Probl. Anal. Issues Anal. Vol. 7 (25), No. 2, 2018, pp. 20–38
DOI: 10.15393/j3.art.2018.4630

UDC 517.983

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GENERALIZED RESOLVENTS OF OPERATORS GENERATED BY INTEGRAL EQUATIONS

Abstract. We define a minimal operator L_0 generated by an integral equation with an operator measure and give a description of the adjoint operator L_0^*. We prove that every generalized resolvent of L_0 is an integral operator and give a description of boundary value problems associated to generalized resolvents.

Key words: *integral equation, Hilbert space, symmetric operator, generalized resolvent, boundary value problem*

2010 Mathematical Subject Classification: *46G12, 45N05, 47A10*

1. Introduction. In [13], A.V. Straus described generalized resolvents of a symmetric operator generated by formally selfadjoint differential expression in the scalar case. In [4] these results were extended to the operator case. Further, the generalized resolvents of differential operators were studied in many works (a detailed bibliography is available, for example, in [10], [12]).

In this paper, we consider the integral equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}(s)y(s) - iJ\int_a^t f(s)ds, \qquad (1)$$

where y is an unknown function; $f \in L_2(H; a, b)$, J is an operator in a separable Hilbert space H, $J = J^*$, $J^2 = E$ (E is the identical operator); $\mathbf{p}$ is an operator-valued measure defined on Borel sets $\Delta \subset [a, b]$ and taking values in the set of linear bounded operators acting in H; $\int_{t_0}^t$ stands for $\int_{[t_0 t)}$ if $t_0 < t$, for $-\int_{[t_0 t)}$ if $t_0 > t$, and for 0 if $t_0 = t$. We assume that the measure $\mathbf{p}$ is self-adjoint, and $\mathbf{p}$ has a bounded variation, and the set $\mathcal{S}_{\mathbf{p}}$ of single-point atoms of measure $\mathbf{p}$ can be arranged in the form of an increasing sequence.

We define the minimal operator L_0 generated by equation (1) and give a description of the adjoint operator L_0^*. We prove that every generalized resolvent of L_0 is an integral operator. Unlike differential operators, the domain and the range of the characteristic function of a generalized resolvent are spaces of sequences. Moreover, we give a description of generalized resolvents in terms of boundary value problems.

2. Preliminary assertions. Let H be a separable Hilbert space with scalar product $(\cdot\,,\cdot)$ and norm $\|\cdot\|$. We consider a function $\Delta \to \mathbf{P}(\Delta)$ defined on Borel sets $\Delta \subset [a,b]$ and taking values in the set of bounded linear operators acting in H. The function $\mathbf{P}$ is called an operator measure on $[a,b]$ (see, e. g., [3, ch. 5]) if it is zero on the empty set and the equality $\mathbf{P}\left(\bigcup_{n=1}^{\infty}\Delta_n\right) = \sum_{n=1}^{\infty}\mathbf{P}(\Delta_n)$ holds for disjoint Borel sets Δ_n, where the series converges weakly. Further, we extend any measure $\mathbf{P}$ on $[a,b]$ to a segment $[a,b_0]$ $(b_0 > b)$ letting $\mathbf{P}(\Delta) = 0$ for all Borel sets $\Delta \subset (b,b_0]$.

By $\mathbf{V}_\Delta(\mathbf{P})$ we denote $\mathbf{V}_\Delta(\mathbf{P}) = \rho(\Delta) = \sup\sum_j \|\mathbf{P}(\Delta_j)\|$, where the supremum is taken over finite sums of disjoint Borel sets $\Delta_j \subset \Delta$. The number $\mathbf{V}_\Delta(\mathbf{P})$ is called variation of the measure $\mathbf{P}$ on the Borel set Δ. Suppose that the measure $\mathbf{P}$ has the bounded variation on $[a,b]$. Then for ρ-almost all $\xi \in [a,b]$ there exists an operator function $\xi \to \Psi_{\mathbf{P}}(\xi)$ such that $\Psi_{\mathbf{P}}$ possesses the values in the set of bounded linear operators acting in H, $\|\Psi_{\mathbf{P}}(\xi)\| = 1$, and the equality $\mathbf{P}(\Delta) = \int_\Delta \Psi_{\mathbf{P}}(\xi)d\rho$ holds for each Borel set $\Delta \subset [a,b]$ ([3, ch. 5]). A function h is integrable with respect to the measure $\mathbf{P}$ on a set Δ if there exists the Bochner integral $\int_\Delta \Psi_{\mathbf{P}}(t)h(t)d\rho = \int_\Delta (d\mathbf{P})h(t)$. Then the function $y(t) = \int_{t_0}^{t}(d\mathbf{P})h(s)$ is continuous from the left.

Denote by $\mathcal{S}_{\mathbf{P}}$ a set of single-point atoms of the measure $\mathbf{P}$ (i. e., a set $t \in [a,b]$ such that $\mathbf{P}(\{t\}) \neq 0$). The set $\mathcal{S}_{\mathbf{P}}$ is at most countable.

In following Lemma 1, $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}$ are operator measures having bounded variations and taking values in the set of linear bounded operators acting in H. Suppose that the measure $\mathbf{q}$ is self-adjoint, i. e., $(\mathbf{q}(\Delta))^* = \mathbf{q}(\Delta)$ for each Borel set $\Delta \subset [a,b]$. We assume that these measures are extended to the segment $[a,b_0] \supset [a,b_0) \supset [a,b]$ in the manner described above.

Lemma 1. [8] *Let f, g be functions integrable on $[a,b_0]$ with respect to the measure $\mathbf{q}$. Then any functions*

$$y(t)=y_0-iJ\int\limits_{t_0}^{t}d\mathbf{p}_1(s)y(s)-iJ\int\limits_{t_0}^{t}d\mathbf{q}(s)f(s),$$

$$z(t)=z_0-iJ\int\limits_{t_0}^{t}d\mathbf{p}_2(s)z(s)-iJ\int\limits_{t_0}^{t}d\mathbf{q}(s)g(s)\ \ (a\leqslant t_0< b_0,\ t_0\leqslant t\leqslant b_0)$$

satisfy the following formula (*analogous to the Lagrange one*):

$$\int\limits_{c_1}^{c_2}(d\mathbf{q}(t)f(t),z(t))-\int\limits_{c_1}^{c_2}(y(t),d\mathbf{q}(t)g(t))=(iJy(c_2),z(c_2))-$$

$$-(iJy(c_1),z(c_1))+\int_{c_1}^{c_2}(y(t),d\mathbf{p}_2(t)z(t))-\int\limits_{c_1}^{c_2}(d\mathbf{p}_1(t)y(t),z(t))-$$

$$-\sum_{t\in\mathcal{S}_{\mathbf{p}_1}\cap\mathcal{S}_{\mathbf{p}_2}\cap[c_1,c_2)}(iJ\mathbf{p}_1(\{t\})y(t),\mathbf{p}_2(\{t\})z(t))-$$

$$-\sum_{t\in\mathcal{S}_{\mathbf{q}}\cap\mathcal{S}_{\mathbf{p}_2}\cap[c_1,c_2)}(iJ\mathbf{q}(\{t\})f(t),\mathbf{p}_2(\{t\})z(t))-$$

$$-\sum_{t\in\mathcal{S}_{\mathbf{p}_1}\cap\mathcal{S}_{\mathbf{q}}\cap[c_1,c_2)}(iJ\mathbf{p}_1(\{t\})y(t),\mathbf{q}(\{t\})g(t))-$$

$$-\sum_{t\in\mathcal{S}_{\mathbf{q}}\cap[c_1,c_2)}(iJ\mathbf{q}(\{t\})f(t),\mathbf{q}(\{t\})g(t)),\quad t_0\leqslant c_1<c_2\leqslant b_0.\quad (2)$$

Let a segment $[l_1,l_2]\subset[a,b_0]$. We consider a set of Borel measurable functions, ranging in H, bounded on $[l_1,l_2]$, continuous from the left, and constant on $[l_1,l_2]\cap(b,b_0]$. We introduce the norm $\|u\|_{[l_1,l_2]}=\sup\limits_{t\in[l_1,l_2]}\|u(t)\|$ on this set and obtain a Banach space denoted by $\widetilde{C}[l_1,l_2]$.

