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INEQUALITIES VIA GENERALIZED h-CONVEX FUNCTIONS

Abstract. In the paper, we establish some new Hermite-Hadamard-Fejér type inequalities via generalized h-convex functions, Toader-like convex functions and their variant forms. Several special cases are also discussed. Results proved in this paper can be viewed as significant new contributions in this field.

Key words: *Generalized convex functions, generalized h-convex functions, Hermite-Hadamard-Fejér type inequalities, Toader-like convex function*

2010 Mathematical Subject Classification: *26D15, 26D10*

1. Introduction. Let I be an interval and $f : I = [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Then the functions f is convex, if and only if, it satisfies the inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int\limits_a^b f(x)\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2} \quad \forall a, b \in I, \tag{1}$$

which is known as the Hermite-Hadamard inequality (see [10, 11]). It provides estimates of the mean value of continuous convex function. In recent years, it has triggered huge amount of interest and is the one of the most investigated inequality (see [1, 3, 5, 13, 26]).

Inequalities are one of the most important tools in many areas of mathematics. Convex analysis is closely related to inequalities and plays a significant role in pure and applied mathematics especially in optimization theory and nonlinear programming due to its symmetry in shape and properties of convex sets and functions. This unique quality allows us to study different problems related to pure and applied sciences. Hence variant new classes has been introduced and investigated by using the concept of convexity (see [1, 3, 6, 8, 14–17, 23, 25]).

A significant generalization of convex functions was the introduction of h-convex functions by Varosanec [28]. She studied the basic properties and proved that h-convex functions include s-convex [2], p-convex [6] and Godunova-Levine [7] functions as special cases. For different properties and other aspects of h-convex functions see [12, 24, 28].

Gordji et al. [8] introduced an important class of convex functions, which is called φ-convex functions or generalized convex functions. These generalized convex (φ-convex) functions are nonconvex functions. For recent developments see [4, 9, 18–22] and the references therein.

Inspired and motivated by this ongoing research, we consider a new class of generalized convex functions relative to non-negative function h, which is called generalized h-convex and Toader-like convex functions. We derive some new Hermite-Hadamard-Fejér type integral inequalities for generalized h-convex function and generalized Toader-like convex functions. Our results includes a wide class of known and new inequalities as special cases. Results obtained in this paper continue to hold for the various classes of convex functions. The ideas and techniques of this paper inspire further research in this field.

1. Preliminaries. Let $I = [a, b]$ and J be the intervals in real line $\mathbb{R}$, $[0, 1] \subseteq J$. Let $f : I = [a, b] \to \mathbb{R}$ and $h : J \to \mathbb{R}$ be two non negative and continuous functions and $\eta(\cdot\,, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous bifunction. First of all, we recall the following well known results and concepts.

Definition 1. [8] *A function $f : I = [a, b] \to \mathbb{R}$ is said to be a generalized convex function with respect to a bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$f((1-t)a + tb) \leqslant (1-t)f(a) + t[f(a) + \eta(f(b), f(a))], \forall a, b \in I, t \in [0,1].$$

We now consider a new class of generalized convex functions with respect to an arbitrary non-negative function and derive some new integral inequalities. This is the main motivation of this paper.

Definition 2. *Let function $h : J \to \mathbb{R}$ be a non-negative function. A function $f : I = [a, b] \to \mathbb{R}$ is said to be a generalized h-convex function in the first sense with respect to a bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if $\forall\ a, b \in\ I,\ t \in\ [0, 1]$*

$$f((1-t)a + tb) \leqslant h(1-t)f(a) + h(t)[f(a) + \eta(f(b), f(a))]. \tag{2}$$

If $t = \frac{1}{2}$, then (2) reduces to

$$f\left(\frac{a+b}{2}\right) \leqslant h\left(\frac{1}{2}\right) f(a) + h\left(\frac{1}{2}\right) [f(a) + \eta(f(b), f(a))] =$$

$$=h\left(\frac{1}{2}\right)[2f(a)+\eta(f(b),f(a))]. \tag{3}$$

The function f is known as generalized Jensen h-convex function.

We now give an example, which shows that the functions may not be convex, but generalized h-convex functions.

Example. Let $f : [0,1] \to \mathbb{R}$ be the function defined by

$$f(x)=\sqrt{x}, \qquad x\in[0,1].$$

Obviously the function $f(x)=\sqrt{x}$ is a concave function. We show that this function is a generalized h-convex function. For all x, y, $t\in[0,1]$, we have

$$\begin{aligned}f((1-t)x+ty) =&\sqrt{(1-t)x+ty}=\\ =&\sqrt{(1-t)x+t(y-x)+tx}\leqslant\\ \leqslant&\sqrt{(1-t)}\sqrt{x}+\sqrt{t}\left[\sqrt{x}+\sqrt{|\,(\sqrt{y})^2-(\sqrt{x})^2\,|}\right]=\\ =&h(1-t)f(x)+h(t)[f(x)+\eta(f(y),f(x))],\end{aligned}$$

where $h(t)=\sqrt{t}$ and $\eta(f(y),f(x))=\sqrt{|\,(\sqrt{y})^2-(\sqrt{x})^2\,|}$. Therefore, f is generalized h-convex function.

