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ON SOLVABILITY OF THE BOUNDARY VALUE PROBLEMS FOR THE INHOMOGENEOUS ELLIPTIC EQUATIONS ON NONCOMPACT RIEMANNIAN MANIFOLDS

Abstract. We study questions of existence and belonging to a given functional class of solutions of the inhomogeneous elliptic equations $\Delta u - c(x)u = g(x)$, where $c(x) \geq 0$, $g(x)$ are Hölder functions on a noncompact Riemannian manifold M without boundary. In this work we develop an approach to evaluation of solutions to boundary-value problems for linear and quasilinear equations of the elliptic type on arbitrary noncompact Riemannian manifolds. Our technique is essentially based on an approach from the papers by E. A. Mazepa and S. A. Korol'kov connected with an introduction of equivalency classes of functions and representations. We investigate the relationship between the existence of solutions of this equation on M and outside some compact set $B \subset M$ with the same growth "at infinity".

Key words: *Riemannian manifold, nonhomogeneous elliptic equations, boundary-value problems*

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1. Introduction. This article is devoted to the investigation of the behavior of solutions of the inhomogeneous linear elliptic equation in relation to the geometry of the manifold in question. Such problems originate in the classification theory of noncompact Riemannian surfaces and manifolds (see [3]). For a noncompact Riemann surface, the well-known problem of the conformal type identification can be stated as follows: does a nontrivial positive superharmonic function exist on this surface?

Exactly this property served as a basis for the extension of the parabolicity notion for arbitrary Riemannian manifolds. Namely, manifolds on which any lower bounded superharmonic function is constant are called parabolic manifolds.

Many questions of this kind fit into the pattern of a Liouville type theorem saying that the space of bounded solutions of some elliptic equation is trivial.

In works of a number of authors the conditions ensuring the validity of the Liouville property on noncompact Riemannian manifolds are adduced in terms of volume growth, or isoperimetric inequalities, and so on (see [3, 5, 6, 8, 10]). However, the class of manifolds admitting nontrivial solutions of some elliptic equations is wide. For example, conditions ensuring the solvability of the Dirichlet problem with continuous boundary conditions "at infinity" for several noncompact manifolds has been found in many papers (see, e. g., [1, 5, 7, 10, 15]).

Notice that even the formulation of boundary-value problems for elliptic differential equations (in particular, the Dirichlet problem) on noncompact Riemannian manifolds and in unbounded domains of that manifolds can be problematic, since it is unclear how we should interpret the boundary data. Geometric compactification enables us sometimes to define them analogously the the classical statement of the Dirichlet problem in bounded domains of R^n (see, e. g., [1, 10]).

In recent years, a large number of works were devoted to solvability of various boundary-value problems for harmonic functions, to solutions of stationary Schrödinger equation, for some other homogeneous linear, nonlinear and quasilinear elliptic equations. But studies of inhomogeneous elliptic equations are of a single nature [9, 12, 14].

In this article we study some questions of existence and belonging to given functional classes of solutions of the inhomogeneous elliptic equation

$$Lu \equiv \Delta u - c(x)u = g(x), \quad (1)$$

where $c(x)$, $g(x) \in C^{0,\alpha}(\Omega)$ for any subset $\Omega \subset\subset M$, $0 < \alpha < 1$ on a noncompact Riemannian manifold M without boundary, $c(x) \geq 0$.

Throughout the paper, we denote by $C^{k,\alpha}(\Omega)$ the subspace $C^k(\Omega)$ consisting of all functions whose derivatives of order k are locally Hölder continuous functions with index α , $0 < \alpha < 1$, $k = 0, 1, 2 \dots$ (see [2, pp. 57–59]).

In our research we use a new approach which is based on the consideration of the equivalence classes of functions on M . Previously, the

approach described below was used to study the solvability of boundary value problems for the Laplace-Beltrami equation, the Schrodinger equation and also for series of semilinear and quasilinear elliptic equations on arbitrary noncompact Riemannian manifolds (see, e. g., [7, 13]).

