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ON TWO NEW MEANS OF TWO ARGUMENTS III

Abstract. In this paper we establish two sided inequalities for the following two new means

$$X = X(a,b) = Ae^{G/P-1}, \quad Y = Y(a,b) = Ge^{L/A-1},$$

where A, G, L and P are the arithmetic, geometric, logarithmic, and Seiffert means, respectively. As an application, we refine many other well known inequalities involving the identric mean I and the logarithmic mean L.

Key words: *inequalities, means of two arguments, identric mean, logarithmic mean*

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1. Introduction. The study of the inequalities involving the classical means such as arithmetic mean A, geometric mean G, identric mean I and logarithmic mean L have been of the extensive interest for several authors, e.g., see [2, 3, 9, 11, 21, 22, 30, 31, 32, 40].

In 2011, Sándor [27] introduced a new mean $X(a,b)$ for two positive real numbers a and b, defined by

$$X = X(a,b) = Ae^{G/P-1},$$

where $A = A(a,b) = (a+b)/2$, $G = G(a,b) = \sqrt{ab}$, and

$$P = P(a,b) = \frac{a-b}{2\arcsin\left(\dfrac{a-b}{a+b}\right)}, \quad a \neq b, \quad P(a,a) = a,$$

are the arithmetic mean, geometric mean, and Seiffert mean [38], respectively.

For $p \in \mathbb{R}$ and $a, b > 0$ with $a \neq b$, we define the pth power mean $M_p(a,b)$ and the pth power-type Heronian mean $H_p(a,b)$ by

$$M_p = M_p(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases}$$

and

$$H_p = H_p(a,b) = \begin{cases} \left(\dfrac{a^p + (ab)^{p/2} + b^p}{3}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases}$$

respectively.

The present paper contains essentially results on the X mean, in particular, several inequalities involving the X mean and the refinements of the following double inequalities are established.

For all $a, b > 0$ with $a \neq b$

$$M_p < X < M_q \tag{1}$$

holds if and only if $p \leq 1/3$ and $q \geq \log(2)/(1 + \log(2)) \approx 0.4094$, and

$$H_\alpha < X < H_\beta \tag{2}$$

holds if and only if $\alpha \leq 1/2$ and $\beta \geq \log(3)/(1 + \log(2)) \approx 0.6488$.

In the same paper, Sándor [27] introduced another mean $Y(a,b)$ for two positive real a and b by

$$Y = Y(a,b) = Ge^{L/A-1},$$

where

$$L = L(a,b) = \frac{a-b}{\log(a) - \log(y)}, \quad a \neq b, \quad L(a,a) = a,$$

is a logarithmic mean. For two positive real numbers a and b, the identric mean and harmonic mean are defined by

$$I = I(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{1/(a-b)}, \quad a \neq b, \quad I(a,a) = a,$$

and

$$H = H(a,b) = 2ab/(a+b),$$

respectively. For the sharp inequalities of logarithmic and identric means, see ([25, 18]). See also [23], [24], [36]. In 2012, the X mean appeared in [27]. In 2014, X and Y means were published in the journal of Notes on Number Theory and Discrete Mathematics [29]. For the historical background and the generalization of these means we refer the reader to, e.g. [3, 9, 17, 21, 22, 28, 30, 31, 32, 33, 34, 40]. Connections of these means and the trigonometric or hyperbolic inequalities are given in [5, 27, 29, 32].

In [29], Sándor proved inequalities for X and Y means in terms of other classical means as well as their relationship. Let us recall some of the results for easy reference.

Theorem 1. [29] *For $a, b > 0$ with $a \neq b$, the following inequalities*

1) $G < \dfrac{AG}{P} < X < \dfrac{AP}{2P-G} < P,$

2) $H < \dfrac{LG}{A} < Y < \dfrac{AG}{2A-L} < G,$

3) $1 < \dfrac{L^2}{IG} < \dfrac{L \cdot e^{G/L-1}}{G} < \dfrac{PX}{AG},$

4) $H < \dfrac{G^2}{I} < \dfrac{LG}{A} < \dfrac{G(A+L)}{3A-L} < Y$

hold.

