DOI: 10.15393/j3.art.2018.4830

UDC 517.548, 517.552

J. E. Kim

THE CORRESPONDING CAUCHY – RIEMANN SYSTEM FOR DUAL QUATERNION-VALUED FUNCTIONS

Abstract. This paper provides differential operators in dual quaternions and represents the regularity of dual quaternion-valued functions using the dual Cauchy – Riemann system in dual quaternions. Also, we give the corresponding Cauchy theorem of the dual quaternion-valued function in Clifford analysis.

Key words: *quaternion, dual number, Cauchy – Riemann system, Cauchy theorem, Clifford analysis*

2010 Mathematical Subject Classification: *32A99, 32W50, 30G35, 11E88*

1. Introduction. A dual quaternion can be represented in the form $p + \varepsilon q$, where p and q are ordinary quaternions and ε is the so-called dual unit, an element that commutes with every element of the algebra and is such that $\varepsilon^2 = 0$. Unlike quaternions, not every dual quaternion has an inverse. The set of dual quaternions is the following Clifford algebra:

$$\mathbb{D}_q := \{Z = p_1 + \varepsilon p_2 \mid p_1, p_2 \in \mathbb{H}\},$$

where $\mathbb{H}$ is the set of quaternions which are combined by the basis elements 1, i, j, k. It has the product rule for i, j and k given by

$$i^2 = j^2 = k^2 = ijk = -1$$

and

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

For two quaternions $p = z_1 + z_2 j$ and $q = w_1 + w_2 j$, where $z_1 = x_0 + ix_1$, $z_2 = x_2 + ix_3$, $w_1 = y_0 + iy_1$ and $w_2 = y_2 + iy_3$, the rule of addition is:

$$p + q = (z_1 + w_1) + (z_2 + w_2)j$$

and multiplication is:

$$pq = (z_1 w_1 - z_2 \overline{w_2}) + (z_1 w_2 + z_2 \overline{w_1})j.$$

From the above rules, we give a norm for a quaternion as follows:

$$|p|^2 := pp^* = z_1 \overline{z_1} + z_2 \overline{z_2}$$

and the inverse of p as follows:

$$p^{-1} = \frac{p^*}{|p|^2} \quad (p \neq 0).$$

Hamilton [7, 8] introduced quaternions in 1843, and in 1873 Clifford [4, 5] obtained a broad generalization of these numbers, which is now called the Clifford algebra [14]. At the turn of the 20th century, Kotelnikov [12] and Study [15] developed dual vectors and dual quaternions. In 1891 Study realized that this associative algebra was ideal for describing the group of motions of three-dimensional spaces. He further developed the idea in [15]. Kajiwara et al. [9] gave a basic estimate for inhomogeneous Cauchy – Riemann system and applied the theory to a closed densely defined operator in a Hilbert space. Kim et al. [10] obtained a corresponding inverse of functions and their properties and a regularity of functions on the form of multidual complex variables in Clifford analysis. Also, we [11] researched corresponding Cauchy – Riemann systems and properties of functions with values in special quaternions and split quaternions by using a regular function with values in dual split quaternions. Mathematicians has appeared null solutions of the Douglis operator, called hyperanalytic functions theory. Blaya et al. [1] presented the definition of conjugate hyperharmonic Douglis algebra-valued functions which proposed generalization of the classical conjugate harmonic functions in the Complex analysis case. They established an upper bound for the norm of a fractal Hilbert transform in the space of Hölder analytic functions and characterized the monogenicity of functions and generalizations of certain two-sided monogenic extension results in the sense of Douglis operator (see [2, 3]).

This paper investigates the expression of differential operators in dual quaternions. The paper also represents a corresponding Cauchy theorem of dual quaternion-valued functions by using a dual Cauchy – Riemann system in dual quaternions.