The proof of the main results is based on the classical propositions of the theory of partial differential equations: the Maximum Principle, the Comparison and Uniqueness Theorems for solutions to linear elliptic differential equations. Their validity on precompact subsets of manifold M can be shown in just the same way as for bounded domains in R^n (see [2, pp. 39–40]).

2. Main concepts and auxiliaries. Let M be an arbitrary smooth connected noncompact Riemannian manifold without boundary and let $\{B_k\}_{k=1}^\infty$ be an exhaustion of M , i. e., a sequence of precompact open subsets of M such that $\bar{B}_k \subset B_{k+1}$ and $M = \cup_{k=1}^\infty B_k$. Throughout the sequel, we assume that the boundaries ∂B_k are C^1 -smooth submanifolds.

Let f_1 and f_2 be arbitrary continuous functions on M .

Definition 1. [13] Say that f_1 and f_2 are equivalent on M and write $f_1 \sim f_2$ if for some exhaustion $\{B_k\}_{k=1}^\infty$ of M we have

$$\lim_{k \rightarrow \infty} \sup_{M \setminus B_k} |f_1 - f_2| = 0.$$

It is easy to verify that the relation " \sim " is an equivalence which does not depend on the choice of the exhaustion of the manifold and so it partitions the set of all continuous functions on M into equivalence classes. Denote the equivalence class of a function f by $[f]$.

Let $B \subset M$ be an arbitrary connected compact subset and the boundary of B be a C^1 -smooth submanifold. Assume that the interior of B is non-empty and $B \subset B_k$ for all k .

Definition 2. We say that the boundary-value problem for equation (1) is solvable on M with boundary data from the class $[f]$, if (1) has a solution $u \in [f]$ on M .

Definition 3. Let $\Phi(x) \in C(\partial B)$ be any function continuous on ∂B . We say that the boundary-value problem for equation (1) is solvable on $M \setminus B$ with boundary data $(\Phi, [f])$ if (1) has a solution $u(x)$ on $M \setminus B$ such that $u \in [f]$ and $u|_{\partial B} = \Phi|_{\partial B}$.

Note that if the manifold M has compact boundary or there is a natural geometric compactification of M (for example, on manifolds of negative

sectional curvature or spherically symmetric manifolds) which adds the boundary at infinity, then this approach naturally leads to the classical statement of the Dirichlet problem (see, for instance, [10]).

Now we formulate without proofs some auxiliary assertions. Detailed proofs of these statements can be found in [11, 13, 14].

The Comparison Principle. *Let $Lv \leq Lu$ on $M \setminus B$, $v|_{\partial B} \geq u|_{\partial B}$, $v \sim u$. Then $v \geq u$ on $M \setminus B$. If $Lv \leq Lu$ on M and $v \sim u$, then $v \geq u$ on M .*

The Uniqueness Theorem. *Let $Lv = g(x)$, $Lu = g(x)$ on $M \setminus B$ and $v|_{\partial B} = u|_{\partial B}$, $v \sim u$. Then $w = u$ on $M \setminus B$.*

Let $Lv = g(x)$, $Lu = g(x)$ on M and $v \sim u$. Then $v = u$ on M .

Lemma 1. *Suppose that $G \subset\subset M$ is a precompact subset in M , a function $u \in C(\overline{G}) \cap C^2(G)$ satisfies the equation $Lu = g$ on G , where $g \in C_0(\overline{G})$, $\Omega := \text{supp } g$ and $\Omega \subset\subset G$, $c \geq 0$ on \overline{G} and $c \neq 0$ on Ω . Then*

$$\sup_G |u| \leq \sup_{\partial G} |u| + \sup_{\Omega} \frac{|g|}{c}.$$

Consider, together with equation (1), the homogeneous linear equation

$$Lu \equiv \Delta u - c(x)u = 0, \tag{2}$$

which is the stationary Schrödinger equation.