Note that there is no function $\eta : [0,1]\times[0,1]\to\mathbb{R}$ such that f is η-convex. Indeed, suppose that f is an η-convex function for some $\eta : [0,1]\times[0,1]\to\mathbb{R}$. Then for all x, $y\in[0,1]$, we have

$$\sqrt{(1-t)x+ty}\leqslant(1-t)\sqrt{x}+t\big[\sqrt{x}+\eta(\sqrt{y},\sqrt{x})\big],\quad t\in[0,1].$$

Let $y>0$ be fixed and $x=0$. Therefore, we have

$$\sqrt{t}\sqrt{y}\leqslant t\eta(\sqrt{y},0),\quad t\in[0,1],$$

which implies

$$\sqrt{y}\leqslant\sqrt{t}\eta(\sqrt{y},0),\quad t\in[0,1].$$

Taking limit as $t\to 0$, we have $y=0$. Contradicting the fact that $y>0$. Hence f is not an η-convex function.

Let us discuss some special cases of generalized h-convex function.

(I). If $\eta(f(b),f(a))=f(b)-f(a)$, then

Definition 3. [27] *Let function $h : J \to \mathbb{R}$ be a non-negative function. A non-negative function $f : I \to (0, \infty)$ is said to be h-convex, $f \in SX(h, I)$, if*

$$f((1-t)a+tb) \leqslant h(1-t)f(a)+h(t)f(b), \quad \forall a,b \in I,\ t \in [0,1].$$

(II). If $h(t) = t$, then Definition 2 reduces to Definition 1.
(III). If $h(t) = t^s$, then

Definition 4. [2] *A function $f : I = [a, b] \to \mathbb{R}$ is said to be a generalized s-convex in the second sense for $s \in (0, 1)$ with respect to a bifunction $\eta(\cdot\,,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$f((1-t)a+tb) \leqslant (1-t)^s f(a)+t^s[f(a)+\eta(f(b), f(a))], \forall a, b \in I, t \in [0, 1].$$

(IV). If $h(t) = 1$, then

Definition 5. [8] *A function $f : I = [a, b] \to \mathbb{R}$ is said to be a generalized P-convex with respect to a bifunction $\eta(\cdot\,,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$f((1-t)a+tb) \leqslant [f(a)] + [f(a)+\eta(f(b), f(a))], \forall a, b \in I, t \in [0, 1].$$

(V). If $h(t) = \dfrac{1}{t}$, then

Definition 6. *A function $f : I = [a, b] \to \mathbb{R}$ is said to be a generalized Godunova-Levine convex with respect to a bifunction $\eta(\cdot\,,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, if*

$$f((1-t)a+tb) \leqslant \frac{1}{1-t}[f(a)] + \frac{1}{t}[f(a)+\eta(f(b), f(a))], \forall a, b \in I, t \in (0,1).$$

(VI). If $h(t) + h(1-t) = 1$, then Definition 2 reduces to

Definition 7. *Let $h : J \to \mathbb{R}$ be a non-negative function. A function $f : I = [a, b] \to \mathbb{R}$ is said to be a generalized Toader-like convex function with respect to a bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$f((1-t)a+tb) \leqslant f(a)+h(t)\eta(f(b), f(a)), \quad \forall a, b \in I, t \in [0, 1].$$

For appropriate and suitable choice of function h, one can obtain several new and known classes of convex functions. This shows that the concept of generalized h-convex function is quite general and unifying one.

2. Main results. In this section, we establish several new integral inequalities of Hermite-Hadamard type for generalized h-convex functions.

Theorem 1. *Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function. Let $f : I = [a, b] \to \mathbb{R}$ be a generalized h-convex function. Then*

$$\frac{1}{b-a}\Bigg[2f(a)\int_a^b f(x)h\left(\frac{b-x}{b-a}\right)\mathrm{d}x+$$

$$+\,2[f(a)+\eta\big(f(b),f(a)\big)]\int_a^b f(x)h\left(\frac{x-a}{b-a}\right)\mathrm{d}x\Bigg]\leqslant$$

$$\leqslant\frac{1}{b-a}\Bigg[\int_a^b f^2(x)\mathrm{d}x+$$

$$+\left\{[f^2(a)]+[f(a)+\eta\big(f(b),f(a)\big)]^2\right\}\int_a^b h^2\left(\frac{x-a}{b-a}\right)\mathrm{d}x+$$

$$+\,2[f(a)][f(a)+\eta\big(f(b),f(a)\big)]\int_a^b h\left(\frac{x-a}{b-a}\right)h\left(\frac{b-x}{b-a}\right)\mathrm{d}x\Bigg].$$