Theorem 1. [7] *For any function* $g\in\widetilde{C}[a,b_0]$ *there exists a unique solution of the equation*

$$y(t)=\int\limits_{t_0}^{t}d\mathbf{p}(\xi)y(\xi)+g(t),\quad a\leqslant t_0\leqslant b_0,\qquad (3)$$

belonging to the space $\widetilde{C}[t_0-\delta,b_0]$, *where* $a\leqslant t_0<b_0$, $\delta=\delta(t_0)>0$ *is small enough if* $t_0>a$ *and* $\delta=0$ *if* $t_0=a$, *the measure* $\mathbf{p}$ *has the bounded variation on* $[a,b]$.

Corollary 1. *Suppose $t_0 = a$. Then for any function $g \in \widetilde{C}[a, b_0]$ there exists a unique solution of equation (3) belonging to the space $\widetilde{C}[a, b_0]$.*

Remark 1. *In general, a solution of (3) can be non-extendable to the left (see [7]). However, if the measure $\mathbf{p}$ in (3) is continuous, then a solution can be extended to the left up to the point a and this extension is unique.*

Suppose further that $\mathbf{p}$ is a self-adjoint measure with the bounded variation. We consider the equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}(s)y(s) - iJ\lambda\int_a^t y(s)d\mu(s) - iJ\int_a^t f(s)d\mu(s), \qquad (4)$$

where $\lambda \in \mathbb{C}$; μ is the usual Lebesque measure on $[a, b]$ $(\mu([\alpha, \beta)) = \beta - \alpha$ for all $\alpha, \beta \in [a, b], \alpha < \beta)$ extended to $[a, b_0]$ by the equality $\mu(\Delta) = 0$ for each Borel set $\Delta \subset (b, b_0]$; $x_0 \in H$; $f \in L_2(H; a, b)$ and $f = 0$ on $(b, b_0]$.

We construct the continuous measure $\mathbf{p}_0$ (i. e., a measure without single-point atoms) from the measure $\mathbf{p}$ in the following way. We set $\mathbf{p}_0(\{t_k\}) = 0$ for $t_k \in \mathcal{S}_{\mathbf{p}}$ and we set $\mathbf{p}_0(\Delta) = \mathbf{p}(\Delta)$ for all Borel sets such that $\Delta \cap \mathcal{S}_{\mathbf{p}} = \varnothing$. The measure $\mathbf{p}_0$ is self-adjoint. Replace $\mathbf{p}$ by $\mathbf{p}_0$ in (4) to obtain the equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}_0(s)y(s) - iJ\lambda\int_a^t y(s)d\mu(s) - iJ\int_a^t f(s)d\mu(s). \qquad (5)$$

By Corollary 1, it follows that equations (4), (5) have unique solutions.

Denote by W the operator solution of the equation

$$W(t, \lambda)x_0 = x_0 - iJ\int_a^t d\mathbf{p}_0(s)W(s, \lambda)x_0 - iJ\lambda\int_a^t W(s, \lambda)x_0 d\mu(s),$$

where $x_0 \in H$. In Lemma 1 we take $\mathbf{p}_1 = \mathbf{p}_0 + \lambda\mu$, $\mathbf{p}_2 = \mathbf{p}_0 + \overline{\lambda}\mu$, $\mathbf{q} = \mu$, $f = g = 0$, $y(t) = W(t, \lambda)x_0$, $z(t) = W(t, \overline{\lambda})z_0$, $x_0, z_0 \in H$. Since the measure $\mathbf{p}_0$ is self-adjoint and the equality $\mathcal{S}_{\mathbf{p}_0} = \varnothing$ holds, we obtain

$$(iJW(c_2, \lambda)x_0, W(c_2, \overline{\lambda})z_0) - (iJW(c_1, \lambda)x_0, W(c_1, \overline{\lambda})z_0) = 0$$

for all c_1, c_2 $(a \leqslant c_1 \leqslant c_2 \leqslant b_0)$. In this equality we take $c_2 = t$, $c_1 = a$. Then we get

$$W^*(t, \lambda)JW(t, \overline{\lambda}) = J. \qquad (6)$$

The functions $t \to W(t,\lambda)$ and $t \to W^{-1}(t,26\lambda) = JW^*(t,\overline{\lambda})J$ are continuous with respect to the uniform operator topology. Consequently, there exist constants $\alpha > 0$, $\beta > 0$ such that the inequality

$$\alpha \|x\|^2 \leqslant \|W(t,\lambda)x\|^2 \leqslant \beta \|x\|^2 \tag{7}$$

holds for all $x \in H$, $t \in [a, b_0]$, $\lambda \in C \subset \mathbb{C}$ (C is a compact set). The function $\lambda \to W(t,\lambda)$ is holomorphic for any fixed t.

Lemma 2. [7, 8] *The function y is a solution of the equation (5) if and only if y has the form*

$$y(t) = W(t,\lambda)x_0 - W(t,\lambda)iJ\int_a^t W^*(s,\overline{\lambda})f(s)d\mu(s),$$

where $x_0 \in H$, $a \leqslant t \leqslant b_0$.

3. Linear operators generated by the integral equation. In this section, we introduce a minimal operator L_0 generated by equations (4), (1) and give a description of the adjoint operator L_0^*. Further the following notation is used: $\mathcal{D}(A)$ is the domain of an operator A, $\mathcal{R}(A)$ is the range of A. Since all considered operators are linear, we shall often omit the word «linear».

Let $L_2(H,\mu;a,b_0)$ be the space of μ-measurable functions y with values in H such that $\int_a^{b_0} \|y(t)\|^2 d\mu(t) < \infty$. This space coincides with the space $\mathfrak{H} = L_2(H;a,b)$ since $\mu(\Delta) = 0$ for each Borel set $\Delta \subset (b, b_0]$.

Let us define the minimal operator L_0 in the following way. The domain $\mathcal{D}(L_0)$ consists of functions $y \in \mathfrak{H}$ for each of which there exists a function $f \in \mathfrak{H}$ such that (4) holds with $\lambda = 0$ and y satisfies the conditions

$$y(a) = y(b_0) = y(t_k) = 0, \quad t_k \in \mathcal{S}_{\mathbf{p}}. \tag{8}$$

Then we set $L_0 y = f$. By Lemma 1, the operator L_0 is symmetric. If equalities (4), (8) hold, then $y \in \mathcal{D}(L_0 - \lambda E)$ and $(L_0 - \lambda E)y = f$ ($\lambda \in \mathbb{C}$).

We claim that if $y \in \mathcal{D}(L_0)$ then $y(t) = 0$ for all $t \in [b, b_0]$. Indeed, $\lim_{t \to b+0} y(t) = 0$ since $y(b_0) = 0$. If $b \notin \mathcal{S}_{\mathbf{p}}$, then $y(b) = 0$. If $b \in \mathcal{S}_{\mathbf{p}}$, then equality (8) implies $y(b) = 0$. Since $\mu(\Delta) = 0$ for each Borel set $\Delta \subset (b, b_0]$, we obtain the desired assertion.