Denote by v_k the solution of equation (2) in $B_k \setminus B$ that satisfies the conditions

$$v_k|_{\partial B} = 1, \quad v_k|_{\partial B_k} = 0.$$

We can easily verify that the sequence v_k is uniformly bounded on $M \setminus B$ and so it is compact in the class $C^2(G)$ for every compact subset $G \subset M \setminus B$. Moreover, as $k \rightarrow \infty$ this sequence increases monotonically and converges on $M \setminus B$ to a solution of equation (2)

$$v = \lim_{k \rightarrow \infty} v_k, \quad 0 < v \leq 1, \quad v|_{\partial B} = 1.$$

Also, note that the function v is independent of the choice of an exhaustion $\{B_k\}_{k=1}^{\infty}$ (see, e. g., [7, 13]).

Definition 4. [13] *We call v the L -potential of the compact set B relative to M .*

For the Laplace-Beltrami equation, the function v is the capacity potential of the compact set B relative to the manifold M (see [3]).

Definition 5. [11, 13] *Call the manifold M L -strict if for some compact set $B \subset M$ there exists an L -potential v of B such that $v \in [0]$.*

The L -strictness is shown to be independent of the choice of the compact set B in [11].

Remark 1. *The connections between solvability of boundary-value and exterior boundary-value problems for linear and quasilinear homogeneous equations is investigated in details in [11, 13].*

Remark 2. *In proving the main results, the asymptotic behavior of the solutions of Laplace-Beltrami and Schrödinger equations plays an important role. It is noted that the cases with $c(x) \equiv 0$ and $c(x) \not\equiv 0$ on the noncompact Riemannian manifold M are served by various theorems [3].*

3. The main results for $c(x) \not\equiv 0$.

Theorem 1. *Let $B \subset M$ be some connected compact subset such that $c(x) > 0$ on some neighborhood of B and the boundary-value problem for equation (1) is solvable with boundary data $(A, [f])$ on $M \setminus B$ for any constant A . Then the boundary-value problem for equation (1) with boundary data from the class $[f]$ is solvable on M too.*

Proof. In what follows we assume that the subset $B \subset M$ is chosen so that $c(x) > 0$ on some neighborhood B'' of B .

First, note that the condition of the theorem implies the existence on $M \setminus B$ of a nontrivial capacity potential $v \in [0]$. Let u_0 be a solution of equation (1) on $M \setminus B$ such that $u_0 \in [f]$ and $u|_{\partial B} = 0|_{\partial B}$. Consider the function $U_0 \in C^{2,\alpha}(M)$ such that $U_0 = u_0$ outside of B'' , $U_0 = 0$ on the precompact $B' \subset\subset B$. Then $LU_0 = g_0(x)$ on M , where the function $g_0(x) \in C^\alpha(M)$ and satisfies the following conditions: $g_0(x) = 0$ on the set B' , $g_0(x) \equiv g(x)$ outside of B'' , $g_0(x) \not\equiv g(x)$ on $B'' \setminus B'$.

Consider now the sequence of functions φ_k that are solutions of the problems

$$\begin{cases} L\varphi_k = g(x) & \text{in } B_k, \\ \varphi_k|_{\partial B_k} = u_0|_{\partial B_k} \end{cases}$$

and the sequence of functions $\psi_k = \varphi_k - U_0$. It is clear that ψ_k are solutions of the problems

$$\begin{cases} L\psi_k = g(x) - g_0(x) & \text{in } B_k, \\ \psi_k|_{\partial B_k} = 0. \end{cases}$$

where the function $g(x) - g_0(x) \in C^\alpha(M)$ and satisfies the following conditions: $g(x) - g_0(x) = g(x)$ on the compact set B' , $g(x) - g_0(x) = 0$ outside of B'' . Thus, $\Omega := \text{supp}\{g(x) - g_0(x)\}$ is compact and $\Omega \subset B''$.

