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THE APPROXIMATE CONFORMAL MAPPING ONTO MULTIPLY CONNECTED DOMAINS

Abstract. The method of boundary curve reparametrization is generalized to the case of multiply connected domains. We construct the approximate analytical conformal mapping of the unit disk with N circular slits or an annulus with $(N - 1)$ circular slits onto an arbitrary $(N + 1)$ multiply connected finite domain with a smooth boundary. The method is based on the solution of the Fredholm equation. This solution is reduced to the solution of a linear system with unknown Fourier coefficients. The approximate mapping function has the form of a set of Laurent polynomials in the set of annular regions. The method is easily computable.

Key words: *conformal mapping, multiply connected domain, Fredholm integral equation*

2010 Mathematical Subject Classification: *30C20, 30C30, 45B05*

1. Introduction. Conformal mappings by analytical functions of complex variable play an important role in solving many problems of mechanics and mathematics, particularly in the case of plane potential fields and the Laplace equation [1]. The conformal mapping of circular domains (a disk with circular slits or an annulus with circular slits) onto a multiply connected domain with a complex boundary can be applied to solving plane boundary value problems for the corresponding domains with the help of Schottky-Klein prime functions known for circular domains [2]. Computer progress has stimulated appearance of many numerical methods for constructing conformal mapping [3]. However, all these methods are rather time-consuming. For example, the widely used Wegmann numerical method is based on solving the Riemann–Hilbert problem and involves iteration processes [4, 5].

There are several types of canonical regions for conformal mappings [6]. The five types of canonical slit regions are a disk with concentric circular slits, an annulus with concentric circular slits, unbounded circular slit regions, unbounded radial slit regions, and unbounded parallel slit regions. Recently a few approaches to map bounded and unbounded multiply connected regions onto these five canonical regions by reformulating the mapping problems as Riemann-Hilbert problems have been proposed; these problems are solved by means of boundary integral equations with the generalized Neumann kernel [7–13]. The integral equations were discretized by the Nyström method with the trapezoidal rule used to obtain linear systems. The right-hand side of the integral equation in [7, 8] involves integral with the cotangent singularity which is approximated by Wittich’s method.

Here we present a new method of the approximate conformal mapping of a unit disk with circular slits or an annulus with circular slits onto a multiply connected domain with a smooth boundary. We apply the integral equation with the Neumann kernel obtained from the necessary and sufficient condition for a function defined at the points of a smooth contour to be the boundary values of some function analytical in the correspondent domain. The theory of this method for simply connected domains has been developed in [14, 15], the generalization to the case of a doubly connected domain and the relative numerical implementation can be found in [16]. Here we generalize the method to the case of multiply connected domains whose internal borders are images of circular arcs and an inner circle. The method is based on the boundary curves reparametrization. We do not search for corrections of boundary parametrization with linearization of this process [17, 18]. The advantages of our method are the following: the method does not use any auxiliary constructions (triangulation, circle packing, domain decomposition) or conformal mappings (the zipper algorithm, the Schwarz-Christoffel mapping), it does not use the accessory solutions of boundary value problems (the conjugate function method, the Wegmann method), it does not use iterations. We do not apply the collocation method or the Nyström’s method for solution of the integral equation.

The solution of integral equations in our method is reduced to the solution of an infinite linear system. We obtain an approximate mapping by solving the finite system with a truncated matrix. The smooth solution of the problem has the form of a Laurent polynomial in each annular

region. This form allows us to apply the differentiation techniques for solutions of some physical problems in the corresponding domains. We must note that our method is applicable only for the domains with smooth boundary components or the boundary components with vertices of the angles more than π . The cases of the angles less than π bring to kernels with strong singularities.