It follows from (8), that $y(a) = 0$. In this case y is independent of the condition $a \in \mathcal{S}_{\mathbf{p}}$. Thus the operator L_0 does not change if the measure $\mathbf{p}$ is replaced by a measure $\mathbf{p}_1$ such that $\mathbf{p}_1(\{a\})=\mathbf{p}_1(\{b\})=0$ and $\mathbf{p}_1(\Delta)=\mathbf{p}(\Delta)$ for all Borel sets $\Delta\subset[a,b]\setminus\{a,b\}$. Therefore, without loss of generality, it can be assumed that $b_0=b$, and $\mathbf{p}(\{a\})=\mathbf{p}(\{b\})=0$ (i. e., $a,b\notin\mathcal{S}_{\mathbf{p}}$), and μ is the usual Lebesque measure on $[a,b]$. Further we write ds instead of $d\mu(s)$.

Remark 2. *It is possible that $\mathcal{D}(L_0)=\{0\}$. An example is available in [7]. In this case $L_0^*=\mathfrak{H}\times\mathfrak{H}$, i. e., L_0^* is a linear relation. (The terminology on linear relations can be found, for example, in [2]).*

Lemma 3. [8] *The operator L_0 is closed. The function y belongs to the domain $\mathcal{D}(L_0-\lambda E)$ if and only if the equalities*

$$y(t)=W(t,\lambda)iJ\int\limits_a^t W^*(s,\overline{\lambda})f(s)ds,$$

$$y(s_k)=W(s_k,\lambda)iJ\int\limits_a^{s_k} W^*(s,\overline{\lambda})f(s)ds=0$$

hold, where $s_k\in\mathcal{S}_{\mathbf{p}}\cup\{b\}$, $f=(L_0-\lambda E)y$.

Corollary 2. *The function $f\in\mathfrak{H}$ belongs to the range $\mathcal{R}(L_0-\lambda E)$ if and only if f satisfies the condition*

$$\int\limits_a^{s_k} W^*(s,\overline{\lambda})f(s)ds=0 \tag{9}$$

for all $s_k\in\mathcal{S}_{\mathbf{p}}\cup\{b\}$.

Remark 3. *Condition (9) is equivalent to the following:*

$$\int\limits_{s_{k-1}}^{s_k} W^*(s,\overline{\lambda})f(s)ds=0,\quad s_k\in\mathcal{S}_{\mathbf{p}}\cup\{a,b\}. \tag{10}$$

Further, suppose that the set $\mathcal{S}_{\mathbf{p}}$ of single-point atoms $\{t_k\}$ can be arranged in the ascending order $t_1<t_2<...<t_k<...$ and the limit point is b. By χ_B denote the characteristic function of a set B.

Lemma 4. *The domain $\mathcal{D}(L_0)$ of the operator L_0 is dense in $\mathfrak{H}$.*

Proof. Suppose that there exists a function $h\in\mathfrak{H}$ such that the equality $(h,z)_{\mathfrak{H}}=0$ holds for all $z\in\mathcal{D}(L_0)$. By y denote a solution of equation (5), in which $\lambda=0$ and the function f is replaced by h. Suppose that

$z \in \mathcal{D}(L_0)$ and denote $z_k(t) = \chi_{[t_{k-1};t_k]} z$ ($t_0 = a$, $t_k \in \mathcal{S}_{\mathbf{p}}$, $k \in \mathbb{N}$, $\mathbb{N}$ is the natural number set). It follows from Lemma 3 that $z_k \in \mathcal{D}(L_0)$. We apply Lagrange's formula (2) to the functions y, h and z_k, $L_0 z_k$ for $c_1 = t_{k-1}$, $c_2 = t_k$, $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_0$, $\mathbf{q} = \mu$. Then we obtain $(y, L_0 z_k)_{\mathfrak{H}} = (h, z_k)_{\mathfrak{H}} = 0$. Hence,

$$(y, L_0 z_k)_{\mathfrak{H}} = \int\limits_{t_{k-1}}^{t_k} (y(s), (L_0 z_k)(s))ds = 0$$

for each function $z \in \mathcal{D}(L_0)$. By (7), it follows that a set of functions $t \to W(t,0)c_k$ is closed in the space $L_2(H; [t_{k-1}, t_k])$, where $c_k \in H$. Using corollary 2 and equality (10), we obtain that there exists $c_k \in H$ such that $y(t) = W(t,0)c_k$ ($t_{k-1} \leqslant t \leqslant t_k$). Lemma 2 implies $h(t) = 0$ for $t \in [t_{k-1}, t_k]$. Since k is arbitrary ($k \in \mathbb{N}$), we get $h = 0$. □

We denote $w_k(t, \lambda) = \chi_{[t_{k-1};\, t_k)}(t) W(t, \lambda) W^{-1}(t_{k-1}, \lambda)$, $t_0 = a$, $k \in \mathbb{N}$. Let $\widetilde{W}_n(t, \lambda) = (w_1(t, \lambda), \ldots, w_n(t, \lambda))$ be the operator one-row matrix. For fixed t, λ, the operator $\widetilde{W}_n(t, \lambda)$ maps H^n to H continuously; here H^n is the Cartesian product of n copies of H. It is convenient to treat elements from H^n as one-column matrices, and to assume that

$$\widetilde{W}_n(t, \lambda)\widetilde{\xi}_n = \sum_{k=1}^{n} w_k(t, \lambda)\xi_k,$$

where we denote $\widetilde{\xi}_n = \mathrm{col}(\xi_1, \ldots, \xi_n) \in H^n$, $\xi_k \in H$.

Let $\ker_k(\lambda)$ be a linear space of functions $t \to w_k(t, \lambda)\xi_k$, $\xi_k \in H$. By (7), it follows that $\ker_k(\lambda)$ is closed in $\mathfrak{H}$. The spaces $\ker_k(\lambda)$ and $\ker_j(\lambda)$ are orthogonal for $k \neq j$. We denote $\mathcal{K}_n(\lambda) = \ker_1(\lambda) \oplus \ldots \oplus \ker_n(\lambda)$. Obviously, $\mathcal{K}_n(\lambda) \subset \mathcal{K}_m(\lambda)$ for $n < m$.

Lemma 5. *The set $\cup_n \mathcal{K}_n(\lambda)$ is dense in* $\ker(L_0^* - \lambda E)$.

Proof. It follows from Corollary 2 and (10) that the range $\mathcal{R}(L_0 - \overline{\lambda} E)$ consists of functions $f \in \mathfrak{H}$ orthogonal to functions of the form $w_k(\cdot, \lambda)\xi_k$, where $\xi_k \in H$, $k \in \mathbb{N}$. The equality $\ker(L_0^* - \lambda E) \oplus \mathcal{R}(L_0 - \overline{\lambda} E) = \mathfrak{H}$ implies the desired assertion. The Lemma is proved. □

Denote the operator $\widetilde{\xi}_n \to \widetilde{W}_n(\cdot\,, \lambda)\widetilde{\xi}_n$ ($\widetilde{\xi}_n \in H^n$) by $\mathcal{W}_n(\lambda)$. The operator $\mathcal{W}_n(\lambda)$ maps H^n continuously and one-to-one onto $\mathcal{K}_n(\lambda) \subset \mathfrak{H}$. Consequently, the adjoint operator $\mathcal{W}_n^*(\lambda)$ maps $\mathfrak{H}$ onto H^n continuously. We find the form of the operator $\mathcal{W}_n^*(\lambda)$. For all $\widetilde{\xi}_n \in H^n$, $f \in \mathfrak{H}$, we have

$$(f, \mathcal{W}_n(\lambda)\,\widetilde{\xi}_n)_{\mathfrak{H}} = \int\limits_a^b (f(s), \widetilde{W}_n(s,\lambda)\widetilde{\xi}_n)ds =$$

$$= \int\limits_a^b (\widetilde{W}_n^*(s,\lambda)f(s), \widetilde{\xi}_n)ds = (\mathcal{W}_n^*(\lambda)f, \widetilde{\xi}_n).$$

Therefore,

$$\mathcal{W}_n^*(\lambda)f = \int\limits_a^b \widetilde{W}_n^*(s,\lambda)f(s)ds. \tag{11}$$

