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COEFFICIENT PROBLEMS ON THE CLASS $U(\lambda)$

Abstract. For $0 < \lambda \le 1$, let $\mathcal{U}(\lambda)$ denote the family of functions $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ analytic in the unit disk $\mathbb{D}$ satisfying the condition $\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < \lambda$ in $\mathbb{D}$. Although functions in this family are known to be univalent in $\mathbb{D}$, the coefficient conjecture about a_n for $n \ge 5$ remains an open problem. In this article, we shall first present a non-sharp bound for $|a_n|$. Some members of the family $\mathcal{U}(\lambda)$ are given by

$$\frac{z}{f(z)} = 1 - (1+\lambda)\phi(z) + \lambda(\phi(z))^2$$

with $\phi(z) = e^{i\theta} z$, that solve many extremal problems in $\mathcal{U}(\lambda)$. Secondly, we shall consider the following question: Do there exist functions ϕ analytic in $\mathbb{D}$ with $|\phi(z)| < 1$ that are not of the form $\phi(z) = e^{i\theta} z$ for which the corresponding functions f of the above form are members of the family $\mathcal{U}(\lambda)$? Finally, we shall solve the second coefficient (a_2) problem in an explicit form for $f \in \mathcal{U}(\lambda)$ of the form

$$f(z) = \frac{z}{1 - a_2 z + \lambda z \int\limits_0^z \omega(t)\, dt},$$

where ω is analytic in $\mathbb{D}$ such that $|\omega(z)| \le 1$ and $\omega(0) = a$, where $a \in \overline{\mathbb{D}}$.

Key words: *Univalent function, subordination, Julia's lemma, Schwarz' lemma*

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We denote the unit disk by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{H}$ be the linear space of analytic functions defined on $\mathbb{D}$ endowed with the topology of locally uniform convergence and $\mathcal{A} = \{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\}$. The family $\mathcal{S}$ of univalent functions from $\mathcal{A}$ and many of its subfamilies, for which the image domains have special geometric properties, have been investigated in detail. Among them are convex, starlike, close-to-convex, spirallike and typically real mappings. For the general theory of univalent functions we refer the reader to the books [7, 10, 23]. The class $\mathcal{U}(\lambda)$ defined below seems to have many interesting properties (cf. [21, 22]). For $0 < \lambda \leq 1$, we consider the family

$$\mathcal{U}(\lambda) = \{f \in \mathcal{A} : \ |U_f(z)| < \lambda \text{ in } \mathbb{D}\},$$

where

$$U_f(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 = \frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' - 1, \quad z \in \mathbb{D}. \tag{1}$$

Set $\mathcal{U} := \mathcal{U}(1)$, and observe that $\mathcal{U} \subsetneq \mathcal{S}$ (see [1, 2]).

Before we continue the discussion, it might be appropriate to include a few well-known properties about the family $\mathcal{U}(\lambda)$.

1) Let Σ denote the family of univalent functions F of the form,

$$F(\zeta) = \zeta + \sum_{n=0}^{\infty} c_n \zeta^{-n}, \quad |\zeta| > 1,$$

which satisfies the condition $F(\zeta) \neq 0$ for $|\zeta| > 1$. Then we observe that each $f \in \mathcal{S}$ can be associated with a mapping $F \in \Sigma$ by the correspondence

$$F(\zeta) = \frac{1}{f(1/\zeta)}, \quad |\zeta| > 1.$$

Using the change of variable $\zeta = \dfrac{1}{z}$, the association $f(z) = 1/F\left(\frac{1}{z}\right)$ quickly yields the formula

$$F'(\zeta) - 1 = U_f(z) \tag{2}$$

and

$$\frac{zf'(z)}{f(z)} = \frac{\zeta F'(\zeta)}{F(\zeta)}, \tag{3}$$

where U_f is defined by (1). Consequently, for $0 < \lambda \leq 1$, the formula (2) gives that $f \in \mathcal{U}(\lambda)$ if and only if $|F'(\zeta) - 1| < \lambda$ for $|\zeta| > 1$. The formula (3) could be used to connect the starlikeness of f with that of F.

2) An interesting fact is that each function in

$$\mathcal{S}_{\mathbb{Z}} = \left\{ z,\quad \frac{z}{(1\pm z)^2},\quad \frac{z}{1\pm z},\quad \frac{z}{1\pm z^2},\quad \frac{z}{1\pm z+z^2} \right\}$$

belongs to $\mathcal{U}\cap\mathcal{S}^*$, where $\mathcal{S}^*$ denoted the family of starlike functions f on $\mathbb{D}$, i.e., univalent functions f such that $f(\mathbb{D})$ is a domain starlike with respect to the origin. Also, it is well-known that these are the only functions in $\mathcal{S}$ having integer coefficients in the power series expansions of functions $f\in\mathcal{S}$ (see [9]).

