it as

$$J_{12} = J_{12}^1 + J_{12}^2, \tag{39}$$

where

$$J_{12}^1 \leq c(\alpha,\lambda) n \theta_n^{-\frac{3}{4}} x^{-\frac{\alpha}{2}+\frac{1}{4}} \theta_n^{\frac{\alpha}{2}} \delta \sum_{\substack{t\in\Omega_\delta,\\ 0\leq t\leq \frac{1}{\theta_n}}} \frac{(t+\delta)^\alpha}{x-t} \leq$$
$$\leq c(\alpha,\lambda)\theta_n^{\frac{\alpha}{2}+\frac{1}{4}} x^{-\frac{\alpha}{2}+\frac{1}{4}} \frac{1}{x} \delta \sum_{\substack{t\in\Omega_\delta,\\ 0\leq t\leq \frac{1}{\theta_n}}} (t+\delta)^\alpha \leq$$
$$\leq c(\alpha,\lambda)\theta_n^{\frac{\alpha}{2}+\frac{1}{4}} x^{-\frac{\alpha}{2}-\frac{3}{4}} \theta_n^{-\alpha-1} = c(\alpha,\lambda)\left(x\theta_n\right)^{-\frac{\alpha}{2}-\frac{3}{4}}, \tag{40}$$

$$J_{12}^2 \le c(\alpha,\lambda) n\theta_n^{-\frac{3}{4}} x^{-\frac{\alpha}{2}+\frac{1}{4}}\theta_n^{-\frac{1}{4}}\delta \sum_{\substack{t\in\Omega_\delta,\\ \frac{1}{\theta_n}<t\le x-\sqrt{\frac{x}{\theta_n}}}} \frac{(t+\delta)^\alpha t^{-\frac{\alpha}{2}-\frac{1}{4}}}{x-t} \le$$

$$\le c(\alpha,\lambda) x^{-\frac{\alpha}{2}+\frac{1}{4}} \left(\delta \frac{(\frac{1}{\theta_n})^{\frac{\alpha}{2}-\frac{1}{4}}}{x-(\frac{1}{\theta_n})} + \int\limits_{\frac{1}{\theta_n}}^{x-\sqrt{\frac{x}{\theta_n}}+\delta} \frac{t^{\frac{\alpha}{2}-\frac{1}{4}}}{x-t} dt \right) \le$$

$$\le c(\alpha,\lambda) \int\limits_{\frac{1}{x\theta_n}}^{1-\sqrt{\frac{1}{x\theta_n}}+\delta/x} \frac{y^{\frac{\alpha}{2}-\frac{1}{4}}}{1-y} dy \le c(\alpha,\lambda) \int\limits_{\frac{1}{x\theta_n}}^{\frac{1}{3}} y^{\frac{\alpha}{2}-\frac{1}{4}} dy +$$

$$\le c(\alpha,\lambda) \int\limits_{\frac{1}{3}}^{1-\sqrt{\frac{1}{x\theta_n}}+\delta/x} \frac{1}{1-y} dy \le c(\alpha,\lambda)(1+\log\frac{2}{3}\sqrt{x\theta_n}). \qquad (41)$$

From (39) – (41) it follows

$$J_{12} \le c(\alpha,\lambda)(1+\log\frac{2}{3}\sqrt{x\theta_n}). \qquad (42)$$

Using the same arguments that led to (40)–(42), we obtain

$$J_{13} \le c(\alpha,\lambda)(1+\log\sqrt{x\theta_n}). \qquad (43)$$

From (38), (42) and (43) we have

$$J_1 \le c(\alpha,\lambda)(1+\log\sqrt{x\theta_n}). \qquad (44)$$

Let us estimate J_3. By (2) we have

$$J_3 \le (J_{31}+J_{32}+J_{33}), \qquad (45)$$

where

$$J_{31} = e^{-\frac{x}{2}} \left|m_{n,N}^\alpha(x)\right| \sum_{t\in D_3} e^{-\frac{t}{2}} \frac{\Gamma(Nt+\alpha+1)}{\Gamma(Nt+1)} (1-e^{-\delta})^{\alpha+1} \left|m_{n,N}^\alpha(t)\right|,$$

$$J_{32} = c(\alpha) n e^{-\frac{x}{2}} \left|m_{n+1,N}^\alpha(x) - m_{n-1,N}^\alpha(x)\right| \delta \sum_{t\in D_3} \frac{e^{-\frac{t}{2}}(t+\delta)^\alpha}{t-x} \left|m_{n,N}^\alpha(t)\right|,$$

$$J_{33} = c(\alpha) n e^{-\frac{x}{2}} \left| m_{n,N}^{\alpha}(x) \right| \delta \sum_{t \in D_3} \frac{e^{-\frac{t}{2}} (t+\delta)^{\alpha}}{t-x} \left| m_{n+1,N}^{\alpha}(t) - m_{n-1,N}^{\alpha}(t) \right|.$$