2. Approximate conformal mapping of the circular domains of two types onto a multiply connected domain by means of the boundary reparametrization. Consider a finite multiply connected domain D_z bounded by the outer simple smooth curve L_0 and the inner simple smooth curves L_s given by the equations

$$L_s = \{z = z_s(t), \quad z_s(0) = z_s(2\pi), \quad t \in [0, 2\pi]\}, \quad s = \overline{0, N}.$$

Also, assume that complex representations of the boundary curves L_s are as follows:

$$z_s(t) = \sum_{k=-m_s}^{n_s} d_{ks} e^{ikt}, \quad t \in [0, 2\pi], \quad s = \overline{0, N}.$$

The parametrization traces the domain D_z along L_0 counterclockwise and along the inner contours L_s , $s = \overline{1, N}$, clockwise.

Definition 1. We call the unit disk $|\zeta| < 1$ with N circular slits $\zeta = R_j e^{i\theta}$, $\theta_{1j} < \theta < \theta_{2j}$, with a constant $R_j < 1$, $j = \overline{1, N}$, an $(N + 1)$ -connected circular domain of the first type.

Definition 2. We call the annulus $r < |\zeta| < 1$, with $(N - 1)$ circular slits $\zeta = R_j e^{i\theta}$, $\theta_{1j} < \theta < \theta_{2j}$, with a constant $R_j \in (r, 1)$, $j = \overline{1, N - 1}$, an $(N + 1)$ -connected circular domain of the second type.

Existence theorems for the approximate conformal mappings of given $(N + 1)$ multiply connected finite domain with a smooth boundary onto a circular domain of type I and of type II are well-known (see, e.g., [6, 13, 19]).

Theorem 1. There exists an $(N + 1)$ -connected circular domain D_ζ of the first type and there exists a regular function $z = f(\zeta)$ in this domain, such that the function $f(\zeta)$ maps conformally the domain D_ζ onto the given $(N + 1)$ -connected domain D_z with smooth boundary components. The map is unique under the following conditions: $f(0) = Z_0$, $f(1) = Z_1$, $Z_0 \in D_z$, $Z_1 \in L_0$. The corresponding approximate function can be constructed in the form of a Laurent polynomial.

Proof. We assume that $0 \in D_z$ and $Z_0 = 0$, without loss of generality. The proof is constructive. We construct the conformal map of the circular domain of the first type onto the domain D_z by reparametrization of the given boundary representations. So, we search for the function $t_0(\theta)$, $\theta \in [0, 2\pi]$, and for $t_s(\theta)$, $s = \overline{1, N}$, $\theta \in [\theta_{1s}, \theta_{2s}]$, such that the values $z_s(t_s(\theta))$, $s = \overline{0, N}$, are the boundary values of an analytic function in the corresponding circular domain. The parameters R_s , θ_{1s} , θ_{2s} , are also unknown and will be found within the solution process.

Let us consider the analytic in the domain D_z function $\zeta(z)$ that maps conformally the domain D_z onto D_ζ with the correspondence $\zeta(0) = 0$ and the analytic in D_z function $\log \frac{z}{\zeta}$. According to [14], the necessary and sufficient condition for $\log \frac{z}{\zeta}$ to be analytic in D_z is the boundary relation

$$\log \frac{z_s(t)}{R_s e^{i\theta_s(t)}} = \sum_{\sigma=0}^N \frac{1}{\pi i} \int_0^{2\pi} \log \left(\frac{z_\sigma(\tau)}{R_\sigma e^{i\theta_\sigma(\tau)}} \right) \frac{z'_\sigma(\tau)}{z_\sigma(\tau) - z_s(t)} d\tau, \quad (1)$$

where $t \in [0, 2\pi]$, $s = \overline{0, N}$, $R_0 = 1$.