So we obtain the following statement:

Lemma 6. *The operator $\mathcal{W}_n(\lambda)$ maps H^n continuously and one-to-one onto $\mathcal{K}_n(\lambda)$. The adjoint operator $\mathcal{W}_n^*(\lambda)$ maps $\mathfrak{H}$ continuously onto H^n and acts by (11). Moreover, $\mathcal{W}_n^*(\lambda)$ maps $\mathcal{K}_n(\lambda)$ one-to-one onto H^n.*

Lemma 7. *There exist $\alpha, \beta > 0$ such that the inequalities*

$$\alpha \sum_{k=1}^{n} \Delta_k \|\tau_k\|^2 \leqslant \|\mathcal{W}_n(\lambda)\widetilde{\tau}_n\|_{\mathfrak{H}}^2 \leqslant \beta \sum_{k=1}^{n} \Delta_k \|\tau_k\|^2,\ \widetilde{\tau}_n = (\tau_1,\ldots,\tau_n) \in H^n, \tag{12}$$

$$\alpha \sum_{k=1}^{n} \Delta_k^{-1} \|\varphi_k\|^2 \leqslant \|\mathcal{W}_n(\lambda)\widetilde{\tau}_n\|_{\mathfrak{H}}^2 \leqslant \beta \sum_{k=1}^{n} \Delta_k^{-1} \|\varphi_k\|^2 \tag{13}$$

hold for all $n \in \mathbb{N}$, where

$$\Delta_k = t_k - t_{k-1}, \qquad \varphi_k = \int\limits_{t_{k-1}}^{t_k} w_k^*(s,\lambda)w_k(s,\lambda)\tau_k ds.$$

Proof. Using (7), we get

$$\alpha\Delta_k \|\tau_k\|^2 \leqslant \int\limits_{t_{k-1}}^{t_k} \|w_k(s,\lambda)\tau_k\|^2 ds \leqslant \beta\Delta_k \|\tau_k\|^2, \ \ \alpha, \beta > 0.$$

Therefore,

$$\alpha \sum_{k=1}^{n} \Delta_k \|\tau_k\|^2 \leqslant \sum_{k=1}^{n} \int\limits_{t_{k-1}}^{t_k} \|w_k(s,\lambda)\tau_k\|^2 ds \leqslant \beta \sum_{k=1}^{n} \Delta_k \|\tau_k\|^2.$$

This implies (12). To prove (13), use (7) to obtain

$$\alpha_1\Delta_k\|\tau_k\|\leqslant\|\varphi_k\|=\left\|\int\limits_{t_{k-1}}^{t_k}w_k^*(s,\lambda)w_k(s,\lambda)\tau_k ds\right\|\leqslant\beta_1\Delta_k\|\tau_k\|,$$

where $\alpha_1,\beta_1>0$. Hence, $\alpha_1\Delta_k\|\tau_k\|^2\leqslant\Delta_k^{-1}\|\varphi_k\|^2\leqslant\beta_1\Delta_k\|\tau_k\|^2$. Thus,

$$\alpha_1\sum_{k=1}^n\Delta_k\|\tau_k\|^2\leqslant\sum_{k=1}^n\Delta_k^{-1}\|\varphi_k\|^2\leqslant\beta_1\sum_{k=1}^n\Delta_k\|\tau_k\|^2.$$

Now, using (12), get (13). The Lemma is proved. □

Let $\mathcal{H}_-$, $\mathcal{H}_+$, $\mathcal{H}_0=l_2(H)$ be linear spaces of sequences, respectively, $\widetilde{\tau}=\{\tau_k\}$, $\widetilde{\varphi}=\{\varphi_k\}$, $\widetilde{\xi}=\{\xi_k\}$ such that the series $\sum_{k=1}^\infty\Delta_k\|\tau_k\|^2$, $\sum_{k=1}^\infty\Delta_k^{-1}\|\varphi_k\|^2$, $\sum_{k=1}^\infty\|\xi_k\|^2$ converge, where τ_k, φ_k, $\xi_k\in H$. These spaces become Hilbert spaces if we introduce scalar products as

$$(\widetilde{\tau},\widetilde{\eta})_-=\sum_{k=1}^\infty(\Delta_k\tau_k,\eta_k),\ \ \widetilde{\tau},\widetilde{\eta}\in\mathcal{H}_-,\ \ (\widetilde{\varphi},\widetilde{\psi})_+=\sum_{k=1}^\infty(\Delta_k^{-1}\varphi_k,\psi_k),\ \ \widetilde{\varphi},\widetilde{\psi}\in\mathcal{H}_+,$$

$$(\widetilde{\xi},\widetilde{\zeta})_0=(\widetilde{\xi},\widetilde{\zeta})=\sum_{k=1}^\infty(\xi_k,\zeta_k),\ \ \widetilde{\xi},\widetilde{\zeta}\in\mathcal{H}_0.$$

In these spaces, the norms are defined by the equalities

$$\|\widetilde{\tau}\|_-^2=\sum_{k=1}^\infty\Delta_k\|\tau_k\|^2,\quad\|\widetilde{\varphi}\|_+^2=\sum_{k=1}^\infty\Delta_k^{-1}\|\varphi_k\|^2,\quad\|\widetilde{\xi}\|_0^2=\sum_{k=1}^\infty\|\xi_k\|^2.$$

The spaces $\mathcal{H}_+$, $\mathcal{H}_-$ can be treated as spaces with positive and negative norms with respect to $\mathcal{H}_0$ (see [3, ch.1], [9, ch.2]). So, $\mathcal{H}_+\subset\mathcal{H}_0\subset\mathcal{H}_-$ and $\alpha\|\widetilde{\varphi}\|_-\leqslant\|\widetilde{\varphi}\|_0\leqslant\beta\|\widetilde{\varphi}\|_+$, where $\widetilde{\varphi}\in\mathcal{H}_+$, $\alpha,\beta>0$, i. e., the space $\mathcal{H}_0$ is equipped with the spaces $\mathcal{H}_+$, $\mathcal{H}_-$. The "scalar product" $(\widetilde{\varphi},\widetilde{\tau})=(\widetilde{\varphi},\widetilde{\tau})_0$ is defined for $\widetilde{\varphi}\in\mathcal{H}_+$, $\widetilde{\tau}\in\mathcal{H}_-$. If $\widetilde{\tau}\in\mathcal{H}_0$, then $(\widetilde{\varphi},\widetilde{\tau})_0$ coincides with the scalar product in $\mathcal{H}_0$.

Let $\mathcal{T}\subset\mathcal{H}_-$ be a set of sequences vanishing starting from a certain number (its own for each sequence). The set $\mathcal{T}$ is dense in the space $\mathcal{H}_-$. The operator $\mathcal{W}_n(\lambda)$ is the restriction of $\mathcal{W}_{n+1}(\lambda)$ to H^n. By $\mathcal{W}'(\lambda)$ denote an operator in $\mathcal{T}$, such that $\mathcal{W}'(\lambda)\widetilde{\tau}=\mathcal{W}_n(\lambda)\widetilde{\tau}_n$ for all $n\in\mathbb{N}$, where $\widetilde{\tau} = (\widetilde{\tau}_n, 0, \dots)$. It follows from (12) that the operator $\mathcal{W}'(\lambda)$ admits an extension by continuity to the space $\mathcal{H}_-$. By $\mathcal{W}(\lambda)$ denote the extended operator. Moreover, we denote $\widetilde{W}(t,\lambda)\widetilde{\tau} = (\mathcal{W}(\lambda)\widetilde{\tau})(t)$, where $\widetilde{\tau} = \{\tau_k\} \in \mathcal{H}_-$. For a fixed t, the operator $\widetilde{W}(t,\lambda)$ maps $\mathcal{H}_-$ into H. Lemmas 5, 6 imply the following assertion.