3) The family $\mathcal{U}$ is not a subset of the starlike family $\mathcal{S}^*$ as the function

$$f_1(z) = \frac{z}{1+\frac{1}{2}z+\frac{1}{2}z^3}$$

demonstrates. Indeed, it is easy to see that $f_1\in\mathcal{U}$ and

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1-z^3}{1+\frac{1}{2}z+\frac{1}{2}z^3}$$

and at $z_0=(-1+i)/\sqrt{2}$, $|z_0|=1$, we obtain that

$$\mathrm{Re}\left\{\frac{z_0f_1'(z_0)}{f_1(z_0)}\right\} = \frac{2-2\sqrt{2}}{3} < 0,$$

from which it follows that the function f_1 is not starlike in $\mathbb{D}$. See also [16].

4) One of the sufficient conditions for a function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ to be in $\mathcal{S}^*$ is that $\sum_{n=2}^{\infty}n|a_n|\le 1$ and this result is sharp. In spite of the fact that neither $\mathcal{S}^*$ is included in $\mathcal{U}$ nor includes $\mathcal{U}$, it is known that (see also [8]) *the condition* $\sum_{n=2}^{\infty}n|a_n|\le 1$ *implies that* $f\in\mathcal{U}$. *The result is sharp.* On the other hand, if $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$ is in $\mathcal{S}^*$, then $f\in\mathcal{U}$. See [17].

5) Since $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$ is in $\mathcal{S}^*$ if and only if $\sum_{n=2}^{\infty}n|a_n|\le 1$ (see [25, Theorem 2]), this result can be used to generate univalent functions $f\in\mathcal{U}$ that are not starlike.

6) Since functions in $\mathcal{S}$ are not necessarily in $\mathcal{U}$, it is natural to consider the largest value r_0 so that for each $f \in \mathcal{S}$ the function $\frac{1}{r}f(rz) \in \mathcal{U}$ for $0 < r \leq r_0$. In this case we say that $r_0 := r_{\mathcal{U}}(\mathcal{S})$ is the $\mathcal{U}$ radius (or the radius of $\mathcal{U}$-property) in the class $\mathcal{S}$. It is known that ([15]), $r_{\mathcal{U}}(\mathcal{S}) = \frac{1}{\sqrt{2}}$. More generally, $r_{\mathcal{U}(\lambda)}(\mathcal{S}) = \sqrt{\frac{\lambda}{1+\lambda}}$. There has been a long history in determining radii problems in the theory of univalent functions, see [10].
7) In [18], it was shown that *if* $f \in \mathcal{U}(\lambda)$, $a := |f''(0)|/2 \leq 1$ *and* $0 \leq \lambda \leq \dfrac{\sqrt{2-a^2}-a}{2}$, *then* $f \in \mathcal{S}^*$. Later Fournier and Ponnusamy [8] have proved that the upper bound on λ is sharp. Moreover, they have shown that there exist non-starlike functions $f \in \mathcal{U}$ such that

$$0 < \frac{\sqrt{2-a^2}-a}{2} < \sup_{z\in\mathbb{D}} \left| f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| \leq 1-a.$$

Recently, a number of useful properties of the family $\mathcal{U}(\lambda)$ were established in [19, 20, 21, 22]. However, the coefficient problem for $\mathcal{U}(\lambda)$ remains open. This article supplements the earlier investigations in this topic.

Let $\mathcal{B} = \{\omega \in \mathcal{H} : |\omega(z)| < 1 \text{ on } |z| < 1\}$ and $\mathcal{B}_0 = \{\omega \in \mathcal{B} : \omega(0) = 0\}$. In addition, for $f, g \in \mathcal{H}$, we use the symbol $f(z) \prec g(z)$, or in short $f \prec g$, to mean that there exists an $\omega \in \mathcal{B}_0$ such that $f(z) = g(\omega(z))$. We now recall the following results from [19] which we need in the sequel.