We introduce the functions $q_s(t) = \arg z_s(t) - \theta_s(t)$, $s = \overline{0, N}$, where $\theta_s(t)$ is the polar angle of the image of the point of $z_s(t)$ and separate the imaginary part of both sides of equation (1):

$$\begin{aligned} q_s(t) &= \sum_{\sigma=0}^N \frac{1}{\pi} \int_0^{2\pi} q_\sigma(\tau) (\arg[z_\sigma(\tau) - z_s(t)])'_\tau d\tau - \\ &\quad - \sum_{\sigma=0}^N \frac{1}{\pi} \int_0^{2\pi} \log \frac{|z_\sigma(\tau)|}{R_\sigma} (\log |z_\sigma(\tau) - z_s(t)|)'_\tau d\tau, \quad s = \overline{0, N}. \end{aligned}$$

After differentiating this relation and integrating the result by parts, we obtain the following relations on the functions $q'_s(t)$:

$$\begin{aligned} q'_s(t) &= \sum_{\sigma=0}^N \frac{1}{\pi} \int_0^{2\pi} q'_\sigma(\tau) K_{\sigma s}(\tau, t) d\tau + p_s(t), \quad s = \overline{0, N}, \quad (2) \\ K_{\sigma s}(\tau, t) &= -(\arg[z_\sigma(\tau) - z_s(t)])'_t, \\ p_s(t) &= \sum_{\sigma=0}^N \frac{1}{\pi} \int_0^{2\pi} [\log |z_\sigma(\tau)|]' (\log |z_\sigma(\tau) - z_s(t)|)'_t d\tau. \end{aligned}$$

The kernel $(\log |z_\sigma(\tau) - z_s(t)|)'_t$ has a singularity in the form of $\cot \frac{\tau - t}{2}$ for $\sigma = s$:

$$\begin{aligned} (\log |z_s(\tau) - z_s(t)|)'_t &= \operatorname{Re} \left(\log \sum_{k=-m_s}^{n_s} d_{ks} [e^{ik\tau} - e^{ikt}] \right)'_t = \\ &= -\frac{1}{2} \cot \frac{\tau - t}{2} + \left(\log \left| \sum_{k=1}^{n_s} d_{ks} e^{ikt} \sum_{l=0}^{k-1} e^{il(\tau-t)} - \right. \right. \\ &\quad \left. \left. - \sum_{k=1}^{m_s} d_{(-k)s} e^{-ik\tau} \sum_{l=0}^{k-1} e^{il(\tau-t)} \right| \right)'_t. \end{aligned}$$

The Cauchy principal value integral

$$\frac{1}{\pi} \int_0^{2\pi} [\log |z_\sigma(\tau)|]' \cot \frac{\tau - t}{2} d\tau$$

can be calculated via the Hilbert formula [20] as in [16].

We search for the approximate solution of (2) in the form of the Fourier polynomial

$$q'_s(t) = \sum_{j=1}^M \alpha_{js} \cos jt + \beta_{js} \sin jt, \quad t \in [0, 2\pi]. \quad (3)$$

Existence of exact solution to equation (2) and convergence of the approximate solution to the exact one as $M \rightarrow \infty$ were proved in [15] for simply connected domains. This proof can be applied to the case of multiply connected domains if we replace the corresponding space l^2 by the space $l^2 \times l^2 \times \dots \times l^2$.

Now the integral Fredholm equations of the second kind (2) can be reduced to the linear system for the Fourier coefficients α_{js} and β_{js} :

$$\begin{pmatrix} A_{00} & B_{00} & A_{01} & B_{01} & \cdots & B_{0N} \\ C_{00} & D_{00} & C_{01} & D_{01} & \cdots & D_{0N} \\ A_{10} & B_{10} & A_{11} & B_{11} & \cdots & B_{1N} \\ C_{10} & D_{10} & C_{11} & D_{11} & \cdots & D_{1N} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{N0} & D_{N0} & C_{N1} & D_{N1} & \cdots & D_{NN} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \alpha_1 \\ \beta_1 \\ \vdots \\ \beta_N \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \\ a_1 \\ b_1 \\ \vdots \\ b_N \end{pmatrix},$$

where $\alpha_s = (\alpha_{0s}, \dots, \alpha_{Ms})^T$, $\beta_s = (\beta_{0s}, \dots, \beta_{Ms})^T$. The vectors $a_s = (a_{0s}, \dots, a_{Ms})^T$, $b_s = (b_{0s}, \dots, b_{Ms})^T$ on the right-hand side of the system consist of the elements

$$a_{js} = \frac{1}{\pi} \int_0^{2\pi} p_s(t) \cos jt \, dt, \quad b_{js} = \frac{1}{\pi} \int_0^{2\pi} p_s(t) \sin jt \, dt, \quad j = \overline{1, M}, s = \overline{0, N}.$$