In [5] a series expansion of X and Y was presented.

Theorem 2. [5] *For $a, b > 0$ with $a \neq b$, the following inequalities*

1) $\dfrac{1}{e}(G+H) < Y < \dfrac{1}{2}(G+H),$

2) $G^2 I < IY < IG < L^2,$

3) $\dfrac{G-Y}{A-L} < \dfrac{Y+G}{2A} < \dfrac{3G+H}{4A} < 1,$

4) $L < \dfrac{2G+A}{3} < X < L(X, A) < P < \dfrac{2A+G}{3} < I,$

5) $2\left(1 - \dfrac{A}{P}\right) < \log\left(\dfrac{X}{A}\right) < \left(\dfrac{P}{A}\right)^2$

are true.

Chu et al. [10] and Zhou et al. [41] proved the double inequalities (1) and (2), respectively.

This paper is organized as follows: in Section 1, we give the introduction. Section 2 consists of main results and remarks. In Section 3, some connections of X, Y and other means are given with trigonometric and hyperbolic functions. Some lemmas are also proved in this section which will be used in the proof of the main result. Section 4 deals with the proof of the main result and corollaries. In the computations we have used also the Mathematica software (see e.g.[26]).

2. Main result and motivation. Making contribution to the topic, we refine some previous results appeared in the literature [1, 2, 5, 10, 41, 29] as well as establish new results involving the X mean.

Theorem 3. *For $a, b > 0$*

$$\alpha G + (1-\alpha)A < X < \beta G + (1-\beta)A, \tag{3}$$

with the best possible constants $\alpha = 2/3 \approx 0.6667$ and $\beta = (e-1)/e \approx \approx 0.6321$, and

$$A + G - \alpha_1 P < X < A + G - \beta_1 P, \tag{4}$$

with the best possible constants $\alpha_1 = 1$ and $\beta_1 = \pi(e-1)/(2e) \approx 0.9929$.

Remark. In [29, Theorem 2.7], Sándor proved that for $a \neq b$,

$$X < A\left[\frac{1}{e} + \left(1 - \frac{1}{e}\right)\frac{G}{P}\right], \tag{5}$$

and

$$Y < G\left[\frac{1}{e} + \left(1 - \frac{1}{e}\right)\frac{L}{A}\right]. \tag{6}$$

As $A/P > 1$, the right side of (3) gives a slight improvement to (5). From (6), as clearly $G \cdot L/A < A$, we get a similar inequality. The second inequality in (4) could be a counterpart of the inequality $L + G - A < Y$ studied in [5, Theorem 20].

H. Alzer [1] proved the following inequalities:

$$1 < (A+G)/(L+I) < e/2, \tag{7}$$

where the constants 1 and $2/e$ are the best possible ones. The following result improves among others the right side of (7).

Theorem 4. *For* $a \neq b$

$$(A+G)/e < X < M_q < (L+I)/2 < (A+G)/2, \tag{8}$$

where $q = \log(2)/(1+\log(2)) \approx 0.4094$ *is the best possible constant.*

Remark. Particularly, (8) implies that

$$X < (L+I)/2, \tag{9}$$

which is new. Since $L < X < I$ (see Theorems 1 and 2), X is less than the arithmetic mean of L and I. In fact, by left side of (1), and by $L < M_{1/3}$ (see [25], [16]), and $L < I < M_{2/3}$ (see [25]; see also [30], for other references), we get also

$$L < M_{1/3} < X < M_q < (L+I)/2 < I < M_{2/3}. \tag{10}$$

Theorem 5. *For* $a \neq b$

$$A+G-P < X < P^2/A < (A+G)/2. \tag{11}$$

Remark. The right hand side of (11) offers another refinement to $X < (A+G)/2$. An improvement of $P^2 > XA$ appears in [29, Theorem 2.9]:

$$P^2 > (A^2((A+G)/2)^4)^{1/3} > AX,$$

so (11) could be further refined. For the following inequalities

$$L < \frac{2G+A}{3} < A+G-P < X < \sqrt{PX} < \frac{A+G}{2} < \tag{12}$$

$$< \frac{P+X}{2} < P < \frac{2A+G}{3} < I,$$

one can see that the first inequality is Carlson's inequality, while the second written in the form $P < (2A+G)/3$ is due to Sándor [33]. The third inequality is Theorem 2.10 in [29], while the fourth, written as $PX < ((A+G)/2)^2$ is Theorem 2.11 of [29]. The inequality $(P+X)/2 < P$ follows by $X < P$, while the last two inequalities are due to Sándor ([33, 31]).

Theorem 6. *For $a \neq b$*

$$M_{1/2} < (P+X)/2 < M_k, \tag{13}$$

where $k = (5\log 2 + 2)/(6(\log 2 + 1)) \approx 0.5380$.

Remark. One has

$$L < \frac{2G+A}{3} < X < \frac{L+I}{2} < \frac{A+G}{2} < \frac{P+X}{2} < P < \frac{2A+G}{3} < I \tag{14}$$

and

$$\sqrt{AG} < \sqrt{PX} < \frac{A+G}{2}. \tag{15}$$

Inequalities (15) show that $\sqrt{PX}$ lies between the geometric and arithmetic means of A and G, while (12) shows among others that $(A+G)/2$ lies between the geometric and arithmetic means of P and X.

Theorem 7. *The following inequalities*

$$M_p \leq M_{1/3} < (2G+A)/3 < X, \quad \text{for} \quad p \leq 1/3, \tag{16}$$

$$H_\alpha \leq H_{1/2} < (2G+A)/3 < X, \quad \text{for} \quad \alpha \leq 1/2, \tag{17}$$

hold.

Theorem 8. *For $a \neq b$*

$$(AX)^{1/\alpha_2} < P < (AX^{\beta_2})^{1/(1+\beta_2)}$$

with the best possible constants $\alpha_2 = 2$ *and* $\beta_2 = \log(\pi/2)/\log(2e/\pi) \approx$ ≈ 0.8234.

3. Preliminaries and lemmas. We use the following result by Biernacki and Krzyż [8] in studying the monotonicity of certain power series.

Lemma 1. *Let* $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $C(x) = \sum_{n=0}^{\infty} c_n x^n$ *be two real power series converging on the interval* $(-R, R)$, $0 < R \leq \infty$. *If the sequence* $\{a_n/c_n\}$ *is increasing (decreasing) and* $c_n > 0$ *for all* n, *then the function* $A(x)/C(x)$ *is also increasing (decreasing) on* $(0, R)$.

For $|x| < \pi$, the following power series expansions

$$x \cot x = 1 - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n}, \tag{18}$$

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}, \tag{19}$$

$$\coth \mathrm{x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}, \tag{20}$$

can be found in [13, 1.3.1.4 (2)–(3)]. Here B_{2n} are the even-indexed Bernoulli numbers (see [12, p. 231]). We get the following expansions directly from (19) and (20)

$$\frac{1}{(\sin x)^2} = -(\cot x)' = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|(2n-1)x^{2n-2}, \tag{21}$$

$$\frac{1}{(\sinh x)^2} = -(\coth x)' = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}(2n-1)|B_{2n}|x^{2n-2}. \tag{22}$$

For the following expansion formula

$$\frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n} \tag{23}$$

see [15].

For easy reference we recall the following lemma from [5, 6].