Theorem A. *Suppose that* $f \in \mathcal{U}(\lambda)$ *for some* $\lambda \in (0,1]$ *and* $a_2 = f''(0)/2$. *Then we have the following:*

(a) *If* $|a_2| = 1+\lambda$, *then* f *must be of the form*

$$f(z) = \frac{z}{(1+e^{i\theta}z)(1+\lambda e^{i\theta}z)}.$$

(b) $\dfrac{z}{f(z)} + a_2 z \prec 1 + 2\lambda z + \lambda z^2$ *and* $\dfrac{f(z)}{z} \prec \dfrac{1}{(1-z)(1-\lambda z)}$, $z \in \mathbb{D}$.

As an analogue to the famous estimate for the Taylor coefficients of univalent functions proved by de Branges [5] (see also [3]), the following conjecture was proposed in [19].

Conjecture 1. *Suppose that $f \in \mathcal{U}(\lambda)$ for some $0 < \lambda \le 1$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then $|a_n| \le \sum_{k=0}^{n-1} \lambda^k$ for $n \ge 2$.*

This conjecture has been verified for $n = 2$ first in [26] and a simpler proof was given in [19]. More recently, in [21], Obradović et al. proved the conjecture for $n = 3, 4$ with an alternate proof for the case $n = 2$, but it remains open for all $n \ge 5$. Because $\mathcal{U}(1) \subsetneq \mathcal{S}$ and the Koebe function belongs to $\mathcal{U}(1)$, this conjecture obviously holds for $\lambda = 1$, in view of the de Branges theorem. Since no bound has been obtained for $|a_n|$ for $n \ge 5$, it seems useful to obtain a reasonable estimate. This attempt gives the following theorem and at the same time the proof for the case $\lambda = 1$ does not require the use of de Branges theorem that $|a_n| \le n$ for $f \in \mathcal{S}$ with equality for the Koebe function and its rotations.

Theorem 1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belong to $\mathcal{U}(\lambda)$ for some $0 < \lambda \le 1$. Then*

$$|a_n| \le 1 + \lambda\sqrt{n-1}\sqrt{\sum_{k=0}^{n-2} \lambda^{2k}}, \quad \text{for } n \ge 2.$$

Proof. Let $f \in \mathcal{U}(\lambda)$. Then the second subordination relation in Theorem A(b) shows that

$$\frac{f(z)}{z} \prec \frac{1}{1-\lambda z}\,\frac{1}{1-z} = f_1(z)\, f_2(z), \quad z \in \mathbb{D}.$$

This means that there exists a function $\phi \in \mathcal{B}_0$ such that

$$\frac{f(z)}{z} = f_1(\phi(z))\, f_2(\phi(z)), \quad z \in \mathbb{D}.$$

Define $g_1(z) = f_1(\phi(z))$ and $g_2(z) = f_2(\phi(z))$. Then

$$g_1(z) = \sum_{n=0}^{\infty} b_n z^n \prec f_1(z) = \frac{1}{1-\lambda z} \quad \text{and} \quad g_2(z) = \sum_{n=0}^{\infty} c_n z^n \prec f_2(z) = \frac{1}{1-z},$$

where $b_0 = c_0 = 1$, Rogosinski's theorems [24] (see also [7, Theorems 6.2 and 6.4]) give that

$$\sum_{k=1}^{n} |b_k|^2 \le \sum_{k=1}^{n} \lambda^{2k} \quad \text{and} \quad |c_n| \le 1 \quad \text{for } n \ge 1. \tag{4}$$

Moreover, the relation $\dfrac{f(z)}{z} = g_1(z)g_2(z)$ gives

$$a_{n+1} = \sum_{k=0}^{n} b_k c_{n-k}.$$

Consequently, by (4), it follows from the classical Cauchy-Schwarz inequality that

$$|a_{n+1}| \le 1 + \sum_{k=1}^{n} |b_k| \le 1 + \sqrt{n}\sqrt{\sum_{k=1}^{n} |b_k|^2} \le 1 + \sqrt{n}\sqrt{\sum_{k=1}^{n} \lambda^{2k}},$$

which implies the desired assertion. □

Suppose that $f \in \mathcal{U}(\lambda)$. Then the second subordination relation in Theorem A(b) shows that there exists a function $\phi \in \mathcal{B}_0$ such that

$$\frac{z}{f(z)} = 1 - (1+\lambda)\phi(z) + \lambda(\phi(z))^2, \quad z \in \mathbb{D}. \tag{5}$$

From Theorem A(a), we see that there is a member in the family $\mathcal{U}(\lambda)$ in the above form with $\phi(z) = e^{i\theta}z$. In this type of functions, we have $|a_2| = 1 + \lambda$. A natural question is whether there exist functions $\phi \in \mathcal{B}_0$ that are not of the form $\phi(z) = e^{i\theta}z$ of the above type for which the corresponding f of the form (5) belongs to $\mathcal{U}(\lambda)$. In order to prove the next result, we need the classical Julia lemma which is often quoted as Jack's lemma [12, Lemma 1] or Clunie-Jack's lemma [6] although this fact was known much before the work of Jack. See the article of Boas [4] for a historical commentary.