The block matrices $A_{\sigma s}, B_{\sigma s}, C_{\sigma s}, D_{\sigma s}$ of size $M \times M$ consist of the elements

$$\begin{aligned} A_{\sigma sjk} &= \delta_{\sigma s} \delta_{jk} - \frac{1}{\pi^2} \int_0^{2\pi} \cos k\tau d\tau \int_0^{2\pi} K_{\sigma s}(\tau, t) \cos jt \, dt, \quad \sigma = \overline{0, N}, \\ B_{\sigma sjk} &= -\frac{1}{\pi^2} \int_0^{2\pi} \sin k\tau d\tau \int_0^{2\pi} K_{\sigma s}(\tau, t) \cos jt \, dt, \quad s = \overline{0, N}, \\ C_{\sigma sjk} &= -\frac{1}{\pi^2} \int_0^{2\pi} \cos k\tau d\tau \int_0^{2\pi} K_{\sigma s}(\tau, t) \sin jt \, dt, \quad j = \overline{1, M}, \\ D_{\sigma sjk} &= \delta_{\sigma s} \delta_{jk} - \frac{1}{\pi^2} \int_0^{2\pi} \sin k\tau d\tau \int_0^{2\pi} K_{\sigma s}(\tau, t) \sin jt \, dt, \quad k = \overline{1, M}, \end{aligned}$$

where $\delta_{\sigma s}$ and δ_{jk} are the Kronecker delta functions.

The functions $q_s(t)$, $s = \overline{0, N}$, can be restored via their derivatives (3) with an arbitrary constant summand

$$q_s(t) = q_{0s} + \tilde{q}_s(t), \quad \tilde{q}_s(t) = \sum_{j=1}^M \frac{\alpha_{js}}{j} \sin jt - \frac{\beta_{js}}{j} \cos jt, \quad t \in [0, 2\pi]. \quad (4)$$

We choose the constant summand q_{00} in accordance with the condition $f(1) = Z_1$ in the following way. We find the value of the parameter \hat{t} such that $z_0(\hat{t}) = Z_1$. Now $q_{00} = \arg(Z_1) - \tilde{q}_0(\hat{t})$.

We obtain the values of the other constant summands q_{0s} , $s = \overline{1, N}$, and also the values of R_s , $s = \overline{1, N}$, in the following way. We take N points in each of the N finite components of the set complement of D_z . Let us denote these points z_j^* , $j = \overline{1, N}$. The function $\log(z_s(t)/R_s e^{i\theta_s(t)})$, $s = \overline{0, N}$, is the boundary value of the analytical in D_z function, so the Cauchy integral with the corresponding density along the boundary of D_z

vanishes at the points z_j^* , $j = \overline{1, N}$. Therefore, we have the linear complex system

$$\sum_{\sigma=0}^N \int_0^{2\pi} (iq_{0s} - \log R_s + \log |z_s(\tau)| + i\tilde{q}_s(\tau)) [\log(z_\sigma(\tau) - z_j^*)]'_\tau d\tau = 0,$$

$j = \overline{1, N}$, with the unknown real q_{0s} and $\log R_s$, $s = \overline{1, N}$.

We restore the values of θ_{1j} and θ_{2j} , $j = \overline{1, N}$, after we have restored q_{0j} . Indeed,

$$\theta_{1j} = \min_{t \in [0, 2\pi]} [\arg z_i(t) - \tilde{q}_j(t)] - q_{0j}, \quad \theta_{2j} = \max_{t \in [0, 2\pi]} [\arg z_i(t) - \tilde{q}_j(t)] - q_{0j}.$$

So, all parameters of the circular domain of the first type D_ζ are found.