Proof. Let f be a generalized h-convex function. Then

$$f((1-t)a+tb))\leqslant h(1-t)[f(a)]+h(t)[f(a)+\eta\big(f(b),f(a)\big)].$$

Using the classical inequality $G(a, b) \leqslant A(a, b)$, we have

$$\sqrt{f((1-t)a+tb))\left\{h(1-t)[f(a)]+h(t)[f(a)+\eta\big(f(b),f(a)\big)]\right\}}\leqslant$$
$$\leqslant\frac{1}{2}\Big[f((1-t)a+tb))+\left\{h(1-t)[f(a)]+h(t)[f(a)+\eta\big(f(b),f(a)\big)]\right\}\Big].$$

which implies that

$$4\Bigg[f((1-t)a+tb))\left\{h(1-t)[f(a)]+h(t)\big[f(a)+\eta\big(f(b),f(a)\big)\big]\right\}\Bigg]\leqslant$$
$$\leqslant\Bigg[f((1-t)a+tb))+\left\{h(1-t)[f(a)]+h(t)\big[f(a)+\eta\big(f(b),f(a)\big)\big]\right\}\Bigg]^2.$$

From this, we have

$$2\big[f((1-t)a+tb))\left\{h(1-t)[f(a)]+h(t)\big[f(a)+\eta\big(f(b),f(a)\big)\big]\right\}\big]\leqslant$$

$$\leqslant \left[f^2((1-t)a+tb))+\left\{h(1-t)[f(a)]+h(t)\big[f(a)+\eta\big(f(b),f(a)\big)\big]\right\}^2\right]=$$
$$= f^2((1-t)a+tb)+h^2(1-t)[f^2(a)]+h^2(t)\big[f(a)+\eta\big(f(b),f(a)\big)\big]^2+$$
$$+2[f(a)]\big[f(a)+\eta\big(f(b),f(a)\big)\big]h(t)h(1-t). \tag{4}$$

Integrating (4) over t on [0, 1], we have

$$2[f(a)]\int_0^1 f((1-t)a+tb))h(1-t)\mathrm{d}t+$$
$$+2\big[f(a)+\eta\big(f(b),f(a)\big)\big]\int_0^1 f((1-t)a+tb))h(t)\mathrm{d}t\leqslant$$
$$\leqslant\int_0^1 f^2((1-t)a+tb)\mathrm{d}t+[f^2(a)]\int_0^1 h^2(1-t)\mathrm{d}t+$$
$$+[f(a)+\eta\big(f(b),f(a)\big)]^2\int_0^1 h^2(t)\mathrm{d}t+$$
$$+2[f(a)]\big[f(a)+\eta\big(f(b),f(a)\big)\big]\int_0^1 h(t)h(1-t)\mathrm{d}t=$$
$$=\int_0^1 f^2((1-t)a+tb)\mathrm{d}t+\left\{[f^2(a)]+\big[f(a)+\eta\big(f(b),f(a)\big)\big]^2\right\}\times$$
$$\times\int_0^1 h^2(t)\mathrm{d}t+2[f(a)]\big[f(a)+\eta\big(f(b),f(a)\big)\big]\int_0^1 h(t)h(1-t)\mathrm{d}t.$$

By making the change of variable $x=(1-t)a+tb$, we have

$$\frac{1}{b-a}\left[2f(a)\int_a^b f(x)h\left(\frac{b-x}{b-a}\right)\mathrm{d}x+2[f(a)+\eta\big(f(b),f(a)\big)]\times\right.$$
$$\left.\times\int_a^b f(x)h\left(\frac{x-a}{b-a}\right)\mathrm{d}x\right]\leqslant$$

$$\leqslant \frac{1}{b-a}\Bigg[\int_a^b f^2(x)\mathrm{d}x + \big\{f^2(a) + \big[f(a) + \eta\big(f(b), f(a)\big)\big]^2\big\}\times$$

$$\times \int_a^b h^2\left(\frac{x-a}{b-a}\right)\mathrm{d}x +$$

$$+ 2[f(a)]\big[f(a) + \eta\big(f(b), f(a)\big)\big] \int_a^b h\left(\frac{x-a}{b-a}\right) h\left(\frac{b-x}{b-a}\right)\mathrm{d}x\Bigg],$$

which is the required result. □

Corollary 1. *If $\eta(f(b), f(a)) = f(b) - f(a)$, then, under the assumptions of Theorem 1, we have*

$$\frac{1}{b-a}\left[2f(a)\int_a^b f(x)h\left(\frac{b-x}{b-a}\right)\mathrm{d}x + 2f(b)\int_a^b f(x)h\left(\frac{x-a}{b-a}\right)\mathrm{d}x\right] \leqslant$$

$$\leqslant \frac{1}{b-a}\Bigg[\int_a^b f^2(x)\mathrm{d}x + \big\{f^2(a) + f^2(b)\big\}\int_a^b h^2\left(\frac{x-a}{b-a}\right)\mathrm{d}x +$$

$$+ 2[f(a)f(b)]\int_a^b h\left(\frac{x-a}{b-a}\right)h\left(\frac{b-x}{b-a}\right)\mathrm{d}x\Bigg].$$