Lemma B. *Let* $|z_0| < 1$ *and* $r_0 = |z_0|$. *Let* $f(z) = \sum_{k=n}^{\infty} a_k z^k$ *be continuous on* $|z| \le r_0$ *and analytic on* $\{z : |z| < r_0\} \cup \{z_0\}$ *with* $f(z) \not\equiv 0$ *and* $n \ge 1$. *If* $|f(z_0)| = \max_{|z| \le r_0} |f(z)|$, *then* $z_0 f'(z_0)/f(z_0)$ *is real number and* $z_0 f'(z_0)/f(z_0) \ge n$.

Theorem 2. *Let* $f \in \mathcal{U}(\lambda)$ *be given by* (5), *with a function* ϕ *analytic on the closed unit disk* $\overline{\mathbb{D}}$ *and a point* $e^{i\theta_0}$ *such that* $\phi(e^{i\theta_0}) = -1$. *Then* ϕ *is of the form* $\phi(z) = e^{i\theta}z$.

Proof. We observe that $f \in \mathcal{U}(\lambda)$ if and only if

$$\left| \frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' - 1 \right| < \lambda, \quad z \in \mathbb{D},$$

which according to (1) and (5) implies that there exists a function $\phi \in \mathcal{B}_0$ such that

$$L(\phi)(z) = |-(1+\lambda)(\phi(z) - z\phi'(z)) + \lambda\phi(z)(\phi(z) - 2z\phi'(z))| < \lambda, \; z \in \mathbb{D}. \tag{6}$$

Let us consider now a function ϕ analytic in $\overline{\mathbb{D}}$ such that there exists θ_0 with $\phi(e^{i\theta_0}) = -1$. Examples of such functions are the Blaschke products. Now, we let $\tilde{\phi}(z) = \phi(rz)$ for $r > 1$ and sufficiently close to 1 such that $\tilde{\phi}$ is analytic in $\mathbb{D}$. If we apply Julia's lemma with $n = 1$ to $\tilde{\phi}$ and $z_0 = e^{i\theta_0}/r$, we see that

$$\frac{z_0\tilde{\phi}'(z_0)}{\tilde{\phi}(z_0)} = \frac{e^{i\theta_0}\phi'\left(e^{i\theta_0}\right)}{\phi\left(e^{i\theta_0}\right)} = m(\theta_0) \geq 1.$$

If we let $\phi(z) = z\psi(z)$, then we see that $\psi(\mathbb{D}) \subset \overline{\mathbb{D}}$ and $\psi(e^{i\theta_0}) = -e^{-i\theta_0}$. Now, we assume that $m(\theta_0) = 1$. Since

$$\frac{z\phi'(z)}{\phi(z)} = 1 + \frac{z\psi'(z)}{\psi(z)},$$

this means that $\psi'(e^{i\theta_0}) = 0$. If ψ' is not a constant, an angle with width less than π, sufficiently close to π and vertex $e^{i\theta_0}$ would be mapped by ψ onto an angle with width sufficiently close to 2π or more and a vertex $-e^{-i\theta_0}$. This contradicts the fact that $\psi(\mathbb{D}) \subset \overline{\mathbb{D}}$. Hence, $m(\theta_0) > 1$ or ϕ is of the form $\phi(z) = e^{i\theta}z$. From the above we get

$$e^{i\theta_0}\phi'(e^{i\theta_0}) = -m(\theta_0),$$

and therefore,

$$\begin{aligned} L(\phi)(z_0) = |-(1+\lambda)(\phi(z_0) - z_0\phi'(z_0)) + \lambda\phi(z_0)(\phi(z_0) - 2z_0\phi'(z_0))| = \\ = \lambda + (1+3\lambda)(m(\theta_0) - 1). \end{aligned}$$

If $m(\theta_0) > 1$, then $L(\phi)(z_0) > \lambda$. This contradicts (6) and hence, $\phi(z) = e^{i\theta}z$. The proof is complete. □

In [19, Theorem 5], under a mild restriction on $f \in \mathcal{U}(\lambda)$, the region of variability of a_2 is established as in the following form.