Now we have the functions $q_s(t)$, $t \in [0, 2\pi]$, $s = \overline{0, N}$, and, therefore, we can restore the relations between the boundary parameters of the domains D_z and D_ζ via the formula $\theta_s(t) = \arg z_s(t) - q_s(t)$. Note that $\theta_0(t)$ grows monotonically when t grows from 0 to 2π , $\theta_0(2\pi) - \theta_0(0) = 2\pi$, while each of the functions $\theta_s(t)$, $s = \overline{1, N}$, is 2π -periodic with one interval of increase and one interval of decrease. We can restore the inverse to $\theta_0(t)$, monotonically increasing the function $t_0(\theta)$, and we can restore the single-valued functions $t_s^\pm(\theta)$, $\theta \in [\theta_{1s}, \theta_{2s}]$, where t^\pm denotes the parameter of a twice traversed circular slit.

The approximate analytical function that maps D_ζ onto D_z now has the form of the Cauchy integral

$$f(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{z_0(t_0(\theta)) e^{i\theta} d\theta}{e^{i\theta} - \zeta} + \sum_{s=1}^N \frac{1}{2\pi} \int_{\theta_{1s}}^{\theta_{2s}} \frac{[z_s(t_s^+(\theta)) - z_s(t_s^-(\theta))] R_s e^{i\theta}}{R_s e^{i\theta} - \zeta} d\theta.$$

We can apply the Cauchy integral in the form

$$f(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{z_0(t) e^{i\theta_0(t)} \theta_0'(t) dt}{e^{i\theta_0(t)} - \zeta} + \sum_{s=1}^N \frac{1}{2\pi} \int_0^{2\pi} \frac{z_s(t) R_s e^{i\theta_s(t)} \theta_s'(t)}{R_s e^{i\theta_s(t)} - \zeta} dt \quad (5)$$

in order not to deal with the functions $t_s^\pm(\theta)$ and not to integrate along the different borders of the same slit.

It is worthy to apply the Laurent series expansions instead of the Cauchy integral for the representation of the function $z(\zeta)$. We have, for

$\zeta \in D_\zeta$, $|\zeta| \neq R_s$, $s = \overline{0, N}$, the series representation of the integrals in (5):

$$\begin{aligned} f(\zeta) &= \sum_{\substack{s=0 \\ |\zeta| < R_s}}^N \sum_{k=0}^{\infty} \frac{\zeta^k}{2\pi R_s^k} \int_0^{2\pi} z_s(t) e^{-ik\theta_s(t)} \theta'_s(t) dt - \\ &- \sum_{\substack{s=1 \\ |\zeta| > R_s}}^N \sum_{k=0}^{\infty} \frac{R_s^k}{2\pi \zeta^k} \int_0^{2\pi} z_s(t) e^{ik\theta_s(t)} \theta'_s(t) dt. \end{aligned}$$

We have, for $\zeta \in D_\zeta$, $\zeta = R_j e^{i\theta}$, $j \in \{1, 2, \dots, N\}$, $\theta \notin [\theta_{1j}, \theta_{2j}]$, the following representation

$$\begin{aligned} f(R_j e^{i\theta}) &= \frac{1}{4\pi i} \int_0^{2\pi} \frac{z_j(t) e^{i(\theta_j(t) - \theta)/2} \theta'_j(t) dt}{\sin((\theta_j(t) - \theta)/2)} + \\ &+ \sum_{\substack{s=0 \\ R_j < R_s}}^N \sum_{k=0}^{\infty} \frac{R_j^k e^{ik\theta}}{2\pi R_s^k} \int_0^{2\pi} z_s(t) e^{-ik\theta_s(t)} \theta'_s(t) dt - \\ &- \sum_{\substack{s=1 \\ R_j > R_s}}^N \sum_{k=0}^{\infty} \frac{R_s^k e^{-ik\theta}}{2\pi R_j^k} \int_0^{2\pi} z_s(t) e^{ik\theta_s(t)} \theta'_s(t) dt. \end{aligned}$$