Corollary 2. *If $h(t) = t$, then, under the assumptions of Theorem 1, we have*

$$\frac{2f(a)}{(b-a)^2}\left[\int_a^b f(x)(b-x)\mathrm{d}x + \frac{2[f(a) + \eta\big(f(b), f(a)\big)]}{(b-a)^2}\int_a^b f(x)(x-a)\mathrm{d}x\right] \leqslant$$

$$\leqslant \frac{1}{b-a}\left[\int_a^b f^2(x)\mathrm{d}x + \frac{M(a,b)}{3}\right],$$

where

$$M(a,b) = \big[f^2(a) + \big[f(a) + \eta\big(f(b),\ f(a)\big)\big]^2 + f(a)\big[f(a) + \eta\big(f(b), f(a)\big)\big]\big].$$

Corollary 3. *If $h(t) = t^s$, then, under the assumptions of Theorem 1, we have*

$$\frac{2f(a)}{(b-a)^{s+1}}\Big[\int\limits_a^b f(x)(b-x)^s \mathrm{d}x + \frac{2[f(a)+\eta\big(f(b),f(a)\big)]}{(b-a)^{s+1}}\int\limits_a^b f(x)(x-a)^s \mathrm{d}x\Big]$$

$$\leqslant \frac{1}{b-a}\Big[\int\limits_a^b f^2(x)\mathrm{d}x + \frac{f^2 + [f(a)+\eta\big(f(b),f(a)\big)]}{(2s+1)} +$$

$$+ f(a)[f(a)+\eta\big(f(b),f(a)\big)]\beta(s+1,s+1)\Big].$$

Corollary 4. *If $h(t) = 1$, then, under the assumptions of Theorem 1, we have*

$$\frac{2f(a)}{(b-a)}\Big[\int\limits_a^b f(x)\mathrm{d}x + \frac{2\big[f(a)+\eta\big(f(b),f(a)\big)\big]}{(b-a)}\int\limits_a^b f(x)\mathrm{d}x\Big] \leqslant$$

$$\leqslant \frac{1}{b-a}\Big[\int\limits_a^b f^2(x)\mathrm{d}x + \{f(a) + [f(a)+\eta\big(f(b),f(a)\big)]\}^2\Big].$$

Theorem 2. *Let $f : I = [a, b] \to \mathbb{R}$ be a generalized h-convex function. such that $\eta(\cdot\,,\cdot)$ is bounded above from $f([a, b]) \times f([a, b])$. Then*

$$\left[\frac{f\left(\frac{a+b}{2}\right)}{2h\left(\frac{1}{2}\right)} - \frac{M_n}{2}\right] \leqslant \frac{1}{b-a}\int\limits_a^b f(x)\mathrm{d}x \leqslant$$

$$\leqslant\Big[\Big[f(a)+f(b)+\frac{\eta(f(a),f(b))+\eta(f(b),f(a))}{2}\Big]\Big(\int\limits_0^1 h(t)\mathrm{d}t\Big)\Big] \leqslant$$

$$\leqslant\Big[[f(a)+f(b)+M_n]\Big(\int\limits_0^1 h(t)\mathrm{d}t\Big)\Big],$$

where M_n is an upper bound of $\eta(\cdot\,,\cdot)$.

Proof. Let f be a generalized h-convex function, Then

$$f\left(\frac{a+b}{2}\right) \leqslant f\left(\frac{1}{2}\left(\frac{a+b-t(b-a)}{2}\right) + (1-\frac{1}{2})\left(\frac{a+b+t(b-a)}{2}\right)\right) \leqslant$$

$$\leqslant h\left(\frac{1}{2}\right)\left[2f\left(\frac{a+b-t(b-a)}{2}\right)\right]+$$
$$+h\left(\frac{1}{2}\right)\eta\left(f\left(\frac{a+b+t(b-a)}{2}\right),f\left(\frac{a+b-t(b-a)}{2}\right)\right),$$

as M_n is an upper bound of $\eta(\cdot\,,\cdot)$, we obtain

$$f\left(\frac{a+b}{2}\right)-h\left(\frac{1}{2}\right)M_n\leqslant h\left(\frac{1}{2}\right)2\left[f\left(\frac{a+b-t(b-a)}{2}\right)\right].$$

Also with same argument, we have

$$f\left(\frac{a+b}{2}\right)-h\left(\frac{1}{2}\right)M_n\leqslant h\left(\frac{1}{2}\right)2\left[f\left(\frac{a+b+t(b-a)}{2}\right)\right].$$

Now using the change of variable $x=\frac{1}{2}(a+b+t(b-a))$, we have

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x=\frac{1}{b-a}\left[\int_a^{(a+b)/2} f(x)\mathrm{d}x+\int_{(a+b)/2}^b f(x)\mathrm{d}x\right]\leqslant$$
$$\leqslant\frac{1}{2}\int_0^1\left[f\left(\frac{a+b-t(b-a)}{2}\right)+f\left(\frac{a+b+t(b-a)}{2}\right)\right]\mathrm{d}t\leqslant$$
$$\leqslant\int_0^1\left[\frac{f\left(\frac{a+b}{2}\right)}{2h\left(\frac{1}{2}\right)}-\frac{M_n}{2}\right]\mathrm{d}t=\left[\frac{f\left(\frac{a+b}{2}\right)}{2h\left(\frac{1}{2}\right)}-\frac{M_n}{2}\right].$$