By Lemma 1, for all k for $x \in B_k$ we have:

$$|\psi_k| \leq \sup_{B_k} |\psi_k| \leq \sup_{\partial B_k} |\psi_k| + \sup_{\Omega} \frac{|g(x) - g_0(x)|}{c(x)} = \sup_{\Omega} \frac{|g(x) - g_0(x)|}{c(x)},$$

which implies the uniform boundedness of the family of functions $\{\psi_k\}_{k=1}^\infty$ on M .

Hence, we obtain compactness of this family in the class $C^2(G)$ for an arbitrary compact subset $G \subset M$.

This in turn implies existence of the limit function $\psi = \lim_{k \rightarrow \infty} \psi_k$ on M such that $L\psi = g(x) - g_0(x)$ on M .

Now we shall show that $\psi \in [0]$. It is clear that ψ is a solution of the equation $L\psi = 0$ on $M \setminus B''$. Since $\partial B''$ is compact by continuity of the function ψ , there exists $A = \max_{\partial B''} |\psi|$ and we have

$$-A \leq \psi|_{\partial B''} \leq A$$

and also

$$-(A+1) \leq \psi_k|_{\partial B''} \leq A+1$$

for sufficiently large values k .

Consider the functions

$$\bar{\psi} = (A+1) \cdot v \quad \text{and} \quad \underline{\psi} = -(A+1) \cdot v$$

on $M \setminus B''$, where v is the L -potential of the compact set \bar{B}'' , $v \in [0]$. The functions $\bar{\psi}$ and $\underline{\psi}$ are solutions to equation (1) and satisfy the conditions

$$\bar{\psi}|_{\partial B''} = A+1, \quad 0 \leq \bar{\psi} \leq A+1, \quad \bar{\psi} \in [0],$$

$$\underline{\psi}|_{\partial B''} = -(A+1), \quad -(A+1) \leq \underline{\psi} \leq 0, \quad \underline{\psi} \in [0].$$

Then $\underline{\psi} \leq \bar{\psi}$ on $M \setminus B''$. Since

$$L\underline{\psi} = L\psi_k = L\bar{\psi} = 0,$$

$$\underline{\psi}|_{\partial B_k} \leq \psi_k|_{\partial B_k} \leq \bar{\psi}|_{\partial B_k}$$

and

$$\underline{\psi}|_{\partial B''} \leq \psi_k|_{\partial B''} \leq \bar{\psi}|_{\partial B''}$$

on $B_k \setminus B''$, the Comparison Principle implies

$$\underline{\psi} \leq \psi_k \leq \bar{\psi}$$

on $B_k \setminus B''$ for sufficiently large values k . Passing to the limit as $k \rightarrow \infty$ we obtain $\underline{\psi} \leq \psi \leq \bar{\psi}$. Since $\underline{\psi} \sim \bar{\psi} \sim 0$ we have $\psi \in [0]$.

Finally, the existence of the function $\psi = \lim_{k \rightarrow \infty} \psi_k$ implies the existence of the limit function $u = \lim_{k \rightarrow \infty} \varphi_k$ such that $Lu = g(x)$ on M and $u \sim u_0$. The proof of Theorem 1 is over. \square

Corollary 1. *Let for any continuous function $\Phi(x) \in C(\partial B)$ the boundary-value problem for equation (1) be solvable on $M \setminus B$ with boundary data $(\Phi, [f])$. Then the boundary-value problem for equation (1) with boundary data from the class $[f]$ is solvable on M too.*

Theorem 2. *Let M be an L -strict manifold and the boundary-value problem for equation (1) is solvable on M with boundary data from the class $[f]$. Then for any continuous function $\Phi(x) \in C(\partial B)$ the boundary-value problem for equation (1) is solvable on $M \setminus B$ with boundary data $(\Phi, [f])$.*