Lemma 8. *The operator $\mathcal{W}(\lambda)$ maps $\mathcal{H}_-$ continuously and one-to-one onto* $\ker(L_0^* - \lambda E)$. *A function u belongs to* $\ker(L_0^* - \lambda E)$ *if and only if there exists $\widetilde{\tau} = \{\tau_k\} \in \mathcal{H}_-$ such that $u(t) = (\mathcal{W}(\lambda)\widetilde{\tau})(t) = \widetilde{W}(t,\lambda)\widetilde{\tau}$.*

The adjoint operator $\mathcal{W}^*(\lambda)$ maps $\mathfrak{H}$ continuously onto $\mathcal{H}_+$. Let us find the form of $\mathcal{W}^*(\lambda)$. Suppose $f \in \mathfrak{H}$, $\widetilde{\xi} \in \mathcal{T}$, $\widetilde{\xi} = \{\widetilde{\xi}_n, 0, \dots\}$. Then

$$(\widetilde{\xi}, \mathcal{W}^*(\lambda)f) = (\mathcal{W}(\lambda)\widetilde{\xi}, f)_{\mathfrak{H}} = \int_a^b (\widetilde{W}(t,\lambda)\widetilde{\xi}, f(t))dt = \int_a^b (\widetilde{\xi}, \widetilde{W}^*(t,\lambda)f(t))dt.$$

Since $\mathcal{W}^*(\lambda)f \in \mathcal{H}_+$ and the set $\mathcal{T}$ is dense in $\mathcal{H}_-$, we obtain

$$\mathcal{W}^*(\lambda)f = \int_a^b \widetilde{W}^*(t,\lambda)f(t)dt. \tag{14}$$

Thus we obtain the following statement.

Lemma 9. *The operator $\mathcal{W}^*(\lambda)$ maps $\mathfrak{H}$ continuously onto $\mathcal{H}_+$ and acts by formula (14). Moreover, $\mathcal{W}^*(\lambda)$ maps* $\ker(L_0^* - \lambda E)$ *one-to-one onto $\mathcal{H}_+$ and* $\ker \mathcal{W}^*(\lambda) = \mathcal{R}(L_0 - \overline{\lambda}E)$.

Lemma 10. *Suppose that $f \in \mathfrak{H}$ and functions $\widetilde{F}_{an}$, $\widetilde{F}_{bn}$ are defined as*

$$\widetilde{F}_{an}(t) = -2^{-1}\widetilde{W}_n(t,\lambda)i\widetilde{J}_n \int_a^t \widetilde{W}_n^*(s,\overline{\lambda})f(s)ds, \tag{15}$$

$$\widetilde{F}_{bn}(t) = 2^{-1}\widetilde{W}_n(t,\lambda)i\widetilde{J}_n \int_t^b \widetilde{W}_n^*(s,\overline{\lambda})f(s)ds.$$

Then $\widetilde{F}_{an}$, $\widetilde{F}_{bn} \in \mathcal{D}(L_0^*)$ for all $n \in \mathbb{N}$. If the function f vanishes on $[t_n, b]$, then $L_0^*(\widetilde{F}_{an}) - \lambda\widetilde{F}_{an} = L_0^*(\widetilde{F}_{bn}) - \lambda\widetilde{F}_{bn} = 2^{-1}f$. Here $\widetilde{J}_n$ is an operator in H^n acting by the formula $\widetilde{J}_n\widetilde{\xi}_n = (J\xi_1, \dots, J\xi_n)$.

Proof. Using (15), we get

$$\widetilde{F}_{an}(t)=\sum_{k=1}^{n}F_k(t),\quad F_k(t)=-2^{-1}w_k(t,\lambda)iJ\int_{t_{k-1}}^{t}w_k^*(s,\overline{\lambda})f(s)ds.$$

The function F_k is continuous on the interval $[t_{k-1},t_k)$ and vanishes outside this interval. The function F_k does not change in the space $\mathfrak{H}$ if changed at one point. Therefore, without loss of generality, the function F_k can be assumed to be continuous from the left at the point t_k. Then, taking into account Lemma 2, we obtain that F_k is a solution of equation (5) (in which $a=t_{k-1}$ and f is replaced by $2^{-1}f$) on the segment $[t_{k-1},t_k]$. In [8] it is proved that every function $y\in\mathcal{D}(L_0)$ is a solution of equality (5) in which f is replaced by $g=L_0y$. Therefore, we can apply Lagrange's formula (2) to the functions $y\in\mathcal{D}(L_0)$, F_k for $c_1=t_{k-1}$, $c_2=t_k$, $\mathbf{q}=\mu$, $\mathbf{p}_1=\mathbf{p}_2=\mathbf{p}_0$. Since the measure $\mathbf{p}_0$ is continuous and the equality $y(t_{k-1})=y(t_k)=0$ holds, we obtain

$$\int_{t_{k-1}}^{t_k}(2^{-1}f(s)+\lambda F_k(s),y(s))ds=\int_{t_{k-1}}^{t_k}(F_k(s),g(s))ds.$$

This implies that $\widetilde{F}_{an}\in\mathcal{D}(L_0^*)$ and $L_0^*(\widetilde{F}_{an})-\lambda\widetilde{F}_{an}=2^{-1}f$ if $f(t)=0$ for $t>t_n$. We denote

$$\widetilde{\vartheta}_n=2^{-1}i\widetilde{J}_n\int_a^b\widetilde{W}_n^*(t,\overline{\lambda})f(t)dt=2^{-1}i\widetilde{J}_n\mathcal{W}_n^*(\overline{\lambda})f;\;\;u_n(t)=\widetilde{W}_n(t,\lambda)\widetilde{\vartheta}_n.$$

By Lemma 6, it follows that $u_n\in\ker_n(\lambda)$. Now the equality $\widetilde{F}_{bn}(t)=u_n(t)+\widetilde{F}_{an}(t)$ implies $\widetilde{F}_{bn}\in\mathcal{D}(L_0^*)$ and $L_0^*\widetilde{F}_{bn}-\lambda\widetilde{F}_{bn}=2^{-1}f$ if $f(t)=0$ for $t>t_n$. The Lemma is proved. □

Theorem 2. *A function $y\in\mathfrak{H}$ belongs to $\mathcal{D}(L_0^*)$ if and only if there exists a function $f\in\mathfrak{H}$ such that*

$$y(t)=\widetilde{W}(t,\lambda)\widetilde{\tau}-\sum_{k=1}^{\infty}w_k(t,\lambda)iJ\int_a^t w_k^*(s,\overline{\lambda})f(s)ds,\quad\widetilde{\tau}=\{\tau_k\}\in\mathcal{H}_-;\quad(16)$$

in this case $L_0^*y-\lambda y=f$. The series in (16) converges in $\mathfrak{H}$.