To prove the right hand side of inequality, consider

$$f((1-t)a+tb)\leqslant h(1-t)f(a)+h(t)[f(a)+\eta(f(b),f(a))]. \tag{5}$$

Integrating (5) on [0, 1], we have

$$U=\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\leqslant f(a)\int_0^1 h(1-t)\mathrm{d}t+[f(a)+\eta(f(b),f(a))]\int_0^1 h(t)\mathrm{d}t=$$
$$=[2f(a)+\eta(f(b),f(a))]\int_0^1 h(t)\mathrm{d}t. \tag{6}$$

Similarly

$$V=\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\leqslant f(b)\int_0^1 h(1-t)\mathrm{d}t+[f(b)+\eta(f(a),f(b))]\int_0^1 h(t)\mathrm{d}t=$$

$$=[2f(b)+\eta(f(a),f(b))]\int_0^1 h(t)\mathrm{d}t. \tag{7}$$

Therefore , we have

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\leqslant \min\{U,V\}\leqslant$$

$$\leqslant\left[f(a)+f(b)+\frac{\eta(f(a),f(b))+\eta(f(b),f(a))}{2}\right]\left(\int_0^1 h(t)\mathrm{d}t\right)\leqslant$$

$$\leqslant\left[[f(a)+f(b)+M_n]\left(\int_0^1 h(t)\mathrm{d}t\right)\right],$$

which is the required result. □

Theorem 3. *Let $f:I=[a,b]\to\mathbb{R}$ be a generalized h-convex function. If $\eta(\cdot\,,\cdot)$ is bounded above from $f([a,b])\times f([a,b])$ and $w:[a,b]\to\mathbb{R}$ is integrable and symmetric with respect to $\frac{a+b}{2}$, then*

$$\left[\frac{f\left(\frac{a+b}{2}\right)}{2h\left(\frac{1}{2}\right)}-\frac{M_n}{2}\right]\int_a^b w(x)\mathrm{d}x\leqslant\int_a^b f(x)w(x)\mathrm{d}x\leqslant$$

$$\leqslant\frac{[f(a)+f(b)]}{2}\left\{\int_a^b h\left(\frac{b-x}{b-a}\right)w(x)\mathrm{d}x+\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x\right\}+$$

$$+M_n\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x,$$

where M_n is an upper bound of $\eta(\cdot,\cdot)$.

Proof. Let f be a generalized h-convex function. Then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) =& f\left(\frac{1}{2}\left(\frac{a+b-t(b-a)}{2}\right)+\left(1-\frac{1}{2}\right)\left(\frac{a+b+t(b-a)}{2}\right)\right) \leqslant \\ &\leqslant h\left(\frac{1}{2}\right)\left[2f\left(\frac{a+b-t(b-a)}{2}\right)\right]+ \\ &\quad + h\left(\frac{1}{2}\right)\eta\left(f\left(\frac{a+b+t(b-a)}{2}\right), f\left(\frac{a+b-t(b-a)}{2}\right)\right), \end{aligned}$$

as M_n is an upper bound of $\eta(\cdot\,,\cdot)$, we obtain

$$f\left(\frac{a+b}{2}\right) - h\left(\frac{1}{2}\right)M_n \leqslant h\left(\frac{1}{2}\right)\left[2f\left(\frac{a+b-t(b-a)}{2}\right)\right]. \tag{8}$$

Similarly,

$$f\left(\frac{a+b}{2}\right) - h\left(\frac{1}{2}\right)M_n \leqslant h\left(\frac{1}{2}\right)\left[2f\left(\frac{a+b+t(b-a)}{2}\right)\right]. \tag{9}$$

Adding (8) and (9), we have

$$\begin{aligned} &\left[f\left(\frac{a+b}{2}\right) - h\left(\frac{1}{2}\right)M_n\right] \leqslant \\ &\quad \leqslant h\left(\frac{1}{2}\right)\left\{\left[f\left(\frac{a+b+t(b-a)}{2}\right)+f\left(\frac{a+b-t(b-a)}{2}\right)\right]\right\}. \end{aligned} \tag{10}$$

Multiplying (10) by $w\left(\frac{a+b+t(b-a)}{2}\right)$ and then integrating with respect to t on [0,1], we have

$$\begin{aligned} &\left[f\left(\frac{a+b}{2}\right) - h\left(\frac{1}{2}\right)M_n\right]\int_0^1 w\left(\frac{a+b+t(b-a)}{2}\right)\mathrm{d}t \leqslant \\ &\leqslant h\left(\frac{1}{2}\right)\left\{\int_0^1 \left[f\left(\frac{a+b+t(b-a)}{2}\right)+f\left(\frac{a+b-t(b-a)}{2}\right)\right]\times \right. \\ &\left. \times \left(\frac{a+b+t(b-a)}{2}\right)\mathrm{d}t\right\}. \end{aligned}$$

Using the change of variable $x = \frac{1}{2}(a+b+t(b-a))$, we have

$$\left[f\left(\frac{a+b}{2}\right) - h\left(\frac{1}{2}\right)M_n\right] \int_{(a+b)/2}^{b} w(x)\mathrm{d}x \leqslant h\left(\frac{1}{2}\right)\int_{a}^{b} f(x)w(x)\mathrm{d}x. \quad (11)$$