2. Preliminaries. We consider the following set

$$\mathbb{D}_q = \{Z = p_1 + \varepsilon p_2 \mid p_r \in \mathbb{H},\ \varepsilon^2 = 0,\ r = 1, 2\},$$

which is isomorphic to $\mathbb{H}^2$ and $\mathbb{R}^8$. For $Z = p_1 + \varepsilon p_2$ and $W = q_1 + \varepsilon q_2$, we have the following rules of addition and multiplication on $\mathbb{D}_q$:

$$Z + W = (p_1 + q_1) + \varepsilon(p_2 + q_2)$$

and

$$ZW = p_1 q_1 + \varepsilon(p_1 q_2 + p_2 q_1),$$

respectively. We give a complex conjugate element of $\mathbb{D}_q$ as follows:

$$Z^* = p_1^* + \varepsilon p_2^*$$

and then, the norm of Z, denoted by $|Z|$, is described by

$$|Z|^2 = \frac{1}{2}(ZZ^* + Z^*Z) = p_1 p_1^* + 2\varepsilon \mathrm{S}(p_1 p_2^*),$$

where $\mathrm{S}(p_1 p_2^*)$ is the scalar part of $p_1 p_2^*$ such that

$$\mathrm{S}(p_1 p_2^*) = x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

The elements of the set $\{\varepsilon p \mid p \in \mathbb{H}\}$ are not invertible; for a dual quaternion $Z = p_1 + \varepsilon p_2$ outside this set, the inverse is given by

$$Z^{-1} = \frac{Z^\dagger}{p_1 p_1^*} \quad (p_1 \neq 0),$$

where

$$Z^\dagger = p_1^* - \varepsilon p_1^{-1} p_2 p_1^*,$$

called the dual conjugate of Z with $ZZ^\dagger = Z^\dagger Z = p_1 p_1^*$.

3. Hyperholomorphic function in dual quaternions. Let Ω be a bounded open set in $\mathbb{D}_q$. A function is given by

$$F : \Omega \to \mathbb{D}_q; \quad F(Z) = f_1(p_1, p_2) + \varepsilon f_2(p_1, p_2),$$

where

$$f_1 = g_1(z_1, z_2, w_1, w_2) + j\overline{g_2}(z_1, z_2, w_1, w_2)$$

and

$$f_2 = h_1(z_1, z_2, w_1, w_2) + j\overline{h_2}(z_1, z_2, w_1, w_2)$$

are quaternion-valued functions and $g_1 = u_0 + iu_1$, $\overline{g_2} = u_2 - iu_3$, $h_1 = v_0 + iv_1$ and $\overline{h_2} = v_2 - iv_3$ are complex-valued functions with real-valued functions u_r and v_r $(r = 0, 1, 2, 3)$.

We consider the corresponding differential operators:

$$D := \Big(\frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}\Big) + \varepsilon\Big(\frac{\partial}{\partial y_0} - i\frac{\partial}{\partial y_1} - j\frac{\partial}{\partial y_2} - k\frac{\partial}{\partial y_3}\Big)$$

and

$$D^* = \Big(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\Big) + \varepsilon\Big(\frac{\partial}{\partial y_0} + i\frac{\partial}{\partial y_1} + j\frac{\partial}{\partial y_2} + k\frac{\partial}{\partial y_3}\Big).$$

Using the properties of the basis elements 1, i, j, and k, we also write as follows:

$$D := \Big(\frac{\partial}{\partial z_1} - j\frac{\partial}{\partial z_2}\Big) + \varepsilon\Big(\frac{\partial}{\partial w_1} - j\frac{\partial}{\partial w_2}\Big)$$

and

$$D^* = \Big(\frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial z_2}\Big) + \varepsilon\Big(\frac{\partial}{\partial \overline{w_1}} + j\frac{\partial}{\partial w_2}\Big),$$

where $\frac{\partial}{\partial z_r}$, $\frac{\partial}{\partial w_r}$, $\frac{\partial}{\partial \overline{z_r}}$ and $\frac{\partial}{\partial \overline{w_r}}$ $(r = 1, 2)$ are the usual differential operators in complex analysis. We let

$$\frac{\partial}{\partial p_1} := \frac{\partial}{\partial z_1} - j\frac{\partial}{\partial z_2}, \quad \frac{\partial}{\partial p_2} := \frac{\partial}{\partial w_1} - j\frac{\partial}{\partial w_2}$$

and

$$\frac{\partial}{\partial p_1^*} = \frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial z_2}, \quad \frac{\partial}{\partial p_2^*} = \frac{\partial}{\partial \overline{w_1}} + j\frac{\partial}{\partial w_2}.$$