Proof. We first prove that for every continuous function Φ on ∂B there is a solution w to equation (2) on $M \setminus B$, such that $w|_{\partial B} = \Phi$ and $w \in [0]$. Consider the sequence of functions w_k that are solutions to the boundary value problems:

$$\begin{cases} Lw_k = 0 & \text{in } B_k \setminus B, \\ w_k|_{\partial B} = \Phi, \\ w_k|_{\partial B_k} = 0. \end{cases}$$

By the Maximum Principle, for every k we have

$$|w_k| \leq \sup_{\partial(B_k \setminus B)} |w_k| = \sup_{\partial B} |\Phi|,$$

i. e., the sequence $\{w_k\}_{k=1}^\infty$ is uniformly bounded on M and so it is compact in the class of twice continuously differentiable functions on every compact subset of M . Let $w(x)$ be a limit function. It is clear that $w|_{\partial B} = \Phi$.

Put $U = \max_{\partial B} |\Phi|$ and show that $w \in [0]$. It is obvious that

$$\begin{aligned} -(U + 1) &\leq \Phi \leq U + 1, \\ -(U + 1) &\leq w|_{\partial B} \leq U + 1 \end{aligned}$$

and for every k

$$-(U + 1) \leq w_k|_{\partial B} \leq U + 1.$$

Consider the functions $u_1 = -(U + 1) \cdot v$ and $u_2 = (U + 1) \cdot v$ on $M \setminus B$, where v is the L -potential of the compact set B , $v \in [0]$. The functions u_1 and u_2 are solutions to equation (1) and satisfy the conditions

$$\begin{aligned} u_1|_{\partial B} &= -(U + 1), \quad -(U + 1) \leq u_1 \leq 0, \quad u_1 \in [0], \\ u_2|_{\partial B} &= U + 1, \quad 0 \leq u_2 \leq U + 1, \quad u_2 \in [0]. \end{aligned}$$

Then $u_1 \leq u_2$ on $M \setminus B$ and, by the Comparison Principle, for all k we have

$$u_1 \leq w_k \leq u_2$$

on $B_k \setminus B$. Taking the limit as $k \rightarrow \infty$, we obtain $u_1 \leq w \leq u_2$. Since $u_1 \sim u_2 \sim 0$, we have $w \in [0]$.

Now, let $u_0 \in [f]$ be a solution to the boundary value problem for equation (1) on M and Φ be an arbitrary continuous function on ∂B . As has been shown above, there exists a solution w of equation (2) on $M \setminus B$ such that $w|_{\partial B} = u_0|_{\partial B} - \Phi$ and $w \in [0]$. Then the function $u = u_0 - w$ is a sought solution to the exterior boundary value problem for equation (1) on $M \setminus B$ such that $u \in [f]$ and $u|_{\partial B} = \Phi$. \square

4. The case of harmonic functions. Let $B \subset M$ be an arbitrary connected compact subset, ∂B be an C^1 -smooth submanifold. The fact that $c(x) \neq 0$ on some compact set $B \subset M$ was crucial in the proof of Theorem 1. However, the condition $c \equiv 0$ on M does not violate Theorem 1 and Theorem 2 for the case of the Poisson equation

$$\Delta u = g(x), \tag{3}$$

where $g(x) \in C^{0,\alpha}(\Omega)$ for any subset $\Omega \subset\subset M$, $0 < \alpha < 1$.

Similarly to the concept of L -strictness introduced above we can define the concept of Δ -strictness of the manifold M .

Definition 6. [13] *A manifold M is called Δ -strict if there exists a nontrivial capacity potential $v \in [0]$ for some compact set $B \subset M$.*

Note that the Δ -strictness of the manifold M implies the non-parabolicity of M (see, e. g., [3]). The following result holds.