To estimate J_{31}, we note that the inequality (19) yields

$$J_{31} \leq e^{-\frac{x}{2}} \left| m_{n,N}^{\alpha}(x) \right| W,$$

hence, from (22) we obtain

$$J_{31} \leq c(\alpha,\lambda) \theta_n^{\frac{\alpha+1}{2}} \theta_n^{-\frac{1}{4}} x^{-\frac{\alpha}{2}-\frac{1}{4}} = c(\alpha,\lambda) \left(\frac{\theta_n}{x} \right)^{\frac{\alpha}{2}+\frac{1}{4}}. \tag{46}$$

To estimate J_{32}, we represent it as

$$J_{32} = J_{32}^1 + J_{32}^2 + J_{32}^3, \tag{47}$$

where

$$J_{32}^1 \leq c(\alpha,\lambda) n \theta_n^{-\frac{3}{4}} x^{-\frac{\alpha}{2}+\frac{1}{4}} \delta \sum_{\substack{t \in \Omega_\delta, \\ x+\sqrt{\frac{x}{\theta_n}} < t \leq \frac{\theta_n}{2} + \sqrt{\frac{x}{\theta_n}}}} \frac{t^{\alpha} \theta_n^{-\frac{1}{4}} t^{-\frac{\alpha}{2}-\frac{1}{4}}}{t-x} \leq$$

$$\leq c(\alpha,\lambda) x^{-\frac{\alpha}{2}+\frac{1}{4}} \delta \sum_{\substack{t \in \Omega_\delta, \\ x+\sqrt{\frac{x}{\theta_n}} < t \leq \frac{\theta_n}{2} + \sqrt{\frac{x}{\theta_n}}}} \frac{t^{\frac{\alpha}{2}-\frac{1}{4}}}{t-x}.$$

Let us examine three cases:

1) If $\alpha = \frac{1}{2}$, then

$$J_{32}^1 \leq c(\lambda) \left(\delta \frac{1}{\sqrt{\frac{x}{\theta_n}}} + \int\limits_{x+\sqrt{\frac{x}{\theta_n}}}^{\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}} \frac{1}{t-x} dt \right) \leq c(\lambda) \log \left(\frac{\frac{\theta_n}{2} - x}{\sqrt{\frac{x}{\theta_n}}} + 1 \right). \tag{48}$$

2) If $-1 < \alpha < \frac{1}{2}$, then

$$J_{32}^1 \leq c(\alpha,\lambda) x^{-\frac{\alpha}{2}+\frac{1}{4}} \left(\delta \frac{(x+\sqrt{\frac{x}{\theta_n}})^{\frac{\alpha}{2}-\frac{1}{4}}}{\sqrt{\frac{x}{\theta_n}}} + \int\limits_{x+\sqrt{\frac{x}{\theta_n}}}^{\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}} \frac{t^{\frac{\alpha}{2}-\frac{1}{4}}}{t-x} dt \right) \leq$$

$$\leq c(\alpha,\lambda)\int\limits_{x+\sqrt{\frac{x}{\theta_n}}}^{2x}\frac{dt}{t-x}+c(\alpha,\lambda)x^{-\frac{\alpha}{2}+\frac{1}{4}}\int\limits_{2x}^{\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}}t^{\frac{\alpha}{2}-\frac{5}{4}}dt\leq$$

$$\leq c(\alpha,\lambda)\log\sqrt{\theta_n x}. \tag{49}$$

3) If $\alpha>\frac{1}{2}$ then

$$J_{32}^1\leq c(\alpha,\lambda)\int\limits_{x+\sqrt{\frac{x}{\theta_n}}}^{2x}\frac{dt}{t-x}+c(\alpha,\lambda)x^{-\frac{\alpha}{2}+\frac{1}{4}}\int\limits_{2x}^{\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}}t^{\frac{\alpha}{2}-\frac{5}{4}}dt\leq$$

$$\leq c(\alpha,\lambda)\left(\log\sqrt{\theta_n x}+\left(\frac{\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}}{x}\right)^{\frac{\alpha}{2}-\frac{1}{4}}\right). \tag{50}$$