Then

$$D := \frac{\partial}{\partial p_1} + \varepsilon\frac{\partial}{\partial p_2} \qquad \text{and} \qquad D^* = \frac{\partial}{\partial p_1^*} + \varepsilon\frac{\partial}{\partial p_2^*}.$$

Let $C^1(\Omega, \mathbb{D}_q)$ be the set of continuous functions from Ω to $\mathbb{D}_q$. The corresponding Laplace operator on $C^1(\Omega, \mathbb{D}_q)$ is

$$\begin{aligned}\triangle &:= \frac{1}{2}(DD^* + D^*D) = \frac{1}{2}(D^*D + DD^*) = \\ &= \Big(\frac{\partial}{\partial p_1}\frac{\partial}{\partial p_1^*} + \varepsilon 2\mathrm{S}\Big(\frac{\partial}{\partial p_1}\frac{\partial}{\partial p_2^*}\Big)\Big) = \\ &= \Big(\frac{\partial}{\partial p_1^*}\frac{\partial}{\partial p_1} + \varepsilon 2\mathrm{S}\Big(\frac{\partial}{\partial p_1^*}\frac{\partial}{\partial p_2}\Big)\Big).\end{aligned}$$

Consider the following calculations with the differential operators defined above:

Remark 1. *On a bounded open set Ω in $\mathbb{H}^2$, we have*

$$\begin{aligned}DF &= \frac{\partial f_1}{\partial p_1} + \varepsilon\Big(\frac{\partial f_2}{\partial p_1} + \frac{\partial f_1}{\partial p_2}\Big), \\ D^*F &= \frac{\partial f_1}{\partial p_1^*} + \varepsilon\Big(\frac{\partial f_2}{\partial p_1^*} + \frac{\partial f_1}{\partial p_2^*}\Big) = \\ &= \Big(\frac{\partial g_1}{\partial \overline{z_1}} - \frac{\partial \overline{g_2}}{\partial \overline{z_2}} + j\Big(\frac{\partial \overline{g_2}}{\partial z_1} + \frac{\partial g_1}{\partial z_2}\Big)\Big) + \\ &\quad + \varepsilon\Big\{\Big(\frac{\partial h_1}{\partial \overline{z_1}} - \frac{\partial \overline{h_2}}{\partial \overline{z_2}} + \frac{\partial g_1}{\partial \overline{w_1}} - \frac{\partial \overline{g_2}}{\partial \overline{w_2}}\Big) + \\ &\quad + j\Big(\frac{\partial \overline{h_2}}{\partial z_1} + \frac{\partial h_1}{\partial z_2} + \frac{\partial \overline{g_2}}{\partial w_1} + \frac{\partial g_1}{\partial w_2}\Big)\Big\}.\end{aligned}$$

Definition 1. *Let Ω be a bounded open set in $\mathbb{H}^2$. A function F is said to be left (resp. right) regular on Ω if the components f_1 and f_2 of F are both continuously differentiable functions and F satisfies the following equation*

$$D^*F = 0 \quad (\textit{resp.}\ \ FD^* = 0). \tag{1}$$

From Remark 1, the left equation of (1) is equivalent to

$$\frac{\partial f_1}{\partial p_1^*} = 0 \qquad \text{and} \qquad \frac{\partial f_2}{\partial p_1^*} = -\frac{\partial f_1}{\partial p_2^*}.$$