Similarly

$$\left[f\left(\frac{a+b}{2}\right) - h\left(\frac{1}{2}\right)M_n\right] \int_{a}^{(a+b)/2} w(x)\mathrm{d}x \leqslant h\left(\frac{1}{2}\right)\int_{a}^{b} f(x)w(x)\mathrm{d}x. \quad (12)$$

Adding (11) and (12), we have

$$\left[f\left(\frac{a+b}{2}\right) - h\left(\frac{1}{2}\right)M_n\right] \int_{a}^{b} w(x)\mathrm{d}x \leqslant 2h\left(\frac{1}{2}\right)\int_{a}^{b} f(x)w(x)\mathrm{d}x.$$

Now for the right hand side of the inequality, consider a generalized h - convex function f and get

$$f((1-t)a+tb) \leqslant h\left(\frac{b-x}{b-a}\right)f(a) + h\left(\frac{x-a}{b-a}\right)[f(a)+\eta(f(b),f(a))]. \quad (13)$$

Multiplying (13) by $w(x)$, and then integrating with respect to x on [a, b], we have

$$\frac{1}{b-a}\int_{a}^{b} f(x)w(x)\mathrm{d}x \leqslant f(a)\int_{a}^{b} h\left(\frac{b-x}{b-a}\right)w(x)\mathrm{d}x+$$
$$+[f(a)+\eta(f(b),f(a))]\int_{a}^{b} h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x. \quad (14)$$

With a similar argument, we have

$$\frac{1}{b-a}\int_{a}^{b} f(x)w(x)\mathrm{d}x \leqslant f(b)\int_{a}^{b} h\left(\frac{b-x}{b-a}\right)w(x)\mathrm{d}x+$$

$$+\left[f(b)+\eta(f(a),f(b))\right]\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x. \qquad (15)$$

Adding (14) and (15), we have

$$\frac{2}{b-a}\int_a^b f(x)w(x)\mathrm{d}x \leqslant [f(a)+f(b)]\int_a^b h\left(\frac{b-x}{b-a}\right)w(x)\mathrm{d}x+$$

$$+\Big\{[f(b)+\eta(f(a),f(b))]+[f(a)+\eta(f(b),f(a))]\Big\}\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x=$$

$$=[f(a)+f(b)]\left\{\int_a^b h\left(\frac{b-x}{b-a}\right)w(x)\mathrm{d}x+\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x\right\}+$$

$$+\Big\{\eta(f(a),f(b))+\eta(f(b),f(a))\Big\}\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x \leqslant$$

$$\leqslant[f(a)+f(b)]\left\{\int_a^b h\left(\frac{b-x}{b-a}\right)w(x)\mathrm{d}x+\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x\right\}+$$

$$+M_n\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x,$$

which is the required result. □

Corollary 1. *If $h(t)+h(1-t)=1$, then, under the assumptions of Theorem 3, we have*

$$\left[f\left(\frac{a+b}{2}\right)-h\left(\frac{1}{2}\right)M_n\right]\int_a^b w(x)\mathrm{d}x \leqslant \int_a^b f(x)w(x)\mathrm{d}x \leqslant$$

$$\leqslant \frac{[f(a)+f(b)]}{2}\left\{\int_a^b w(x)\mathrm{d}x\right\}+M_n\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x.$$

Corollary 2. [4] *If $h(t)=t$, then, under the assumptions of Theorem 3,* *we have*

$$\left[f\left(\frac{a+b}{2}\right)-\frac{M_n}{2}\right]\int_a^b w(x)\mathrm{d}x \leqslant \int_a^b f(x)w(x)\mathrm{d}x \leqslant$$

$$\leqslant \frac{[f(a)+f(b)]}{2}\left\{\int_a^b w(x)\mathrm{d}x\right\}+\frac{M_n}{(b-a)}\int_a^b (x-a)w(x)\mathrm{d}x.$$

Corollary 3. [4] *If $w(x)=1$ in Corollary 2, then, under the assumptions of Theorem 3, we have*

$$\left[f\left(\frac{a+b}{2}\right)-\frac{M_n}{2}\right] \leqslant \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leqslant \frac{[f(a)+f(b)]}{2}+\frac{M_n}{2}.$$

3. Integral Inequalities. In this section, we obtain some results for the Toader-like convex functions.

Theorem 4. *Let $h : J \to \mathbb{R}$ be an integrable and non-negative function such that $\lim_{t\to 0}(\frac{h(t)}{t} = k;\ k \neq 0)$, and let $f : I = [a, b] \to \mathbb{R}$ be differentiable generalized Toader-like convex function. If $u \in I$ is the minimum of Toader-like convex function, then*

$$\langle f'(u), v-u\rangle \geqslant 0 \ \Rightarrow\ k\eta(f(v), f(u)) \geqslant 0, \quad \forall\, v \in I.$$