Consequently, from (48) – (50) we deduce the estimate:

$$J_{32}^1\leq c(\alpha,\lambda)\begin{cases}\log\left(\frac{\frac{\theta_n}{2}-x}{\sqrt{\frac{x}{\theta_n}}}+1\right), & \alpha=\frac{1}{2},\\ \log\sqrt{x\theta_n}, & \alpha\in(-1,\frac{1}{2}),\\ \log\sqrt{\theta_n x}+\left(\frac{\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}}{x}\right)^{\frac{\alpha}{2}-\frac{1}{4}}, & \alpha>\frac{1}{2}.\end{cases} \tag{51}$$

Further

$$J_{32}^2\leq c(\alpha,\lambda)nx^{-\frac{\alpha}{2}+\frac{1}{4}}\theta_n^{-\frac{3}{4}}\delta\times$$

$$\times\sum_{\substack{t\in\Omega_\delta,\\ \frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}<t\leq\frac{3\theta_n}{2}}}\frac{(t+\delta)^\alpha\theta_n^{-\frac{\alpha}{2}}}{t-x}\left[\theta_n\left(\theta_n^{\frac{1}{3}}+|t-\theta_n|\right)\right]^{-\frac{1}{4}}\leq$$

$$\leq c(\alpha,\lambda)x^{-\frac{\alpha}{2}+\frac{1}{4}}\theta_n^{\frac{\alpha}{2}}\int\limits_{\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}-\delta}^{\frac{3\theta_n}{2}}\left(\theta_n^{\frac{1}{3}}+|t-\theta_n|\right)^{-\frac{1}{4}}\frac{dt}{t-x}=$$

$$=c(\alpha,\lambda)x^{-\frac{\alpha}{2}+\frac{1}{4}}\theta_n^{\frac{\alpha}{2}}\left[\int\limits_{\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}-\delta}^{\theta_n-\theta_n^{\frac{1}{3}}}+\int\limits_{\theta_n-\theta_n^{\frac{1}{3}}}^{\frac{3\theta_n}{2}}\right]\left(\theta_n^{\frac{1}{3}}+|t-\theta_n|\right)^{-\frac{1}{4}}\frac{dt}{t-x}\leq$$

$$\leq c(\alpha,\lambda)\left(\frac{\theta_n}{x}\right)^{\frac{\alpha}{2}-\frac{1}{4}}\log\frac{\theta_n-x}{\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}-x}, \tag{52}$$

$$J_{32}^3\leq c(\alpha,\lambda)nx^{-\frac{\alpha}{2}+\frac{1}{4}}\theta_n^{-\frac{3}{4}}\delta\sum_{\substack{t\in\Omega_\delta,\\ \frac{3\theta_n}{2}<t<\infty}}\frac{e^{-\frac{t}{4}}(t+\delta)^\alpha\theta_n^{-\frac{\alpha}{2}}}{t-x}\leq$$

$$\leq c(\alpha,\lambda)x^{-\frac{\alpha}{2}+\frac{1}{4}}\theta_n^{-\frac{\alpha}{2}+\frac{1}{4}}\delta\sum_{\substack{t\in\Omega_\delta,\\ \frac{3\theta_n}{2}<t<\infty}}e^{-\frac{t}{4}}t^{\alpha-1}\leq c(\alpha,\lambda)e^{-\frac{3\theta_n}{8}}. \tag{53}$$

From (47) and (51) – (53) we obtain the estimate

$$J_{32}\leq c(\alpha,\lambda)\begin{cases}\log(nx+1), & \alpha\in(-1,\frac{1}{2}),\\ \left(\frac{\theta_n}{x}\right)^{\frac{\alpha}{2}-\frac{1}{4}}\log\frac{\theta_n-x}{\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}-x}, & \alpha>\frac{1}{2}.\end{cases} \tag{54}$$

Now we estimate J_{33} using the same scheme as for J_{32}. We have

$$J_{33}=J_{33}^1+J_{33}^2+J_{33}^3, \tag{55}$$

where

$$J_{33}^1\leq c(\alpha,\lambda)n\theta_n^{-\frac{1}{4}}x^{-\frac{\alpha}{2}-\frac{1}{4}}\delta\sum_{\substack{t\in\Omega_\delta,\\ x+\sqrt{\frac{x}{\theta_n}}<t\leq\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}}}\frac{t^\alpha\theta_n^{-\frac{3}{4}}t^{-\frac{\alpha}{2}+\frac{1}{4}}}{t-x}\leq$$

$$\leq c(\alpha,\lambda)x^{-\frac{\alpha}{2}-\frac{1}{4}}\delta\sum_{\substack{t\in\Omega_\delta,\\ x+\sqrt{\frac{x}{\theta_n}}<t\leq\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}}}\frac{t^{\frac{\alpha}{2}+\frac{1}{4}}}{t-x}\leq$$