In detail, it is also equivalent to the following system:

$$\begin{cases} \dfrac{\partial g_1}{\partial \overline{z_1}} = \dfrac{\partial \overline{g_2}}{\partial \overline{z_2}}, \\[2ex] \dfrac{\partial \overline{g_2}}{\partial z_1} = -\dfrac{\partial g_1}{\partial z_2}, \\[2ex] \dfrac{\partial h_1}{\partial \overline{z_1}} + \dfrac{\partial g_1}{\partial \overline{w_1}} = \dfrac{\partial h_2}{\partial \overline{z_2}} + \dfrac{\partial g_2}{\partial \overline{w_2}}, \\[2ex] \dfrac{\partial h_2}{\partial z_1} + \dfrac{\partial g_2}{\partial w_1} = -\dfrac{\partial h_1}{\partial z_2} - \dfrac{\partial g_1}{\partial w_2}. \end{cases} \tag{2}$$

Clearly, the properties and progresses of the theory of left regular functions are equivalent to that of right regular functions. For the sake of the convenience we consider left regular functions, which are called just regular.

Let U be a compact oriented C^∞-manifold with boundary ∂U contained in a domain Ω of $\mathbb{H}^2$. For $r \quad (0 \leq r \leq 3)$, let

$$d\widehat{x_r} := dx_0 \wedge \cdots \wedge dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_3 \wedge dy_0 \wedge dy_1 \wedge dy_2 \wedge dy_3,$$

$$d\widehat{y_r} := dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dy_0 \wedge \cdots \wedge dy_{r-1} \wedge dy_{r+1} \wedge \cdots \wedge dy_3$$

and

$$d\sigma = d\sigma(p_1, p_2) := d\sigma_1 + \varepsilon d\sigma_2,$$

where

$$d\sigma_1 := d\widehat{x_0} - id\widehat{x_1} + jd\widehat{x_2} - kd\widehat{x_3}$$

and

$$\sigma_2 := d\widehat{y_0} - id\widehat{y_1} + jd\widehat{y_2} - kd\widehat{y_3}.$$

Hence, for $F = F(p_1, p_2) = g(z, w) + \varepsilon h(z, w)$ in $C^\infty(\Omega, \mathbb{D}_q)$, where

$$g(z, w) = u_0 + u_1 i + u_2 j + u_3 k$$

and

$$h(z, w) = v_0 + v_1 i + v_2 j + v_3 k$$

with

$$(z, w) = (x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3),$$

the corresponding dual quaternion-valued 7-form is

$$\begin{aligned}\omega &:= d\sigma(p_1,p_2)F(p_1,p_2) = (d\sigma_1+\varepsilon d\sigma_2)(g(z,w)+\varepsilon h(z,w)) = \\ &= d\sigma_1 g(z,w)+\varepsilon d\sigma_1 h(z,w)+\varepsilon d\sigma_2 g(z,w)\end{aligned}$$

and the exterior derivative is

$$\begin{aligned}d\omega &:= \frac{\partial F}{\partial x_0}dx_0\wedge d\widehat{x_0} - i\frac{\partial F}{\partial x_1}dx_1\wedge d\widehat{x_1} + j\frac{\partial F}{\partial x_2}dx_2\wedge d\widehat{x_2} - \\ &\quad -k\frac{\partial F}{\partial x_3}dx_3\wedge d\widehat{x_3} + \varepsilon\Big(\frac{\partial g}{\partial y_0}dy_0\wedge d\widehat{y_0} - i\frac{\partial g}{\partial y_1}dy_1\wedge d\widehat{y_1} + \\ &\quad + j\frac{\partial g}{\partial y_2}dy_2\wedge d\widehat{y_2} - k\frac{\partial g}{\partial y_3}dy_3\wedge d\widehat{y_3}\Big) = \\ &= (D^*F)dV(p_1,p_2),\end{aligned}$$

where $dV(p_1,p_2) = dx_0\wedge dx_1\wedge dx_2\wedge dx_3\wedge dy_0\wedge dy_1\wedge dy_2\wedge dy_3$.