Theorem 1 is proved. \square

Theorem 2. *There exists an $(N+1)$ -connected circular domain D_ζ of the second type and there exists a regular function $z = f(\zeta)$ in this domain such that the function $f(\zeta)$ maps conformally the domain D_ζ onto the given $(N+1)$ -connected domain D_z with smooth boundary components. The map is unique under the following conditions: the image of the inner circle $|\zeta| = r$ is the boundary component L_j , $j \in \{1, 2, \dots, N\}$, $f(1) = Z_1$, $Z_1 \in L_0$. The corresponding approximate function can be constructed in the form of a Laurent polynomial.*

Proof. We assume that $j = N$ and $\int_0^{2\pi} (\arg z_N(t))' dt = -2\pi$ without loss of generality. We construct the conformal map of the circular domain of the first type onto the domain D_z by reparametrization of the given boundary

representations. So, we search for functions $t_s(\theta)$, $\theta \in [0, 2\pi]$, $s = \overline{0, N}$, and for functions $t_s(\theta)$, $\theta \in [\theta_{1s}, \theta_{2s}]$, $s = \overline{1, N-1}$. The construction is as that for mapping of the circular domain of the first type.

We consider the analytic in the domain D_z function $\zeta(z)$ that maps conformally the domain D_z onto D_ζ , and the analytic in D_z function $\log \frac{z}{\zeta}$. We apply the boundary relation (1) as this has been done above. We introduce the functions $q_s(t) = \arg z_s(t) - \theta_s(t)$, $s = \overline{0, N}$. After separation of the imaginary parts of the both sides of the previous equation, differentiation, and integration by parts, we get equation (2). We reduce the solution of the integral equation to the solution of a linear system with truncated matrices if we consider $q'_s(t)$ representation (3). Now we have representation (4) for the functions $q_s(t)$. The constant summand q_{00} can be restored in the same way as for the previous case. The values of q_{0s} and R_s , $s = \overline{1, N}$, can be also restored as has been done above with the help of the additional points z_s^* , $s = \overline{1, N}$, located in the exterior of the domain D_z . Note that $z_N^* = 0$.

Finally, we have the mapping function in the series form

$$f(\zeta) = \sum_{\substack{s=0 \\ |\zeta| < R_s}}^{N-1} \sum_{k=0}^{\infty} \frac{\zeta^k}{2\pi R_s^k} \int_0^{2\pi} z_s(t) e^{-ik\theta_s(t)} \theta'_s(t) dt - \\ - \sum_{\substack{s=1 \\ |\zeta| > R_s}}^N \sum_{k=0}^{\infty} \frac{R_s^k}{2\pi \zeta^k} \int_0^{2\pi} z_s(t) e^{ik\theta_s(t)} \theta'_s(t) dt,$$

for $\zeta \in D_\zeta$, $|\zeta| \neq R_s$, $s = \overline{0, N}$. \square

We offer a new method of finding the values of q_{0s} , R_s , $s = \overline{1, N}$, in order to avoid using many additional points z_j^* , $j = \overline{1, N-1}$, and to deal only with the point $z_N^* = 0$. Let $\tilde{\theta}_s(t) = \arg z_s(t) - \tilde{q}_s(t)$. Now we have the following integral representation of the mapping function for $\zeta \in D_\zeta$:

$$f(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{z_0(t) e^{i\theta_0(t)} \theta'_0(t) dt}{e^{i\theta_0(t)} - \zeta} + \sum_{s=1}^N \frac{1}{2\pi} \int_0^{2\pi} \frac{z_s(t) Q_s e^{i\tilde{\theta}_s(t)} \tilde{\theta}'_s(t) dt}{Q_s e^{i\tilde{\theta}_s(t)} - \zeta}, \quad (6)$$