$$\leq c(\alpha,\lambda)\begin{cases}\log\left(\frac{\frac{\theta_n}{2}-x}{\sqrt{\frac{x}{\theta_n}}}+1\right), & \alpha=-\frac{1}{2},\\ \log\sqrt{x\theta_n}, & \alpha\in(-1,-\frac{1}{2}),\\ \log\sqrt{\theta_n x}+\left(\frac{\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}}{x}\right)^{\frac{\alpha}{2}+\frac{1}{4}}, & \alpha>-\frac{1}{2}.\end{cases} \tag{56}$$

$$J_{33}^2\leq c(\alpha,\lambda)n\theta_n^{-\frac{1}{4}}x^{-\frac{\alpha}{2}-\frac{1}{4}}\delta\sum_{\substack{t\in\Omega_\delta,\\ \frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}<t\leq\frac{3\theta_n}{2}}}\frac{t^{\frac{\alpha}{2}}\theta_n^{-\frac{3}{4}}\left(\theta_n^{1/3}+|t-\theta_n|\right)^{1/4}}{t-x}\leq$$

$$\leq c(\alpha,\lambda)x^{-\frac{\alpha}{2}-\frac{1}{4}}\theta_n^{\frac{\alpha}{2}+\frac{1}{4}}\delta \sum_{\substack{t\in\Omega_\delta,\\ \frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}<t\leq\theta_n}} \left(\frac{\theta_n^{1/3}+\theta_n-t}{t}\right)^{1/4}\frac{1}{t-x}+$$

$$+c(\alpha,\lambda)x^{-\frac{\alpha}{2}-\frac{1}{4}}\theta_n^{\frac{\alpha}{2}+\frac{1}{4}}\delta \sum_{\substack{t\in\Omega_\delta,\\ \theta_n<t\leq\frac{3\theta_n}{2}}} \frac{\left(\theta_n^{1/3}+t-\theta_n\right)^{1/4}}{t^{5/4}}\leq$$

$$\leq c(\alpha,\lambda)\left(\frac{\theta_n}{x}\right)^{\frac{\alpha}{2}+\frac{1}{4}}\left(1+\log\frac{\theta_n-x}{\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}-x}\right). \tag{57}$$

$$J_{33}^3\leq c(\alpha,\lambda)n\theta_n^{-\frac{1}{4}}x^{-\frac{\alpha}{2}-\frac{1}{4}}\delta\sum_{\substack{t\in\Omega_\delta,\\ \frac{3\theta_n}{2}<t<\infty}}\frac{(t+\delta)^\alpha e^{-t/4}}{t-x}\leq$$

$$\leq c(\alpha,\lambda)n^{3/4}x^{-\frac{\alpha}{2}-\frac{1}{4}}\delta\sum_{\substack{t\in\Omega_\delta,\\ \frac{3\theta_n}{2}<t<\infty}}t^{\alpha-1}e^{-t/4}\leq c(\alpha,\lambda)e^{-3n/2}. \tag{58}$$

From (55) – (58) we obtain

$$J_{33}\leq c(\alpha,\lambda)\begin{cases}\log(nx+1), & \alpha\in(-1,-\frac{1}{2}),\\ \log(n+1), & \alpha=-\frac{1}{2},\\ \log\sqrt{x\theta_n}+\left(\frac{\theta_n}{x}\right)^{\frac{\alpha}{2}+\frac{1}{4}}\left(1+\log\frac{\theta_n-x}{\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}-x}\right), & \alpha>-\frac{1}{2}.\end{cases} \tag{59}$$

In turn, from (45), (46), (54) and (59) we deduce the estimate

$$J_{3}\leq c(\alpha,\lambda)\begin{cases}\log(nx+1), & \alpha\in(-1,-\frac{1}{2}),\\ \log(n+1), & \alpha=-\frac{1}{2},\\ \log\sqrt{x\theta_n}+\left(\frac{\theta_n}{x}\right)^{\frac{\alpha}{2}+\frac{1}{4}}\left(1+\log\frac{\theta_n-x}{\frac{\theta_n}{2}+\sqrt{\frac{x}{\theta_n}}-x}\right), & \alpha>-\frac{1}{2}.\end{cases} \tag{60}$$

Combining (34), (44) and (60), we see that

$$\lambda_{n,N}^{\alpha}(x) \le c(\alpha,\lambda) \begin{cases} \log(nx+1), & \alpha \in (-1,-\frac{1}{2}), \\ \log(n+1), & \alpha = -\frac{1}{2}, \\ \log(n+1) + \left(\frac{n}{x}\right)^{\frac{\alpha}{2}+\frac{1}{4}}, & \alpha > -\frac{1}{2}. \end{cases}$$

Therefore, (14) is proved. This completes the proof of the theorem.

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