Remark 2. *For each $Z\in\partial U$, let $n = n(p_1,p_2) = N+\varepsilon M$, where $N = n_0+n_1i+n_2j+n_3k$ and $M = m_0+m_1i+m_2j+m_3k$ be the outward unit normal to ∂U at Z. Then we have*

$$d\sigma(p_1,p_2) = n(p_1,p_2)dS(p_1,p_2)$$

and

$$\omega = n(p_1,p_2)F(p_1,p_2)dS(p_1,p_2) = Ng(z,w)+\varepsilon(Nh(z,w)+Mg(z,w)),$$

where $dS(p_1,p_2)$ is the scalar element of a surface area on ∂U.

Following [13], let

$$\Phi(Z-W) = \frac{Z^*-W^*}{\nu|Z-W|^8},$$

where

$$Z^* = \zeta_0-\zeta_1 i-\zeta_2 j-\zeta_3 k \quad \text{and} \quad W^* = \eta_0-\eta_1 i-\eta_2 j-\eta_3 k$$

with $\zeta_r = x_r+\varepsilon y_r$ and $\eta_r = \lambda_r+\varepsilon\mu_r$ ($r = 0,1,2,3$ and $x_r, y_r, \lambda_r, \mu_r\in\mathbb{R}$), and ν is the surface area of the unit sphere in $\mathbb{H}^2$, it is called the Cauchy kernel on $\mathbb{D}_q$.

Remark 3. *For $Z = \zeta_0 + \zeta_1 i + \zeta_2 j + \zeta_3 k$, the norm of Z is*

$$ZZ^* = Z^* Z = \sum_{r=0}^{3} \zeta_r^2.$$

Remark 4. *Let $\Phi(Z - W) = \frac{Z^* - W^*}{\nu|Z-W|^8}$, where ν is the surface of the unit sphere in $\mathbb{R}^4$, is so-called the Cauchy kernel on $\mathbb{D}_q$. The function $\Phi(Z - W)$ is left and right regular in Ω. Indeed,*

$$\begin{aligned}
D^*\Phi(Z-W) &= D^*\Big(\frac{Z^*-W^*}{\nu|Z-W|^8}\Big) = \\
&= \Big(\frac{\partial}{\partial \zeta_0} + i\frac{\partial}{\partial \zeta_1} + j\frac{\partial}{\partial \zeta_2} + k\frac{\partial}{\partial \zeta_3}\Big) \times \\
&\times \Big(\frac{(\zeta_0-\eta_0) - (\zeta_1-\eta_1)i - (\zeta_2-\eta_2)j - (\zeta_3-\eta_3)k}{\nu(\sum_{r=0}^{3}(\zeta_r-\eta_r)^2)^4}\Big) = \\
&= \Big(\frac{Z^*-W^*}{\nu|Z-W|^8}\Big)D^* = 0,
\end{aligned}$$

where

$$\frac{\partial}{\partial \zeta_r} = \frac{\partial}{\partial x_r} + \varepsilon\frac{\partial}{\partial y_r}$$

and

$$\frac{\partial}{\partial \zeta_r}\zeta_r = 1, \quad (r = 0, 1, 2, 3).$$

Lemma 1. [6] *Let Ω be a bounded open set in $\mathbb{D}_q$. Let u and v be smooth scalar-valued functions on Ω. Then for all r and t $(0 \leq r, t \leq 3)$,*

$$\int_U \Big(u\frac{\partial v}{\partial x_r} + \frac{\partial u}{\partial x_r}v\Big)dV = \int_{\partial U} uv\ n_r dS$$

and

$$\int_U \Big(\frac{\partial v}{\partial y_t} + \frac{\partial u}{\partial y_t}v\Big)dV = \int_{\partial U} uv\ m_t dS,$$

where n_r and m_t are defined in Remark 2.