Theorem C. *Let $f \in \mathcal{U}(\lambda)$ for some $0 < \lambda \le 1$, and such that*

$$\frac{z}{f(z)} \neq (1-\lambda)(1+z), \ z \in \mathbb{D}. \tag{7}$$

Then, we have

$$\frac{z}{f(z)} - (1-\lambda)z \prec 1 + 2\lambda z + \lambda z^2 \tag{8}$$

and the estimate $|a_2 - (1-\lambda)| \le 2\lambda$ holds. In particular, $|a_2| \le 1+\lambda$ and the estimate is sharp as the function $f_\lambda(z) = z/((1+\lambda z)(1+z))$ shows.

Certainly, it was not unnatural to raise the question whether the condition (7) is necessary for a function f to belong to the family $\mathcal{U}(\lambda)$. This question was indeed raised in [19]. In the next result, we show that the condition (7) cannot be removed from Theorem C. Before we present the proof, it is worth recalling from [19] that if $f \in \mathcal{U}(\lambda)$, then for each $R \in (0,1)$, the function $f_R(z) = R^{-1}f(Rz)$ also belongs to $\mathcal{U}(\lambda)$.

Theorem 3. *Let $f(z) = z/((1-z)(1-\lambda z))$ and for a fixed $R \in (0,1)$, let $f_R(z) = R^{-1}f(Rz)$. Then we have*

(a) *For $0 < \lambda \le 1/2$ there exists, for any $R \in (0,1)$, an $r \in (0,1)$ such that $F(R,r) = 0$, where*

$$F(R,r) = \frac{r}{f_R(r)} - (1-\lambda)(1+r). \tag{9}$$

(b) *For $1/2 < \lambda < 1$ there exists, for any*

$$1 > R > \frac{1+\lambda - \sqrt{(1-\lambda)(1+7\lambda)}}{2\lambda},$$

an $r \in (0,1)$ such that $F(R,r) = 0$.

Proof. We consider $F(R,r)$ given by (9) and observe that

$$F(R,r) = \lambda R^2 r^2 - r[R(1+\lambda) + 1 - \lambda] + \lambda.$$

We see that in the cases indicated in the statement of the theorem $F(R,0) = \lambda > 0$ and $F(R,1) < 0$. Indeed

$$F(R,1) = \lambda R^2 - R(1+\lambda) + 2\lambda - 1 = -R[(1-R)\lambda + 1] - (1-2\lambda)$$

which is less than zero for any $R \in (0,1)$ and for $0 < \lambda \leq 1/2$. Similarly, for the case $1/2 < \lambda < 1$, one can compute the roots of the equation $F(R,1) = 0$ and obtain the desired conclusion. This proves the assertion of Theorem 3. $\Box$

Because of the characterization of functions in $\mathcal{U}(\lambda)$ via functions in $\mathcal{B}$, the following result is of independent interest. As pointed out in the introduction, it is known that if $f \in \mathcal{U}(\lambda)$, then $|a_2| \leq 1+\lambda$ with equality for $f(z) = z/[(1-z)(1-\lambda z)]$ and its rotations.

Theorem 4. *Let $f \in \mathcal{U}(\lambda)$, $\lambda \in (0,1)$, have the form*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = \frac{z}{1 - a_2 z + \lambda z \int\limits_0^z \omega(t)\,dt} \tag{10}$$

for some $\omega \in \mathcal{B}$ such that $\omega(0) = a \in \mathbb{D}$ and $v(x)$ be defined by

$$v(x) = \int\limits_0^1 \frac{x+t}{1+xt}\,dt = \frac{1}{x} - \frac{1-x^2}{x^2}\log(1+x) < 1 \quad \textit{for } 0 < x < 1,$$

and $v(0) = \lim\limits_{x\to 0^+} v(x) = 1/2$. Then $|a_2| \leq 1 + \lambda v(|a|)$. The result is sharp.