Proof. Let $u \in I$ be the minimum of generalized Toader-like convex function on I. Then

$$f(u) \leqslant f(v), \quad \forall\, v \in I. \tag{16}$$

Since I is an interval, so $\forall\, u, v \in I$, $t \in [0, 1]$, we have

$$v_t = u + t(v-u) \in I.$$

Replace v by v_t in (16), we have

$$f(u) \leqslant f(v_t) = f(u + t(v-u)).$$

Dividing the above inequality by t and taking limit as $t \to 0$, we have

$$\langle f'(u), v-u\rangle \geqslant 0.$$

This shows that $u \in I$ satisfies

$$\langle f'(u), v-u\rangle \geqslant 0 \quad \forall\, u, v \in I. \tag{17}$$

Since f is a generalized Toader-like convex function, we have

$$f((1-t)u+tv) \leqslant f(u) + h(t)\eta(f(v), f(u)), \quad \forall\, u, v \in I, t \in [0,1],$$

which implies that

$$\frac{f(a+t(v-u)) - f(u)}{t} \leqslant \frac{h(t)}{t}\eta(f(v), f(u)). \tag{18}$$

Taking limit as $t \to 0$ on both side of the (18), we have

$$\langle f'(u), v-u\rangle \leqslant k\eta(f(v), f(u)).$$

Using (17), implies

$$k\eta(f(v), f(u)) \geqslant 0.$$

This completes the proof. □

Corollary 1. *If $\eta(f(v), f(u)) = f(v) - f(u)$, then, under the assumptions of Theorem 4, we have*

$$\langle f'(u), v-u\rangle \geqslant 0 \Rightarrow k(f(v) - f(u)) \geqslant 0.$$

Corollary 2. *If $h(t) = t$, then, under the assumptions of Theorem 4, we have*

$$\langle f'(u), v-u\rangle \geqslant 0 \Rightarrow f(v) - f(u) \geqslant 0.$$

Theorem 5. *Let $h : J \to \mathbb{R}$ be an integrable and non-negative function such that $\lim\limits_{t\to 0}(\frac{h(t)}{t} = k;\ k \neq 0)$. Suppose that $f : I = [a,b] \to \mathbb{R}$ be a differentiable generalized Toader-like convex function on I and $\eta(\cdot\,,\cdot)$ is measurable on $f([a,b]) \times f([a,b])$. Then*

1) $\displaystyle \frac{k}{b-a}\int_a^b \eta(f(y), f(x))\mathrm{d}y \geqslant \langle f'(x), \frac{b+a}{2} - x\rangle;$

$$2)\ k\int_a^b \eta(f(y),f(x))\mathrm{d}x+(y-a)f(a)+(b-y)f(b)\geqslant \int_a^b f(x)\mathrm{d}x.$$

Proof. Let f be a generalized Toader-like convex function. Then

$$f((1-t)x+ty)\leqslant f(x)+h(t)\eta(f(y),f(x)),\quad \forall\, x,y\in I,\ t\in[0,1].$$

This implies that

$$\frac{f(x+t(y-x))-f(x)}{t}\leqslant \frac{h(t)}{t}\eta(f(y),f(x)). \tag{19}$$

Taking limit as $t\to 0$ on both side of the (19), we have

$$\langle f'(x),y-x\rangle \leqslant k\eta(f(y),f(x)). \tag{20}$$

Integrating (20) with respect to y on $[a,b]$ and dividing by $(b-a)$, we have

$$\langle f'(x),\frac{b+a}{2}-x\rangle \leqslant \frac{k}{b-a}\int_a^b \eta(f(y),f(x))\mathrm{d}y.$$

Similarly, integrating (20) with respect to x on $[a,b]$, we have

$$\int_a^b f'(x)(y-x)\mathrm{d}x\leqslant k\int_a^b \eta(f(y),f(x))\mathrm{d}x. \tag{21}$$

Now consider

$$\int_a^b f'(x)(y-x)\mathrm{d}y=\int_a^b f(x)\mathrm{d}x-(y-a)f(a)-(b-y)f(b). \tag{22}$$

Substituting the value from (22) in (21), we have

$$\int_a^b f(x)\mathrm{d}x\leqslant k\int_a^b \eta(f(y),f(x))\mathrm{d}x+(y-a)f(a)+(b-y)f(b).$$

□

Corollary 1. *If $\eta(f(y), f(x)) = f(y) - f(x)$, then, under the assumptions of Theorem 5, we have*

$$\frac{2k}{b-a}\int_a^b f(y)\mathrm{d}y \geqslant \langle f'(x), a+b-2x\rangle + 2kf(x)$$

and

$$1 + \frac{bf(b)-af(a)}{k(b-a)} - \frac{y(f(b)-f(a))}{k(b-a)} \geqslant \frac{(1+k)}{k(b-a)}\int_a^b f(x)\mathrm{d}x.$$

Corollary 2. [4] *If $h(t) = t$, then, under the assumptions of Theorem 5, we have*

$$\langle f'(x), \frac{b+a}{2} - x\rangle \leqslant \frac{1}{b-a}\int_a^b \eta(f(y), f(x))\mathrm{d}y$$

and

$$\int_a^b f(x)\mathrm{d}x \leqslant \int_a^b \eta(f(y), f(x))\mathrm{d}x + (y-a)f(a) + (b-y)f(b).$$

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References

[1] Anderson G. D., Vamanamurthy M. K., Vuorinen M. *Generalized convexity and inequalities,* J. Math. Anal. Appl., 2007, vol. 335, pp. 1294–1308.