Lemma 2. [13] *Let Ω be a bounded open set in $\mathbb{D}_q$. Let F and ψ be smooth dual quaternion-valued functions on Ω, where $F = g(z, w) +$*

$+\varepsilon h(z,w)$ and $\psi = \phi(z,w) + \varepsilon\varphi(z,w)$ with $\phi(z,w) = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ and $\varphi(z,w) = \beta_0 + \beta_1 i + \beta_2 j + \beta_3 k$, where α_r and β_r $(r = 0,1,2,3)$ are real-valued functions. Then we have

$$\int_U \{F(D^*\psi) + (FD^*)\psi\} dV = \int_{\partial U} F(n\psi) dS,$$

where n is defined in Remark 2.

Theorem 1. *Let Ω be a bounded open set in $\mathbb{D}_q$ and U be a subset of Ω. For $Z \in \Omega$, if $D^*F = 0$ and $W \in \mathrm{int}(U)$, then we have*

$$F(Z) = \int_{\partial U} \Phi(Z-W)\{d\sigma(p_1,p_2)F(p_1,p_2)\},$$

where $\mathrm{int}(U)$ is the interior of U and $\Phi(Z-W)$ is a regular function expressed in Remark 4. Also, if $D^*F = 0$ and $W \in \Omega \setminus U$, then the above integral is zero.

Proof. For $W \in \mathrm{int}(U)$ and given $\epsilon > 0$, let U_B be U except the open ball of radius ϵ centered at W. Then from Remark 4, the function $\Phi(Z-W)$ is regular in U_B and from Lemma 2, we have

$$\int_{U_B} \{\Phi(Z-W)(D^*F) + (\Phi(Z-W)D^*)F\} dV =$$

$$= \int_{U_B} \{\Phi(Z-W)(D^*F) dV = \int_{U_B} \Phi(Z-W)(n\ F) dS =$$

$$= \int_{\partial U} \Phi(Z-W)(n\ F) dS - \int_{B_\epsilon} \Phi(Z-W)(n\ F) dS,$$

where B_ϵ is the sphere of radius ϵ centered at W. Since

$$\Phi(Z-W)(n\ F) = \frac{Z^* - W^*}{\nu|Z-W|^8} \frac{Z-W}{\nu|Z-W|} = \frac{F}{\nu|Z-W|^7},$$

we have

$$\int_{B_\epsilon} \Phi(Z-W)(n\ F) dS = \int_{U_B} \frac{F}{\nu|Z-W|^7} dS.$$

Since $W \in \mathrm{int}(U)$ and the integral is taken over B_ϵ, as $\epsilon \to 0$, we get:

$$\Big| \int\limits_{B_\epsilon} \frac{F}{\nu |Z-W|^7} dS - F \Big| < \epsilon.$$

Also, for $W \in \Omega \setminus U$, we have

$$\int\limits_{U_B} \Phi(Z-W)(n\ F) dS = 0.$$

Therefore, we obtain

$$\int\limits_{U_B} \Phi(Z-W)(D^* F) dV = \int\limits_{U_B} \Phi(Z-W)(n\ F) dS =$$
$$= \int\limits_{\partial U} \Phi(Z-W)(n\ F) dS - \int\limits_{B_\epsilon} \Phi(Z-W)(n\ F) dS =$$
$$= \int\limits_{\partial U} \Phi(Z-W) d\sigma(p_1, p_2) F(p_1, p_2) - F(p_1, p_2).$$

From the hypothesis $D^* F = 0$, we obtain

$$0 = \int\limits_{\partial U} \Phi(Z-W) d\sigma(p_1, p_2) F(p_1, p_2) - F(p_1, p_2). \qquad \square$$

Acknowledgment. This work was supported by the Dongguk University Research Fund of 2017.

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Received January 9, 2018.
In revised form, April 9, 2018.
Accepted April 11, 2018.
Published online June 19, 2018.

Department of Mathematics, Dongguk University
Gyeongju-si 38066, Republic of Korea
E-mail: jeunkim@pusan.ac.kr