Proof. First we prove that if y has form (16), then $y \in \mathcal{D}(L_0^*)$. It follows from Lemma 8 that $\mathcal{W}(\lambda)\widetilde{\tau} \in \ker(L_0^* - \lambda E)$. The function

$$z_k(t) = w_k(t,\lambda) iJ \int_a^t w_k^*(s,\overline{\lambda}) f(s) ds = w_k(t,\lambda) iJ \int_{t_{k-1}}^t w_k^*(s,\overline{\lambda}) f(s) ds$$

vanishes outside the interval $[t_{k-1}, t_k)$. We denote $f_k(t) = \chi_{[t_{k-1};\, t_k)}(t) f(t)$. By (7), it follows that

$$\|z_k(t)\| \leqslant \beta \int_{t_{k-1}}^{t_k} \|f(s)\|\, ds \leqslant \beta \Delta_k^{1/2} \left\| \chi_{[t_{k-1};\, t_k)} f \right\|_{\mathfrak{H}}.$$

Therefore,

$$\|z_k\|_{\mathfrak{H}}^2 = \int_{t_{k-1}}^{t_k} \|z_k(t)\|^2\, dt \leqslant \beta^2 \Delta_k \left\| \chi_{[t_{k-1};\, t_k)} f \right\|_{\mathfrak{H}}^2. \tag{17}$$

We denote $u_n(t) = \sum_{k=1}^n z_k(t)$ and claim that the sequence $\{u_n\}$ converges in $\mathfrak{H}$. Indeed, using (17), we get

$$\|u_n\|_{\mathfrak{H}}^2 = \sum_{k=1}^n \|z_k\|_{\mathfrak{H}}^2 \leqslant \beta^2 \sum_{k=1}^n \Delta_k \left\| \chi_{[t_{k-1};\, t_k)} f \right\|_{\mathfrak{H}}^2 \leqslant \beta^2 (b-a) \|f\|_{\mathfrak{H}}^2.$$

Therefore the sequence $\{u_n\}$ converges to some function $u \in \mathfrak{H}$ and

$$u(t) = -\sum_{k=1}^{\infty} w_k(t,\lambda) iJ \int_a^t w_k^*(s,\overline{\lambda}) f(s) ds, \quad \|u\|_{\mathfrak{H}} \leqslant \beta_1 \|f\|_{\mathfrak{H}},\ \beta_1 > 0.$$

By Lemma 10, it follows that $u_n = 2\widetilde{F}_{an} \in \mathcal{D}(L_0^*)$ and $L_0^* u_n - \lambda u_n = \sum_{k=1}^n \chi_{[t_{k-1};t_k)} f$. Since the operator L_0^* is closed, we see that $u \in \mathcal{D}(L_0^*)$ and $L_0^* u - \lambda u = f$.

Now suppose that a function $\widehat{y} \in \mathcal{D}(L_0^*)$ and $L_0^* \widehat{y} - \lambda \widehat{y} = f$. If the function y has the form (16), then the function $\widehat{y} - y \in \ker(L_0^* - \lambda E)$. According to Lemma 8, there exists $\widetilde{\xi} \in \mathcal{H}_-$ such that $\widehat{y} - y = \mathcal{W}(\lambda)\widetilde{\xi}$. Therefore, $\widehat{y}$ has form (16). The Theorem is proved. □

By standard transformations, equality (16) is reduced to the form

$$y(t) = \widetilde{W}(t,\lambda)\widetilde{\zeta} - 2^{-1}\sum_{k=1}^{\infty} w_k(t,\lambda)iJ\int_a^t w_k^*(s,\overline{\lambda})f(s)ds +$$
$$+ 2^{-1}\sum_{k=1}^{\infty} w_k(t,\lambda)iJ\int_t^b w_k^*(s,\overline{\lambda})f(s)ds, \quad (18)$$

where $\widetilde{\zeta}=\{\zeta_k\}\in\mathcal{H}_-$, $\zeta_k = \tau_k - 2^{-1}iJ\int_{t_{k-1}}^{t_k} w_k^*(s,\overline{\lambda})f(s)ds.$

Let $\widetilde{J}$ denote an operator in $\mathcal{H}_-$ acting by the formula $\widetilde{J}\{\xi_k\}=\{J\xi_k\}$. Taking into account the convergence of the series in (18), we write equality (18) in the form

$$y(t) = \widetilde{W}(t,\lambda)\widetilde{\zeta} - 2^{-1}\widetilde{W}(t,\lambda)i\widetilde{J}\int_a^t \widetilde{W}^*(s,\overline{\lambda})f(s)ds +$$
$$+ 2^{-1}\widetilde{W}(t,\lambda)i\widetilde{J}\int_t^b \widetilde{W}^*(s,\overline{\lambda})f(s)ds, \quad (19)$$

where $\widetilde{\zeta}\in\mathcal{H}_-$, $f = L_0^*y - \lambda y$.

4. The description of generalized resolvents. Let A be a symmetric operator acting in a Hilbert space $\mathbf{H}$ and $\widetilde{A}$ be a selfadjoint extension of A to $\widetilde{\mathbf{H}}$, where $\widetilde{\mathbf{H}}$ is a Hilbert space, $\widetilde{\mathbf{H}}\supset\mathbf{H}$, and scalar products coincide in $\mathbf{H}$ and $\widetilde{\mathbf{H}}$. By P denote an orthogonal projection of $\widetilde{\mathbf{H}}$ onto $\mathbf{H}$. The function $\lambda\to R_\lambda$ defined as $R_\lambda = P(\widetilde{A}-\lambda E)^{-1}|_{\mathbf{H}}$, $\mathrm{Im}\lambda\neq 0$, is called a generalized resolvent of the operator A (see, e. g., [1, ch.9])

Theorem 3. *Every generalized resolvent R_λ ($\mathrm{Im}\lambda\neq 0$) of the operator L_0 is the integral operator*

$$R_\lambda f = \int_a^b K(t,s,\lambda)f(s)ds.$$

The kernel $K(t,s,\lambda)$ has the form

$$K(t,s,\lambda) = \widetilde{W}(t,\lambda)(M(\lambda) + 2^{-1}\mathrm{sgn}(s-t)i\widetilde{J})\widetilde{W}^*(s,\overline{\lambda}),$$

where $M(\lambda):\mathcal{H}_+\to\mathcal{H}_-$ is the bounded operator such that $M(\overline{\lambda})=M^*(\lambda)$,

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(M(\lambda)\widetilde{x},\widetilde{x})\geqslant 0 \quad (20)$$

for every λ $(\mathrm{Im}\lambda \neq 0)$ *and for every* $\widetilde{x} \in \mathcal{H}_+$. *The function* $\lambda \to M(\lambda)\widetilde{x}$ *is holomorphic for every* $\widetilde{x} \in \mathcal{H}_+$ *in half-planes* $\mathrm{Im}\lambda \neq 0$.

Proof. Suppose $y = R_\lambda f$. Then y has form (19). In this equality, $\widetilde{\zeta} \in \mathcal{H}_-$ is uniquely determined by f and λ, $\mathrm{Im}\lambda \neq 0$, i. e., $\widetilde{\zeta} = \widetilde{\zeta}(f, \lambda)$. Indeed, if $f = 0$, then $\widetilde{W}(t, \lambda)\widetilde{\zeta} = R_\lambda 0 = 0$. It follows from Lemma 8 that $\widetilde{\zeta} = 0$. Moreover, $\widetilde{\zeta}$ depends on f linearly. Consequently $\widetilde{\zeta} = S(\lambda)f$, where $S(\lambda) : \mathfrak{H} \to \mathcal{H}_-$ is a linear operator for fixed λ. We claim that the operator $S(\lambda)$ is bounded. Indeed, if a sequence $\{f_n\}$ converges to zero in the space $\mathfrak{H}$ as $n \to \infty$, then the sequence $\{y_n\} = \{R_\lambda f_n\}$ converges to zero in $\mathfrak{H}$. Hence, the sequence $\{\mathcal{W}(\lambda)\widetilde{\zeta}_n\}$ (where $\widetilde{\zeta}_n = S(\lambda)f_n$) converges to zero in $\mathfrak{H}$. By Lemma 8, it follows that the sequence $\{S(\lambda)f_n\}$ converges to zero in the space $\mathcal{H}_-$. Therefore $S(\lambda)$ is a bounded operator.