where $Q_s = R_s e^{-iq_{0s}}$, $s = \overline{1, N}$ are unknown. The Cauchy integral vanishes at the points outside D_ζ . So, the expression at the right-hand side of (6)

and all derivatives of this expression vanish at the point $\zeta = 0$. Therefore, we have for all $j \in \mathbf{N}$

$$\int_0^{2\pi} z_0(t) e^{i(1-j)\theta_0(t)} \theta_0'(t) dt + \sum_{s=1}^N \int_0^{2\pi} z_s(t) (Q_s e^{i\tilde{\theta}_s(t)})^{(1-j)} \tilde{\theta}_s'(t) dt = 0. \quad (7)$$

We apply relations (7) in order to find the values of $Q_s = R_s e^{-iq_{0s}}$, $s = \overline{1, N}$, and, therefore, the values of q_{0s} , R_s , $s = \overline{1, N}$. The least-squares method can be applied. After we find these values, we apply formula (6) as the representation of the mapping function.

4. Examples of conformal mappings. In the first example, we apply the conformal mapping from [10]; the parametrization of the boundaries is as follows:

$$\begin{aligned} L_0 &= \{z = 4 + 6i + (10 + 3 \cos 3t)e^{it}\}, \\ L_1 &= \{z = 0.5 + 12i + 0.5e^{-i\pi/4}(e^{it} + 4e^{-it})\}, \\ L_2 &= \{z = 9 + 6i + 0.5e^{i\pi/4}(e^{it} + 4e^{-it})\}, \\ L_3 &= \{z = 0.5 + 0.5e^{i\pi/4}(e^{it} + 4e^{-it})\}. \end{aligned}$$

Recall that [10] describes construction of the conformal mapping of a bounded multiply connected region onto an annulus with circular slits and we are solving the inverse problem: we construct an approximate analytical conformal mapping of an annulus with circular slits onto the given multiply connected domain.

The result of the conformal mapping is shown in Figure 1.

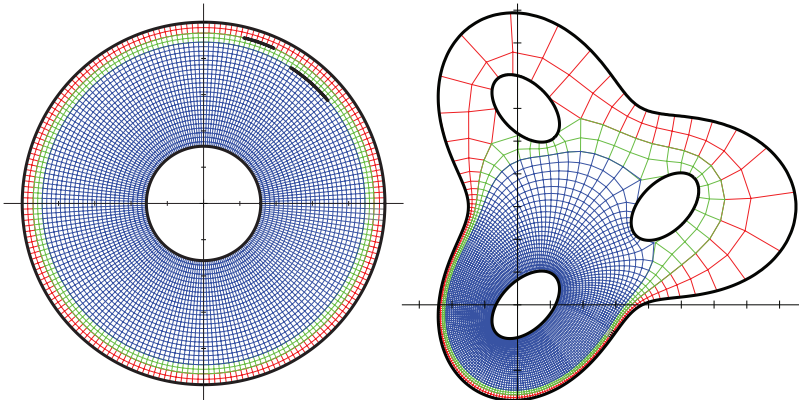


Figure 1: Domains D_ζ and D_z in example 1.

We also investigated the effect of the number M of the Fourier coefficients on the parameters of the problem and accuracy.

Table 1: Accuracy and the elapsed time for example 1.

M	R_1	R_2	R_3	ε	Time (s)
25	0.939091	0.888035	0.315839	$0.4916 \cdot 10^{-3}$	0.28
50	0.939829	0.889457	0.315620	$0.3010 \cdot 10^{-4}$	0.42
100	0.940017	0.889814	0.315564	$0.1862 \cdot 10^{-5}$	1.07
200	0.940063	0.889902	0.315550	$0.1158 \cdot 10^{-6}$	4.07
400	0.940074	0.889924	0.315546	$0.7219 \cdot 10^{-8}$	19.84

Table 1 presents the results; it also shows the elapsed time. Calculations were carried out on a computer Intel Core i5-3330 3Ghz CPU, 4Gb RAM, all the computations were done using Fortran (double precision).