Lemma 2. *For* $x = \arcsin((a-b)/(a+b))$ *and* $y = (1/2)\log(a/b)$, *with* $a > b > 0$, *one has*

$$\frac{P}{A} = \frac{\sin(x)}{x}, \ \frac{G}{A} = \cos(x), \ \frac{H}{A} = \cos(x)^2, \ \frac{X}{A} = e^{x\cot(x)-1},$$

$$\frac{L}{G} = \frac{\sinh(y)}{y}, \ \frac{L}{A} = \frac{\tanh(y)}{y}, \ \frac{H}{G} = \frac{1}{\cosh(y)}, \ \frac{Y}{G} = e^{\tanh(y)/y-1},$$

$$\log\left(\frac{I}{G}\right) = \frac{A}{L} - 1, \quad \log\left(\frac{Y}{G}\right) = \frac{L}{A} - 1.$$

Remark. It is well known that many inequalities involving the means can be obtained from the classical inequalities for trigonometric functions. For example, the following inequality

$$e^{(x/\tanh(x)-1)/2} < \frac{\sinh(x)}{x}, \quad x > 0,$$

recently appeared in [7, Theorem 1.6], is equivalent to

$$\frac{\sinh(x)}{x} > e^{x/\tanh(x)-1}\frac{x}{\sinh(x)}. \tag{24}$$

By Lemma 2, this can be written as

$$\frac{L}{G} > \frac{I}{G}\cdot\frac{G}{L} = \frac{I}{L},$$

or

$$L > \sqrt{IG}. \tag{25}$$

The inequality (25) was proved by Alzer [3].

The following trigonometric inequalities (see [7, Theorem 1.5]) imply an other double inequality for Seiffert mean P,

$$\begin{cases} \exp\left(\frac{1}{2}\left(\frac{x}{\tan x}-1\right)\right) < \frac{\sin x}{x} < \exp\left(\left(\log\frac{\pi}{2}\right)\left(\frac{x}{\tan x}-1\right)\right) & x \in (0,\pi/2), \\ \sqrt{AX} < P < A\left(\frac{X}{A}\right)^{\log(\pi/2)}. \end{cases} \tag{26}$$

The second mean inequality in (26) was also pointed out by Sándor (see [29, Theorem 2.12]). For various related trigonometric and hyperbolic inequalities, see also [14], [19].

Lemma 3. [4, Theorem 2] *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and differentiable on (a,b). Let $g^{'}(x) \neq 0$ on (a,b). If $f^{'}(x)/g^{'}(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \textit{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f^{'}(x)/g^{'}(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 4. *The following function*

$$h(x) = \frac{\log(x/\sin(x))}{\log(e^{1-x/\tan(x)}\sin(x)/x)}$$

is strictly decreasing from $(0, \pi/2)$ *onto* $(\beta_2, 1)$, *where*

$$\beta_2 = \log(\pi/2)/\log(2e/\pi) \approx 0.8234.$$

In particular, for $x \in (0, \pi/2)$ *we have*

$$\left(\frac{e^{1-x/\tan(x)}\sin(x)}{x}\right)^{\beta_2} < \frac{x}{\sin(x)} < \left(\frac{e^{1-x/\tan(x)}\sin(x)}{x}\right). \tag{27}$$

Proof. Let

$$h(x) = \frac{h_1(x)}{h_2(x)} = \frac{\log(x/\sin(x))}{\log(e^{1-x/\tan(x)}\sin(x)/x)}$$

for $x \in (0, \pi/2)$. Differentiating with respect to x we get

$$\frac{h_1'(x)}{h_2'(x)} = \frac{1 - x/\tan(x)}{(x/\sin(x))^2 - 1} = \frac{A_1(x)}{B_1(x)}.$$

Using the expansion formula we have

$$A_1(x) = \sum_{n=1}^{\infty} \frac{2^{2n}2n}{(2n)!}|B_{2n}|x^{2n} = \sum_{n=1}^{\infty} a_n x^{2n}$$

and

$$B_1(x) = \sum_{n=1}^{\infty} \frac{2^{2n}2n}{(2n)!}|B_{2n}|(2n-1)x^{2n} = \sum_{n=1}^{\infty} b_n x^{2n}.$$

Let $c_n = a_n/b_n = 1/(2n-1)$, which is the decreasing in $n \in \mathbb{N}$. Thus, by Lemma 1 $h_1'(x)/h_2'(x)$ is strictly decreasing in $x \in (0, \pi/2)$. In turn, this implies by Lemma 3 that $h(x)$ is strictly decreasing in $x \in (0, \pi/2)$. Applying L'Hôpital rule, we get $\lim_{x\to 0} h(x) = 1$ and $\lim_{x\to\pi/2} h(x) = \beta_2$. This completes the proof. □

Remark. It is observed that the inequalities in (27) coincide with the trigonometric inequalities given in (26). Here Lemma 4 gives a new and an optimal proof for these inequalities.