Now we prove that $\widetilde{\zeta}(f,\lambda)$ is uniquely determined by the element $\mathcal{W}^*(\overline{\lambda})f \in \mathcal{H}_+$. Suppose $\mathcal{W}^*(\overline{\lambda})f = 0$. Consider a function equal to the sum of the last two summands in (19). This function belongs to $\mathcal{D}(L_0 - \lambda E)$. Therefore, $\mathcal{W}(\lambda)\widetilde{\zeta}(f,\lambda)$ belongs to the range $\mathcal{R}(R_\lambda)$ of the operator R_λ. Hence, $\widetilde{\zeta}(f,\lambda) = 0$. Thus, $S(\lambda)f = M(\lambda)\mathcal{W}^*(\overline{\lambda})f$, where $M(\lambda) : \mathcal{H}_+ \to \mathcal{H}_-$ is an everywhere defined operator. Let $\mathcal{W}_0^*(\overline{\lambda})$ be a restriction of $\mathcal{W}^*(\overline{\lambda})$ to $\ker(L_0^* - \overline{\lambda}E)$. By Lemma 9, it follows that $M(\lambda) = S(\lambda)(\mathcal{W}_0^*(\overline{\lambda}))^{-1}$. Hence $M(\lambda)$ is the bounded operator.

Let us prove that the function $\lambda \to M(\lambda)\widetilde{x}$ is holomorphic for every $\widetilde{x} \in \mathcal{H}_+$ $(\mathrm{Im}\lambda \neq 0)$. It follows from (19) and the holomorphicity of the function $\lambda \to R_\lambda$ that the function $\lambda \to \mathcal{W}(\lambda)S(\lambda)f$ is holomorphic. Using (6), we obtain that the function $\lambda \to S(\lambda)f$ is holomorphic. Now the holomorphicity of the function $\lambda \to M(\lambda)$ follows from Lemma 11. This Lemma is formulated after the proof of the Theorem. In Lemma 11 it should be taken that $\mathcal{B}_1 = \mathfrak{H}$, $\mathcal{B}_2 = \mathcal{H}_+$, $\mathcal{B}_3 = \mathcal{H}_-$, $T_1(\lambda) = \mathcal{W}^*(\overline{\lambda})$, $T_2(\lambda) = M(\lambda)$, $T_3(\lambda) = S(\lambda)$.

Note that the equality $R_\lambda^* = R_{\overline{\lambda}}$ implies $M(\overline{\lambda}) = M^*(\lambda)$.

Let us prove that (20) holds. It follows from Lemma 9 that there exists a function $f \in \mathfrak{H}$ such that $\widetilde{x} = \mathcal{W}^*(\overline{\lambda})f$. Let $p_k : \mathcal{H}_- \to H$ be the operator defined by the formula $p_k\widetilde{\xi} = \xi_k$, where $\widetilde{\xi} = \{\xi_k\} \in \mathcal{H}_-$. We denote $M_k(\lambda) = p_k M(\lambda)$ and

$$z(t) = W(t, \lambda)(M(\lambda)\widetilde{x} - 2^{-1}\widetilde{J}\widetilde{x}) = \sum_{k=1}^{\infty} w_k(t, \lambda)(M_k(\lambda)\widetilde{x} - 2^{-1}Jx_k),$$

where $\widetilde{x} = \mathcal{W}^*(\overline{\lambda})f$, $x_k = p_k\widetilde{x}$. We shall apply formula (2) to the functions

$y = R_\lambda f$, z on the interval $[t_{k-1}, t_k]$. Using the argument from the proof of Lemma 10, we can assume that the function $w_k(t, \lambda)$ is continuous from the left at the point t_k. We note that $w_k(t_{k-1}, \lambda) = E$. Hence,

$$y(t_k) = z(t_k) = w(t_k)(M_k(\lambda)\widetilde{x} - 2^{-1}iJx_k),$$

$$y(t_{k-1}) = w(t_k)(M_k(\lambda)\widetilde{x} + 2^{-1}iJx_k),\ \ z(t_{k-1}) = w(t_k)(M_k(\lambda)\widetilde{x} - 2^{-1}iJx_k).$$

Using (2), we get

$$(\lambda - \overline{\lambda})^{-1}\Big(\int_{t_{k-1}}^{t_k}(R_\lambda f, f)dt - \int_{t_{k-1}}^{t_k}(f, R_\lambda f)dt\Big) - \int_{t_{k-1}}^{t_k}(R_\lambda f, R_\lambda f)dt + \\ + \int_{t_{k-1}}^{t_k}\|z(t)\|^2\,dt = (\mathrm{Im}\lambda)^{-1}\mathrm{Im}(M_k(\lambda)\widetilde{x}, \widetilde{x}).$$

Therefore,

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(R_\lambda f, f)_{\mathfrak{H}} - (R_\lambda f, R_\lambda f)_{\mathfrak{H}} + \|z\|^2_{\mathfrak{H}} = (\mathrm{Im}\lambda)^{-1}\mathrm{Im}(M(\lambda)\widetilde{x}, \widetilde{x}).$$

Since $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(R_\lambda f, f)_{\mathfrak{H}} - (R_\lambda f, R_\lambda f)_{\mathfrak{H}} \geqslant 0$, we see that (20) holds. □

The function $\lambda \to M(\lambda)$ is called characteristic function (see [13]).

Lemma 11. [6] *Let $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ be Banach spaces. Let bounded operators $T_3(\lambda) : \mathcal{B}_1 \to \mathcal{B}_3$, $T_1(\lambda) : \mathcal{B}_1 \to \mathcal{B}_2$, $T_2(\lambda) : \mathcal{B}_2 \to \mathcal{B}_3$ satisfy the equality $T_3(\lambda) = T_2(\lambda)T_1(\lambda)$ for every fixed λ belonging to some neighborhood of a point λ_0 and suppose the range of operator $T_1(\lambda_0)$ coincides with $\mathcal{B}_2$. If the functions $T_1(\lambda)$, $T_3(\lambda)$ are strongly differentiable at the point λ_0, then the function $T_2(\lambda)$ is strongly differentiable at λ_0.*

5. Boundary value problems connected with generalized resolvents. To shorten the notation, we shall denote $\widetilde{W}(t, 0) = \widetilde{W}(t)$, $w(t, 0) = w(t)$, $\mathcal{W}(0) = \mathcal{W}$. We put $\lambda = 0$ in formula (19). By Theorem 2, it follows that $y \in \mathcal{D}(L_0^*)$ and $L_0^* y = f$ if and only if y has the form

$$y(t) = \widetilde{W}(t)\widetilde{\zeta} - 2^{-1}\widetilde{W}(t)i\widetilde{J}\int_a^t \widetilde{W}^*(s)f(s)ds + 2^{-1}\widetilde{W}(t)i\widetilde{J}\int_t^b \widetilde{W}^*(s)f(s)ds, \tag{21}$$

where $\widetilde{\zeta} \in \mathcal{H}_-$. Each function $y \in \mathcal{D}(L_0^*)$ represented by (21) is associated with a pair of boundary values $\{Y,Y'\} \in \mathcal{H}_- \times \mathcal{H}_+$, where

$$Y = \Gamma_1 y = \widetilde{\zeta}, \quad Y' = \Gamma_2 y = \mathcal{W}^* f = \int_a^b \widetilde{W}^*(s) f(s) ds.$$

Let Γ denote the operator that takes each $y \in \mathcal{D}(L_0^*)$ to the ordered pair $\{Y, Y'\}$, i. e., $\Gamma y = \{\Gamma_1 y, \Gamma_2 y\}$.

Theorem 4. *The range $\mathcal{R}(\Gamma)$ of the operator Γ coincides with $\mathcal{H}_- \times \mathcal{H}_+$ and "the Green formula"*

$$(L_0^* y, z)_{\mathfrak{H}} - (y, L_0^* z)_{\mathfrak{H}} = (Y', Z) - (Y, Z') \tag{22}$$

holds, where $y, z \in \mathcal{D}(L_0^*)$, $\Gamma y = \{Y, Y'\}$, $\Gamma z = \{Z, Z'\}$.