Theorem 3. *Let the boundary-value problem for equation (3) be solvable with boundary data $(A, [f])$ on $M \setminus B$ for any constant A . Then the boundary-value problem for equation (3) with boundary data from the class $[f]$ is solvable on M too.*

Proof. Let u_0 be a solution of equation (3) on $M \setminus B$ such that $u_0 \in [f]$ and $u_0|_{\partial B} = 0|_{\partial B}$. As in Theorem 1, consider the function $U_0 \in C^{2,\alpha}(M)$ such that $U_0 = u_0$ outside of B'' , $U_0 = 0$ on the precompact $B' \subset\subset B$. Then $\Delta U_0 = g_0(x)$ on M , where the function $g_0(x) \in C^{0,\alpha}(M)$ and satisfies the following conditions: $g_0(x) = 0$ on the set B' , $g_0(x) \equiv g(x)$ outside of B'' , $g_0(x) \not\equiv g(x)$ on $B'' \setminus B'$.

Now consider the sequence of functions φ_k solving the problems

$$\begin{cases} \Delta\varphi_k = g(x) & \text{in } B_k, \\ \varphi_k|_{\partial B_k} = u_0|_{\partial B_k} \end{cases}$$

and the sequence of functions $\psi_k = \varphi_k - U_0$. For these functions we have

$$\begin{cases} \Delta\psi_k = g(x) - g_0(x) & \text{in } B_k, \\ \psi_k|_{\partial B_k} = 0, \end{cases}$$

where the function $g(x) - g_0(x) \in C^\alpha(M)$ and satisfies the following conditions: $g(x) - g_0(x) = g(x)$ on the compact set B' , $g(x) - g_0(x) = 0$ outside of B'' . Thus, $\Omega := \text{supp}\{g(x) - g_0(x)\}$ is compact and $\Omega \subset B''$.

Let $G_k(x,y)$ be a Green's function in every B_k , i. e., the function which satisfies the conditions

$$\Delta_x G_k(x,y) = -\delta_y(x), \quad G_k(x,y)|_{x \in \partial B_k} = 0$$

for every $y \in B_k$; here $\delta_y(x)$ is the Dirac's δ -function.

Therefore, a Green representation

$$\psi_k(x) = \int_{B_k} G_k(x,y)(g(y) - g_0(y))dy$$

exists.

Note that the condition of the theorem implies existence of a nontrivial capacitive potential on the manifold M , and hence the non-parabolicity

of the manifold M [3]. Since the manifold M is non-parabolic, there is a finite limit of the Green's functions $G(x, y) = \lim_{k \rightarrow \infty} G_k(x, y)$ (see, e.g., [3]). The fact implies existence of the limit of the sequence $\{\psi_k\}$. Let $\lim_{k \rightarrow \infty} \psi_k = \psi$. Then $\Delta\psi = g(x) - g_0(x)$ on M . As in Theorem 1, we can prove that $\psi \in [0]$. Hence, there is a limit function $u = \lim_{k \rightarrow \infty} \varphi_k$ of the sequence $\{\varphi_k\}$ and u satisfies $u = \psi + U_0$, $\Delta u = g(x)$ on M and $u \sim u_0$. The proof of the Theorem 3 is over. \square

Corollary 1. *Let the boundary-value problem for equation (3) be solvable with boundary data $(\Phi, [f])$ on $M \setminus B$ for any continuous function $\Phi(x) \in C(\partial B)$. Then the boundary-value problem for equation (3) with boundary data from the class $[f]$ is solvable on M too.*

Theorem 4. *Let M be a Δ -strict manifold and the boundary-value problem for equation (3) is solvable on M with boundary data from the class $[f]$. Then the boundary-value problem for equation (3) is solvable with boundary data $(\Phi, [f])$ on $M \setminus B$ for any continuous function $\Phi(x) \in C(\partial B)$.*

Proof. The proof of this theorem coincides with the proof of the similar statement in Theorem 2. \square

Remark 3. *Note, that the given statements for the bounded continuous function f on M are proved in [14]. The exact conditions for the solvability of the Dirichlet problem for the Poisson equation on model manifolds are obtained in [9].*

The obtained results can find application in the development of functional-analytic methods in the theory of elliptic equations on non-compact Riemannian manifolds. A more detailed description of these methods can be found, for example, in [3, 4].

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