Lemma 5. *The following function*

$$f(x) = \frac{1 - e^{x/\tan(x)-1}}{1 - \cos(x)}$$

is strictly decreasing from $(0,\pi/2)$ *onto* $((e-1)/e, 2/3)$ *where* $(e-1)/e \approx \approx 0.6321$*. In particular, for* $x \in (0,\pi/2)$*, we have*

$$\frac{1}{\log(1+(e-1)\cos(x))} < \frac{\tan(x)}{x} < \frac{1}{1+\log((1+2\cos(x))/3)}.$$

Proof. Write $f(x) = f_1(x)/f_2(x)$, where $f_1(x) = 1 - e^{x/\tan(x)-1}$ and $f_2(x) = 1-\cos(x)$ for all $x \in (0\pi/2)$. Clearly, $f_1(x) = 0 = f_2(x)$. Differentiating with respect to x, we get

$$\frac{f_1'(x)}{f_2'(x)} = \frac{e^{x/\tan(x)-1}}{\sin(x)^3}\left(\frac{x}{\sin(x)^2} - \frac{\cos(x)}{\sin(x)}\right) = f_3(x).$$

Again

$$f_3'(x) = -\frac{e^{x/\tan(x)-1}}{\sin(x)^3}\left(c(x)-2\right),$$

where

$$c(x) = x\left(\frac{\cos(x)}{\sin(x)} + \frac{x}{\sin(x)^2}\right).$$

In order to show that $f_3' < 0$, it is enough to prove that

$$c(x) > 2,$$

which is equivalent to

$$\frac{\sin(x)}{x} < \frac{x+\sin(x)\cos(x)}{2\sin(x)}.$$

Applying the Cusa-Huygens [20] inequality

$$\frac{\sin(x)}{x} < \frac{\cos(x)+2}{3},$$

we get

$$\frac{\cos(x)+2}{3} < \frac{x+\sin(x)\cos(x)}{2\sin(x)},$$

which is equivalent to $(\cos(x)-1)^2 > 0$. Thus $f_3' > 0$, clearly f_1'/f_2' is strictly decreasing in $x \in (0,\pi/2)$. By Lemma 3, we conclude that the function $f(x)$ is strictly decreasing in $x \in (0,\pi/2)$. The limiting values follow easily. This completes the proof of the lemma. □

Lemma 6. *The following function*

$$f_4(x) = \frac{\sin(x)}{x\left(\cos(x) - e^{x\cot(x)-1} + 1\right)}$$

is strictly increasing from $(0, \pi/2)$ *onto* $(1, c)$, *where* $c = 2e/(\pi(e-1)) \approx$ ≈ 1.0071. *In particular, for* $x \in (0, \pi/2)$ *we have*

$$1 + \cos(x) - e^{x/\tan(x)-1} < \frac{\sin(x)}{x} < c(1 + \cos(x) - e^{x/\tan(x)-1}).$$

Proof. Differentiating with respect to x we get

$$f_4'(x) = \frac{e(x - \sin(x))\left(e\cos(x) - (x + \sin(x))e^{x\cot(x)}\csc(x) + e\right)}{x^2\left(e\cos(x) - e^{x\cot(x)} + e\right)^2}.$$

Let $f_5(x) = \log\left((x + \sin(x))e^{x\cot(x)}/\sin(x)\right) - \log(e\cos(x) + e)$ for $x \in (0, \pi/2)$. Differentiation yields

$$f_5'(x) = \frac{2 - x\left(\cot(x) + x\csc^2(x)\right)}{x + \sin(x)},$$

which is negative by the proof of Lemma 5, and $\lim_{x\to 0} f_5(x) = 0$. This implies that $f_4'(x) > 0$, and $f_4(x)$ is strictly increasing. The limiting values follow easily. This implies the proof. □