Proof. The equality $\mathcal{R}(\Gamma) = \mathcal{H}_- \times \mathcal{H}_+$ follows from Lemmas 8, 9. Let us prove (22). Suppose that the function y has form (21) and

$$z(t) = \widetilde{W}(t)\widetilde{\eta} - 2^{-1}\widetilde{W}(t) i\widetilde{J} \int_a^t \widetilde{W}^*(s) g(s) ds + 2^{-1}\widetilde{W}(t) i\widetilde{J} \int_t^b \widetilde{W}^*(s) g(s) ds, \tag{23}$$

where $\widetilde{\eta} \in \mathcal{H}_-$, $g = L_0^* z$. Then

$$(f, \mathcal{W}\widetilde{\eta}) = (\mathcal{W}^* f, \widetilde{\eta}) = (Y', Z); \quad (\mathcal{W}\widetilde{\zeta}, g) = (\widetilde{\zeta}, \mathcal{W}^* g) = (Y, Z'). \tag{24}$$

In (21), we denote

$$\widetilde{F}_a(t) = -2^{-1}\widetilde{W}(t) i\widetilde{J} \int_a^t \widetilde{W}^*(s) f(s) ds = -\sum_{k=1}^{\infty} 2^{-1} w_k(t) iJ \int_a^t w_k^*(s) f(s) ds,$$

$$\widetilde{F}_b(t) = 2^{-1}\widetilde{W}(t) i\widetilde{J} \int_t^b \widetilde{W}^*(s) f(s) ds = \sum_{k=1}^{\infty} 2^{-1} w_k(t) iJ \int_t^b w_k^*(s) f(s) ds.$$

We introduce the similar notation $\widetilde{G}_a$, $\widetilde{G}_b$ for equality (23) by changing f to g. We define functions F_k, G_k by formulas

$$F_k(t) = -2^{-1} w_k(t) iJ \int_{t_{k-1}}^t w_k^*(s) f(s) ds,$$

$$G_k(t) = -2^{-1} w_k(t) iJ \int_{t_{k-1}}^{t} w_k^*(s) g(s) ds.$$

Also, as in the proof of Lemma 10, it can be assumed, without loss of generality, that the functions F_k, G_k are continuous from the left at the point t_k. Arguing as in proof of Lemma 10, we apply Lagrange's formula (2) to the functions F_k, $2^{-1}f$ and G_k, $2^{-1}g$ on the segment $[t_{k-1}, t_k]$. Taking into account (6), we obtain

$$\int_{t_{k-1}}^{t_k} (2^{-1}f(s), G_k(s))ds - \int_{t_{k-1}}^{t_k} (F_k(s), 2^{-1}g(s))ds =$$

$$= 4^{-1}\left(iJW(t_k)iJ \int_{t_{k-1}}^{t_k} W^*(s)f(s)ds,\ W(t_k)iJ \int_{t_{k-1}}^{t_k} W^*(s)g(s)ds \right) =$$

$$= 4^{-1}\left(iJ \int_{t_{k-1}}^{t_k} W^*(s)f(s)ds, \int_{t_{k-1}}^{t_k} W^*(s)g(s)ds \right).$$

Therefore,

$$(2^{-1}f, \widetilde{G}_a)_{\mathfrak{H}} - (\widetilde{F}_a, 2^{-1}g)_{\mathfrak{H}} = 4^{-1}(i\widetilde{J}\mathcal{W}^* f, \mathcal{W}^* g). \tag{25}$$

We denote $u(t) = 2^{-1}\widetilde{W}(t)i\widetilde{J}\mathcal{W}^* f$, $v(t) = 2^{-1}\widetilde{W}(t)i\widetilde{J}\mathcal{W}^* g$. By Lemma 8, it follows that $u, v \in \ker(L_0^*)$ and $\widetilde{F}_b(t) = u(t) + \widetilde{F}_a(t)$, $\widetilde{G}_b(t) = v(t) + \widetilde{G}_a(t)$. Using (25), we get

$$(2^{-1}f, \widetilde{G}_b)_{\mathfrak{H}} - (\widetilde{F}_b, 2^{-1}g)_{\mathfrak{H}} = (2^{-1}f, \widetilde{G}_a)_{\mathfrak{H}} - (\widetilde{F}_a, 2^{-1}g)_{\mathfrak{H}} + (2^{-1}f, v)_{\mathfrak{H}} -$$
$$- (u, 2^{-1}g)_{\mathfrak{H}} = 4^{-1}(i\widetilde{J}\mathcal{W}^* f, \mathcal{W}^* g) - 4^{-1}(i\widetilde{J}\mathcal{W}^* f, \mathcal{W}^* g) -$$
$$- 4^{-1}(i\widetilde{J}\mathcal{W}^* f, \mathcal{W}^* g) = -4^{-1}(i\widetilde{J}\mathcal{W}^* f, \mathcal{W}^* g). \tag{26}$$

From (24), (25), (26), we obtain (22). The Theorem is proved. □

We introduce operators $\delta_- : \mathcal{H}_- \to \mathcal{H}_0$, $\delta_+ : \mathcal{H}_+ \to \mathcal{H}_0$ by the formulas $\delta_- \widetilde{\tau} = \{\Delta_k^{1/2} \tau_k\}$, $\delta_+ \widetilde{\varphi} = \{\Delta_k^{-1/2} \varphi_k\}$, where $\widetilde{\tau} = \{\tau_k\} \in \mathcal{H}_-$, $\widetilde{\varphi} = \{\varphi_k\} \in \mathcal{H}_+$. The operator δ_- (δ_+) maps continuously and one-to-one $\mathcal{H}_-$ onto $\mathcal{H}_0$ ($\mathcal{H}_+$ onto $\mathcal{H}_0$, respectively). Suppose that $y \in \mathcal{D}(L_0^*)$. We put $\mathcal{Y} = \gamma_1 y = \delta_- \Gamma_1 y$; $\mathcal{Y}' = \gamma_2 y = \delta_+ \Gamma_2 y$ and $\gamma y = \{\gamma_1 y, \gamma_2 y\}$. Then $\mathcal{R}(\gamma) = \mathcal{H}_0 \times \mathcal{H}_0$. Using (22), we get

$$(L_0^* y, z)_{\mathfrak{H}} - (y, L_0^* z)_{\mathfrak{H}} = (\mathcal{Y}', \mathcal{Z}) - (\mathcal{Y}, \mathcal{Z}'), \tag{27}$$

where $y, z \in \mathcal{D}(L_0^*)$, $\gamma y = \{\mathcal{Y}, \mathcal{Y}'\}$, $\gamma z = \{\mathcal{Z}, \mathcal{Z}'\}$.

It follows from (27) that the ordered triple $(\mathcal{H}_0, \gamma_1, \gamma_2)$ is the space of boundary values (a boundary triplet in another terminology) for the operator L_0 in the sense of papers [11], [5] (see also [9], [12]).

We consider the boundary value problem

$$L_0^* y = \lambda y + h, \quad (K(\lambda) - E)\mathcal{Y}' - i(K(\lambda) + E)\mathcal{Y} = 0, \tag{28}$$

where $\{\mathcal{Y},\mathcal{Y}'\} = \gamma y$; $h \in \mathfrak{H}$; $\lambda \to K(\lambda)$ is a holomorphic operator function in $\mathcal{H}_0$ such that $\|K(\lambda)\| \leqslant 1$; $\mathrm{Im}\lambda > 0$.

From [5] and (27) we obtain the following statement.

Theorem 5. *There exists a one-to-one mapping between boundary problems (28) and generalized resolvents of the operator L_0. For any solution y of problem (28), a function R_λ defined by the equality $y = R_\lambda h$ is a generalized resolvent and, conversely, for any generalized resolvent R_λ there exists a function $K(\lambda)$ such that $y = R_\lambda h$ is the solution of (28).*

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Received April 04, 2018.
In revised form, August 08, 2018.
Accepted August 11, 2018.
Published online August 31, 2018.

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