Proof. Let $f \in \mathcal{U}(\lambda)$. Then, we may write (cf. [14])

$$f'(z)\left(\frac{z}{f(z)}\right)^2 = -z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} = 1 + \lambda\omega(z), \tag{11}$$

where $\omega\colon \mathbb{D} \to \mathbb{D}$ is analytic with $\omega(0) = \omega'(0) = 0$. By the Schwarz' lemma, we have $|\omega(z)| \leq |z|^2$ in $\mathbb{D}$ and hence, $|U_f(z)| \leq |z|^2$ for $z \in \mathbb{D}$. In view of (11), $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n \in \mathcal{U}(\lambda)$ if and only if

$$\frac{z}{f(z)} = 1 - a_2 z + \lambda z \int\limits_0^z \omega(t)\,dt \neq 0, \quad z \in \mathbb{D}, \tag{12}$$

where $\omega \in \mathcal{B}$. By assumption $\omega(0) = a \in \mathbb{D}$. As in the proof of [19, Theorem 1], assume on the contrary that

$$|a_2| = \frac{1 + \lambda v(|a|)}{r}, \quad r \in (0,1), \tag{13}$$

and consider the function F defined by

$$F(z) = \frac{1}{a_2}\left[1 + \lambda z \int_0^z \omega(t)\,dt\right], \quad z \in \mathbb{D}.$$

Then, according to the Schwarz-Pick lemma applied to $\omega \in \mathcal{B}$, we can easily obtain that

$$|\omega(z)| \leq \frac{|a| + |z|}{1 + |az|}, \quad z \in \mathbb{D},$$

and thus, as in the proof of [19, Theorem 2], it follows that

$$\left|\int_0^z \omega(t)\,dt\right| \leq v(|a|) < 1, \quad z \in \mathbb{D},$$

where $v(x)$ is defined as in the statement. Consequently, for $|z| \leq r$, we get by (13)

$$|F(z)| \leq \frac{1}{|a_2|}\left[1 + \lambda|z|\left|\int_0^z \omega(t)\,dt\right|\right] \leq \frac{1 + r\lambda v(|a|)}{|a_2|} = \frac{(1 + r\lambda v(|a|))r}{1 + \lambda v(|a|)} < r.$$

Hence F is a mapping of the closed disk $\overline{\mathbb{D}}_r$ into itself, where $\mathbb{D}_r = \{z : |z| < r\}$. Secondly, we have for z_1 and z_2 in $\overline{\mathbb{D}}_r$,

$$|F(z_1) - F(z_2)| = \frac{\lambda r}{1 + \lambda v(|a|)}\left|z_1 \int_0^{z_1} \omega(t)\,dt + (-z_1 + z_1 - z_2)\int_0^{z_2} \omega(t)\,dt\right| \leq$$

$$\leq \frac{\lambda r}{1 + \lambda v(|a|)}\left(|z_1|\left|\int_{z_2}^{z_1} \omega(t)\,dt\right| + |z_1 - z_2|\left|\int_0^{z_2} \omega(t)\,dt\right|\right) \leq$$

$$\leq \frac{\lambda r}{1 + \lambda v(|a|)}(|z_1| + v(|a|))|z_1 - z_2| \leq \frac{\lambda r(r + v(|a|))}{1 + \lambda v(|a|)}|z_1 - z_2| < r|z_1 - z_2|.$$

Therefore, F is a contraction of the disk $\overline{\mathbb{D}}_r$ and according to Banach's fixed point theorem, F has a fixed point in $\overline{\mathbb{D}}_r$. This implies that there exists a $z_0 \in \mathbb{D}_r$ such that $F(z_0) = z_0$ which contradicts (12) at $z_0 \in \mathbb{D}$

(and thus, (13) is not true for any $r \in (0,1)$). Hence, we must have $|a_2| \le 1 + \lambda v(|a|)$ for $f \in \mathcal{U}(\lambda)$.

To prove that the second coefficient inequality is sharp, we consider

$$\omega(z) = \frac{z+a}{1+az}, \quad a \in (0,1), \tag{14}$$

and we use the fact that $v(a) = \int_0^1 \omega(t)\,dt$. Hence,

$$1 - (1+\lambda v(a))z + \lambda z \int_0^z \omega(t)\,dt = 1 - z - \lambda z \int_z^1 \omega(t)\,dt =: G(z).$$

We claim that $G(z) \neq 0$ in $\mathbb{D}$. Since $G(0) = 1$, we may assume on the contrary that there exists a $z \in \mathbb{D} \setminus \{0\}$ such that $G(z) = 0$. This is equivalent to

$$\frac{1}{\lambda z} = \frac{1}{1-z} \int_z^1 \omega(t)\,dt.$$

As

$$\left|\frac{1}{\lambda z}\right| > 1 \quad \text{and} \quad \left|\frac{1}{1-z} \int_z^1 \omega(t)\,dt\right| \le 1,$$

we have now proved that $G(z) \neq 0$ for $z \in \mathbb{D}$. In particular, this implies that the function f defined by

$$f(z) = \frac{z}{1 - (1+\lambda v(a))z + \lambda z \int_0^z \omega(t)\,dt}$$

belongs to the family $\mathcal{U}(\lambda)$, where ω is given by (14). This proves the sharpness. □

Moreover, one can show that a similar sharp inequality is valid for any ω as above.