Lemma 7. *For* $a \neq b$, *one has*

$$M_{1/3} < (2G + A)/3. \tag{28}$$

Proof. Let $G = G(a, b)$, etc. Divide both sides with b and put $a/b = x$. Then inequality (28) becomes the following:

$$\left(\frac{x^{1/3} + 1}{2}\right)^3 < 4(x + 4\sqrt{x} + 1). \tag{29}$$

Let $x = t^6$, where $t > 1$. Then raising both sides of (29) to 3th power, after elementary transformations we get,

$$t^6 - 9t^4 + 16t^3 - 9t^2 + 1 > 0,$$

which can be written as $(t-1)^4(t^2 + 4t + 1) > 0$, so it is true. Thus (29) and (28) are proved. □

Since $L < M_{1/3}$, by (28) we get a new proof , as well as a refinement of Carlson's inequality $L < (2G + A)/3$.

Lemma 8. *The inequality*

$$H_{1/2} < (2G + A)/3 \tag{30}$$

holds for $a \neq b$.

Proof. By definition of H_α one has

$$H_{1/2} = ((\sqrt{a} + (ab)^{1/4} + \sqrt{b})/3)^2 = (\sqrt{2(A+G)} + \sqrt{G})^2/9,$$

by remarking that $\sqrt{a} + \sqrt{b} = \sqrt{2(A+G)}$. Therefore, (2) can be written equivalently as

$$(2(A+G) + 2\sqrt{2G(A+G)} + G)/9 < (2G + A)/3. \tag{31}$$

Now, it is immediate that (31) becomes, after elementary computations

$$A + 3G > 2\sqrt{2G(A+G)}, \tag{32}$$

or by raising both sides to the 2th power:

$$A^2 + 6AG + 9G^2 > 8AG + 8G^2,$$

which becomes $(A - G)^2 > 0$, true. Thus (32) and (31) are proved, and (30) follows. □

4. Proof of main result.

Proof of Theorem 3. By Lemma 5

$$\frac{e-1}{e} < \frac{1 - 1/e^{1-x/\tan(x)}}{\cos(x)/e^{1-x/\tan(x)} - 1/e^{1-x/\tan(x)}} < \frac{2}{3}.$$

Now we get the proof of (3) by utilizing the Lemma 2. The proof of (4) follows easily from Lemmas 2 and 5. □

Proof of Theorem 4. The second inequality of (8) is the right hand side of (1). In [2], Alzer and Qiu proved the third inequality of (8). The last inequality is the left side of (7). By [10] and [2], q is the best possible constant in both sides.

Now let us prove the first inequality of (8). By Lemma 2 this becomes equivalent to $1+\cos(x) < e^{x\cot(x)}$, or

$$\log(1+\cos(x)) < x\cot(x), \quad x \in (0, \pi/2). \tag{33}$$

Now, by the classical inequality $\log(1+t) < t \quad (t > 0)$, applied to $t = \cos(x)$, we get $\log(1+\cos(x)) < \cos(x)$. Now $\cos(x) < x\cot(x) = x\cos(x)/\sin(x)$ is true by $\sin(x) < x$. The proof of (33) follows. □

One has the following relation, in analogy with relation (7) of Theorem 2 for the mean Y:

Corollary. *The inequality $(A+G)/e < X < (A+G)/2$ holds. The constants e and 2 are the best possible ones.*

The inequalities $(A+G)/e < X$ and $(2G+A)/3 < X$ are not comparable.