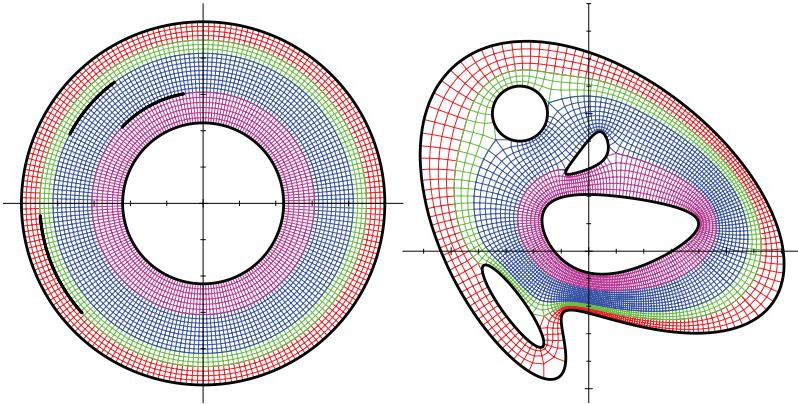


Figure 2: Domains D_ζ and D_z in example 2.

Figure 2 shows an example of a conformal mapping on the domain with non-convex and non-starlike boundary. The boundary parametrization is

$$L_0 = \{z = e^{it} - 0.5ie^{2it} + 0.2e^{-2it}\},$$

$$L_1 = \{z = -0.55 - 0.4i - 0.1(1 - i)e^{it} - 0.1(1 - 2i)e^{-it}\},$$

$$L_2 = \{z = -0.5 + i + 0.2e^{-it}\},$$

$$L_3 = \{z = 0.7i + (1 + i)(0.04e^{it} + 0.1e^{-it}) - 0.02e^{2it}\},$$

$$L_4 = \{z = 0.2 + 0.15i - 0.1(1 + i)e^{it} - 0.3(1 - i)e^{-it} - 0.03(1 - i)e^{2it}\}.$$

In the following example, we consider the case of $0 \in D_z$. The domain

D_z is the disk with four cut out smaller disks:

$$L_0 = \{z = 4e^{it}\}, \quad L_1 = \{z = 2 + e^{-it}\}, \quad L_2 = \{z = -2 + e^{-it}\}, \\ L_3 = \{z = 2i + e^{-it}\}, \quad L_4 = \{z = -2i + e^{-it}\}.$$

An example of a conformal mapping onto the domain is shown by Figure 3.

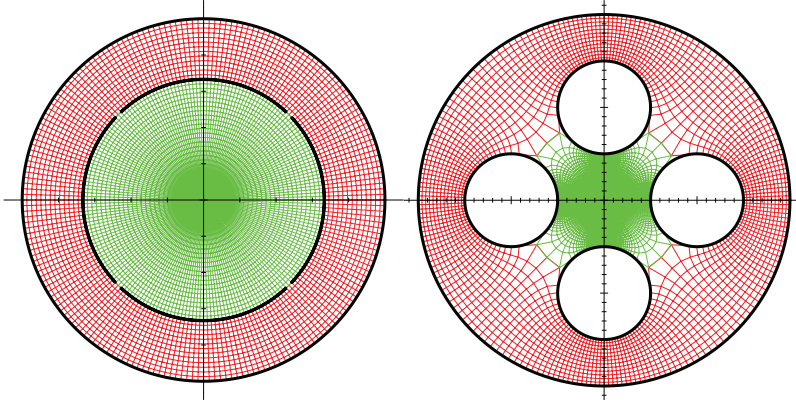


Figure 3: Domains D_ζ and D_z in example 3.

5. Conclusion. It is possible to construct the approximate conformal mapping of the unit disk or an annulus with circular slits onto the given multiply connected domain by Laurent polynomial functions in each annular region. The solution can be reduced to a finite linear system. The method is easily computable.

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