[2] Breckner W. W. *Stetigkeitsaussagen fiir eine Klasse verallgemeinerter convexer funktionen in topologischen linearen Raumen,* Publ. Inst. Math., 1978, vol. 23, pp. 13–20.

[3] Cristescu G., Lupsa L. *Non-connected Convexities and Applications,* Kluwer Academic Publishers, Dordrechet, Holland, 2002.

[4] Delavar M. R., Dragomir S. S. *On η-convexity,* Math. Inequal. Appl., 2017, vol. 20, no. 1, pp. 203–216.

[5] Dragomir S. S., Mond B. *Integral inequalities of Hadamard type for log-convex functions,* Demonstration, 1998, vol. 31, pp. 354–364.

[6] Dragomir S. S., Pearce C. E. *Selected topics on Hermite-Hadamard inequalities and applications,* Victoria University, Australia, 2000.

[7] Godunova E. K., Levin V. I. *Neravenstva dlja funkcii sirokogo klassa soderzascego vypuklye monotonnye i nekotorye drugie vidy funkii,* Vycislitel. Mat. i.Fiz. Mezvuzov. Sb. Nauc. MGPI Moskva, 1985, pp. 138 –142, (in Russian).

[8] Gordji M. E., Delavar M. R., De La Sen M. *On φ convex functions,* J. Math. Inequal., 2016, vol. 10, no. 1, pp. 73–183.

[9] Gordji M. E., Delavar M. R., Dragomir S. S. *An inequality related to η-convex functions (II),* Int. J. Nonl. Anal. Appl., 2015, vol. 6, no. 2, pp. 27–33.

[10] Hadamard J. *Etude sur les proprietes des fonctions entieres e.t en particulier dune fonction consideree par Riemann,* J. Math. Pure. Appl., 1893, vol. 58, pp. 171–215.

[11] Hermite C. *Sur deux limites d'une integrale definie,* Mathesis, 1883, vol. 3, pp. 82.

[12] Hyers D. H., Ulam S. M. *Approximately convex functions,* Proc. Amer. Math. Soc., 1952, vol. 3, pp. 821–828.

[13] Niculescu C. P., Persson L. E. Convex Functions and Their Applications. Springer-Verlag, New York, 2006.

[14] Noor M. A. *General variational inequalities,* Appl. Math. Letters, 1988, vol.1, no. 1, pp. 119 –121.

[15] Noor M. A. *Some developments in general variational inequalities,* Appl. Math. Comput., 2004, vol. 152, pp. 199 –277.

[16] Noor M. A. *Extended general variational inequalities,* Appl. Math. Letters, 2009, vol. 22, no. 2, pp. 182–186.

[17] Noor M. A., Noor K. I. *Harmonic variational inequalities,* Appl. Math. Inform. Sci. 2016, vol. 10, no. 5, pp. 1811–1814.

[18] Noor M. A., Noor K. I., Awan M. U., Safdar F. *On strongly generalized convex functions,* Filomat, 2017, vol. 31, no. 18, pp. 5783–5790.

[19] Noor M. A., Noor K. I., Safdar F. *Generalized geometrically convex functions and inequalities,* J. Inequal. Appl., 2017, vol. 2017, Art. 202.

[20] Noor M. A., Noor K. I., Safdar F. *Integral inequaities via generalized convex functions,* J. Math. Computer, Sci., 2017, vol. 17, no. 4, pp. 465–476.

[21] Noor M. A., Noor K. I., Iftikhar S., Safdar F. *Integral inequaities for relative harmonic (s,η)-convex functions,* Appl. Math. Comp. Sci., 2015, 1(1), pp. 27–34.

[22] Noor M. A., Noor K. I., Safdar F. *Integral inequaities via generalized (α,m)-convex functions,* J. Nonlinear. Func. Anal., 2017, vol. 2017, Article ID: 32.

[23] Pečarić J. E., Proschan F., Tong Y. L. *Convex Functions, Partial Orderings and Statistical Applications,* Academic Press, Boston, 1992.

[24] Sarikaya M. Z., Saglam A., Yildirim H. *On some Hadamard-type inequalities for h-convex functions,* J. Math. Inequal., 2008, vol. 2, no. 3, pp. 335–341.

[25] Toader G. *Some generalizations of the convexity, Proceedings of the Colloquium on Approximation and Optimization,* Univ. Cluj-Napoca, 1985, pp. 329–338.

[26] Tunc M., Kirmaci U. *New inequalities for convex functions,* EÜFBED - Fen Bilimleri Enstitüsü Dergisi Cilt-Sayi: 2010, vol. 3, no. 1, pp. 91–101.

[27] Varosanec S. *On h-convexity,* J. Math. Anal. Appl., 2007, vol. 326, pp. 303–311.

[28] Xi B-Y., Qi F. *Some inequalities of Hermite-Hadamard type for h-convex functions,* Adv. Inequal. Appl., 2013, vol. 2, no. 1, pp. 1–15.

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