Proof of Theorem 5. The second inequality of (11) appeared in [27] in the form $P^2 > AX$. The last inequality follows by $P < (2A+G)/3$. Indeed, one has $((2A+G)/3)^2 < A(A+G)/2$ becomes $2G^2 < A^2 + AG$, and this is true by $G < A$. □

Proof of Theorem 6. By [29, Theorem 2.10], one has $P + X > A + G$, and remarking that $(A+G)/2 = M_{1/2}$, the left side of (13) follows. For the right hand side of (13), we will use $P < M_t$ with $t = 2/3$ (see [33]), and $X < M_q$ ([10]), where $q = (\log 2)/(\log 2 + 1)$. On the other hand the function $f(t) = M_t$ is known to be strictly log-concave for $t > 0$ (see [35]). Particularly, this implies that $f(t)$ is strictly concave. Thus $(M_t + M_q)/2 < M_{(t+q)/2}$. As $(t+q)/2 = k \approx 0.5380$, the result follows. □

Corollary. *One has the following two sets of inequalities:*

1) $PX > PL > AG$,
2) $IL > PL > AG$.

Proof. The first inequality of (1) follows by $X > L$, while the second appears in [33]. The first inequality of (2) follows by $I > P$, while the second one is the same as the second one in (1). □

Remark. Particularly in Corollary , (2) improves Alzer's inequality $IL > AG$. Inequality (1) improves $PX > AG$, which appears in [29].

Corollary. *One has*

1) $X > A(P+G)/(3P-G) > (2G+A)/3 > L$,
2) $P^2/A > X > (P+G)/2$.

Proof. The first two inequalities of (1) appear in [29, Theorem 2.5 and Remark 2.3]. The second inequality of (2) follows by the first inequality of (1) and the remark that $A/(3P-G) > 1/2$, since this is $P < (2A+G)/3$; while the first one is $P^2 > AX$ ([27]). □

Remark. Since it is known that $P > (2/\pi)A$ (due to Seiffert, see [33]). By $X > (P+G)/2$ we get the inequality $X > [(2/\pi)A+G]/2$, which is not comparable with $(A+G)/e < X$.

Proof of Theorem 7. The first inequality of (16) follows, since the function $f(t) = M_t$ is known to be strictly increasing. The second inequality follows by (28), while the third one can be found in Theorem 2.

It is known that H_p is an increasing function of p. Therefore, the proof of (17) follows by (30). □

Corollary. *For $a, b > 0$ with $a \neq b$, we have*

$$\frac{I}{L} < \frac{L}{G} < 1 + \frac{G}{H} - \frac{I}{G}. \tag{34}$$

Proof. The first inequality is due to Alzer [3], while the second inequality follows from the fact that the function

$$x \mapsto \frac{1 - e^{x/\tanh(x)-1}}{1-\cosh(x)} : (0,\infty) \to (0,1)$$

is strictly decreasing. The proof of the monotonicity of the function is the analogue to the proof of Lemma 5. □

The right hand side of (34) may be written as $L + I < G + A$ (by $H = G^2/A$), and this is due to Alzer (see [2, 30] for history of early results).

Proof of Theorem 8. The proof follows easily from Lemma 4. □

In [37] (see also [39]), Seiffert proved that

$$\frac{2}{\pi}A < P \tag{35}$$

for all $a, b > 0$ with $a \neq 0$. As a counterpart of the above result we give the following inequalities.

Corollary. *The following inequalities*

$$\frac{1}{e}A < \frac{\pi}{2e}P < X < P$$

hold true for $a, b > 0$ with $a \neq b$.

Proof. The first inequality follows from (35). For the proof of the second inequality we write by Lemma 2

$$f_5'(x) = \frac{X}{P} = \frac{xe^{x/\tan(x)-1}}{\sin(x)} = f_5(x)$$

for $x \in (0, \pi/2)$. Differentiation gives

$$\frac{e^{x/tan(x)-1}}{\sin(x)}\left(1 - \frac{x^2}{\sin(x)^2}\right) < 0.$$

Hence the function f_5 is strictly decreasing in x, with

$$\lim_{x\to 0} f_5(x) = 1 \quad \text{and} \quad \lim_{x\to\pi/2} f_5(x) = \pi/(2e) \approx 0.5779.$$

This implies the proof. □

We finish this paper by giving the following open problem and a conjecture.

Open problem. *What are the best positive constants a and b, such that*

$$M_a < (P + X)/2 < M_b.$$

Conjecture. *For $a \neq b$, one has $PX > IL$.*

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