Since $\left|\int_{z_1}^{z_2} \omega(t)\,dt\right| \le |z_1 - z_2|$, the function $\int_0^z \omega(t)\,dt$ is uniformly continuous in the open unit disk. Therefore this function can be extended continuously onto the closed unit disk. Hence, the real functional $m(\omega) :=$

$:= \max\left\{\left|\int\limits_0^z \omega(t)\,dt\right| : z \in \overline{\mathbb{D}}\right\}$ is well defined. Suppose that $f \in \mathcal{U}(\lambda)$ is given by

$$f(z) = \frac{z}{1 - a_2 z + \lambda z \int\limits_0^z \omega(t)\,dt}$$

for some $0 \leq \lambda < 1$, where $\omega \in \mathcal{B}$. Then

$$|a_2| \leq 1 + \lambda m(\omega), \tag{15}$$

is valid and this inequality is sharp.

In order to prove this inequality, we assume again that

$$|a_2| = \frac{1 + \lambda m(\omega)}{r}, \quad r \in (0,1),$$

and do similar steps as in the proof of Theorem 4. The inequality (15) can be shown to be sharp in the following way: Consider

$$\tilde{\omega}(z) \;=\; e^{i\varphi}\omega\left(e^{i\theta}z\right),$$

where $\varphi, \theta \in [0, 2\pi)$ are chosen such that

$$m(\omega) \;=\; \int\limits_0^1 \tilde{\omega}(t)\,dt.$$

Next, we may proceed as before to complete the proof. However, we omit the details to avoid a repetition of the arguments.

A more detailed consideration of these cases can give more explicit bounds for $|a_2|$ as follows.

Theorem 5. *Let $f \in \mathcal{U}(\lambda)$, $\lambda \in (0,1)$, have the form* (10) *for some analytic function ω such that $|\omega(z)| \leq 1$ and $\omega(0) = a \in \overline{\mathbb{D}}$. Let further*

$$B_a(z) = \frac{1}{\overline{a}} - \frac{1-|a|^2}{\overline{a}^2 z}\log(1+\overline{a}z) = a \;+\; \left(1-|a|^2\right)\sum_{k=1}^{\infty}(-\overline{a})^{k-1}\frac{z^k}{k+1}$$

for $a \in \overline{\mathbb{D}}$. Then $|a_2| \;\leq 1 + \lambda\max\{\left|B_a\left(e^{i\tau}\right)\right| : \tau \in [0, 2\pi]\}$. The inequality is sharp.

Proof. The function f considered here by (10) is a member of the class $\mathcal{U}(\lambda)$ if and only if $z/f(z) \neq 0$, which is equivalent to

$$a_2 \neq \frac{1}{z} + \lambda \int_0^z \omega(t)\, dt := C_\omega(z), \quad z \in \mathbb{D}.$$

Using the above argument, it is clear that the function C_ω can be extended continuously onto the boundary $\partial\mathbb{D}$. Moreover this function is univalent on $\overline{\mathbb{D}}$. The proof of this assertion is similar to the above arguments. Indeed if $C_\omega(z_1) = C_\omega(z_2)$ for some $z_1 \neq z_2,\ z_1, z_2 \in \overline{\mathbb{D}}$, then

$$\frac{\lambda}{z_1 - z_2} \int_{z_1}^{z_2} \omega(t)\, dt = \frac{1}{z_1 z_2}$$

which is not possible. Thus, C_ω is univalent on $\overline{\mathbb{D}}$ and therefore, for each ω, the curve $C_\omega\left(e^{i\theta}\right)$, $\theta \in [0, 2\pi]$, is a Jordan curve which divides the plane into two components. Let us call the bounded closed component $\overline{\mathbb{C}} \setminus C_\omega(\mathbb{D}) =: A_2(\omega)$. Obviously, the function f is in the class $\mathcal{U}(\lambda)$ if and only if

$$a_2 \in \bigcup_{\omega(0)=a} A_2(\omega).$$

Now, we look at the curves $C_\omega\left(e^{i\theta}\right)$, $\theta \in [0, 2\pi]$. Since $\omega(0) = a$, the modulus of the function

$$\frac{\omega(z) - a}{1 - \overline{a}\omega(z)}$$

is bounded by unity in the unit disk and this function vanishes at the origin. This means that ω can be represented in the form

$$\omega(z) = \frac{a + z\varphi(z)}{1 + \overline{a}z\varphi(z)},$$

where φ is analytic in $\mathbb{D}$ and $|\varphi(z)| \leq 1$ for $z \in \mathbb{D}$. In other words, $\omega(z)$ is subordinate to $(a+z)/(1+\overline{a}z)$, $z \in \mathbb{D}$. Since the function $(a+z)/(1+\overline{a}z)$ maps the unit disk onto the unit disk, a convex domain, we may now use a theorem proved by Hallenbeck and Ruscheweyh in [11] (compare with [13, Theorem 3.1b]). In our case we use the fact that the function

$$p(z) = \frac{1}{z} \int_0^z \omega(t)\, dt$$

satisfies the subordination relation

$$p(z) + zp'(z) = \omega(z) \prec \frac{a + z}{1 + \overline{a}z} = h(z).$$

According to the above theorem, in this case the function p is subordinate to the function

$$\frac{1}{z}\int_0^z h(t)\,dt = \frac{1}{z}\int_0^z \frac{a + t}{1 + \overline{a}t}\,dt = B_a(z).$$

Therefore, we get the representation

$$\int_0^z \omega(t)\,dt = \frac{1}{\varphi(z)}\int_0^{z\varphi(z)} \frac{a + t}{1 + \overline{a}t}\,dt = zB_a(z\varphi(z)),$$

where φ is analytic in $\mathbb{D}$ and $|\varphi(z)| \leq 1$ for $z \in \mathbb{D}$. Since B_a is analytic in the closed unit disk this representation together with the above considerations implies that

$$|a_2| \leq \sup_{z\in\mathbb{D},\theta\in[0,2\pi]} \left|e^{-i\theta} + \lambda e^{i\theta} B_a(z)\right| \leq 1 + \lambda \max\{\left|B_a\left(e^{i\tau}\right)\right| : \tau \in [0, 2\pi]\}.$$

Now, we have to prove the sharpness of the inequality. To that end, let τ_0 be chosen such that

$$\left|B_a\left(e^{i\tau_0}\right)\right| = \max\{\left|B_a\left(e^{i\tau}\right)\right| : \tau \in [0, 2\pi]\}, \text{ and } B_a\left(e^{i\tau_0}\right) = e^{i\alpha}\left|B_a\left(e^{i\tau_0}\right)\right|.$$

We take $2\theta = -\alpha$, $\psi = \tau_0 - \theta$, consider the function

$$\omega(z) = \frac{a + ze^{i\psi}}{1 + \overline{a}ze^{i\psi}},$$

and let $a_2 = e^{-i\theta} + \lambda e^{i\theta} B_a\left(e^{i\tau_0}\right)$. Then we have

$$|a_2| = \left|e^{-2i\theta} + \lambda\, e^{i\alpha}\left|B_a\left(e^{i\tau_0}\right)\right|\right| = 1 + \lambda\left|B_a\left(e^{i\tau_0}\right)\right|.$$

Further, we consider

$$D(z) = 1 - \left(e^{-i\theta} + \lambda e^{i\theta} B_a\left(e^{i\tau_0}\right)\right) z + \lambda z \int_0^z \frac{a + te^{i\psi}}{1 + \overline{a}te^{i\psi}}\,dt.$$

It is easily seen that in our case

$$D(z) = 1 - \left(e^{-i\theta} + \lambda e^{i\theta} B_a\left(e^{i\tau_0}\right)\right) z + \lambda z^2 B_a\left(ze^{i\psi}\right) \quad \text{and} \quad D\left(e^{i\theta}\right) = 0.$$

The assumption that there would exist a second zero w of D in the unit disk, via the equation $D(w) = D\left(e^{i\theta}\right)$ leads to

$$\frac{1}{w} + \lambda \int_0^w \omega(t)\, dt = e^{-i\theta} + \lambda \int_0^{e^{i\theta}} \omega(t)\, dt.$$

Now we proceed similar to a reasoning above. We conclude that this implies

$$\frac{1}{\lambda w e^{i\theta}} = \frac{1}{e^{i\theta} - w} \int_w^{e^{i\theta}} \omega(t)\, dt.$$

But this is impossible, since the left hand side has modulus bigger than 1, whereas the right hand side has modulus less than or equal to 1. Hence, the function $f(z) = z/D(z)$ is a member of the class $\mathcal{U}(\lambda)$. □

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