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ON ENTIRE FUNCTIONS WITH GIVEN ASYMPTOTIC BEHAVIOR

Abstract. We study approximation of subharmonic functions on the complex plane by logarithms of moduli of entire functions. In the theory of series of exponentials these entire functions are the main tool. In questions of decomposition of functions into a series of exponentials, the subharmonic function, as a rule, satisfies the Lipschitz condition. We prove the theorem on approximation of such subharmonic functions. Also we prove the theorem on joint approximation of two subharmonic functions.

Key words: *subharmonic function, approximation, entire function, Riesz measure*

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1. Introduction. Entire functions with prescribed asymptotic behavior serve as the main tool in constructing the exponential series. The problems of existence and constructing entire functions with prescribed asymptotic properties arose as inner problems of the theory of entire functions. This problem was dealt with by many mathematicians. See, for example, [1], [3]–[5], [8]. The next theorem is proved in [8].

Theorem A. [8] *For each subharmonic on the plain function u of finite order greater than zero and for each $\beta < 0$, there exists an entire function f satisfying the relation*

$$|u(\lambda) - \ln|f(\lambda)|| = O(\ln(|\lambda| + 1)),\ \lambda \notin E,\ |\lambda| \longrightarrow \infty,$$

where the exceptional set E can be covered by a set of disks $B(w_k, r_k)$ so that $\sum_{|w_k|\geqslant R} r_k = O(R^\beta)$, $R \longrightarrow \infty$.

By $B(z,t)$ we denote the open disk with center z and radius t.

In applications, approximated subharmonic functions often have some additional properties that can refine the asymptotics. On the other hand, one usually requires not just to estimate the size of the exceptional set, but, to a greater extent, to get its construction. In questions of decomposition of functions into series of exponentials, subharmonic functions, as a rule, satisfy the Lipschitz condition (see, for example, [6]). Therefore, we consider the approximation of such functions separately.

For a measure μ we denote the μ-measure of the disk $B(z,t)$ by $\mu(z,t)$ and let $\mu(t)=\mu(0,t)$. The set of zeros of an entire function L is denoted by $N(L)$. The notation $A(x)\prec B(x),\ x\in X$, means that for some constant $C>0$, for all $x\in X$, the estimate $A(x)\leqslant CB(x)$ holds true.

In the present work we prove the following theorem.

Theorem 1. *Let u be a subharmonic function on the plane, and μ be the Riesz measure of u. If for some $M>0$, for all points $z\in\mathbb{C}$ the condition*

$$\mu(z,t)\leqslant Mt,\ t\in(0;1), \tag{1}$$

holds true, then there exists an entire function f with simple zeros λ_n such that for some $\delta\in(0;1)$ the disks $B_\delta(\lambda_n)=B(\lambda_n,\delta(|\lambda_n|+1)^{-1})$ are pairwise disjoint and the function satisfies the relations

$$|\ln|f(\lambda)|-u(\lambda)|\leqslant A\ln(|\lambda|+1)+C,\ \ \lambda\notin\bigcup_n B_\delta(\lambda_n),$$

$$|\ln|f'(\lambda)|-u(\lambda)|\leqslant A\ln(|\lambda|+1)+C',\ \ \lambda\in N(f).$$

Here the constant $A>0$ is independent on the constant M and the function u, and constants C, C', δ depend on M, but not on u.

Note that if a subharmonic function u satisfies for some constant $K>0$ the Lipschitz condition $|u(z)-u(w)|\leqslant K|z-w|,\ z,w\in\mathbb{C}$, then its Riesz measure satisfies condition (1): $\mu(z,t)\leqslant Ke\,t,\ z\in\mathbb{C},\ t>0$. This follows from Jensen's formula

$$\frac{1}{2\pi}\int\limits_0^{2\pi}u(z+re^{i\varphi})d\varphi=u(z)+\int\limits_0^r\frac{\mu(z,t)}{t}\,dt,\ \ z\in\mathbb{C},\ t>0.$$

In fact, the Lipschitz condition implies the estimate

$$\mu(z,\frac{r}{e})\leqslant\int\limits_{\frac{r}{e}}^r\frac{\mu(z,t)}{t}\,dt\leqslant\frac{1}{2\pi}\int\limits_0^{2\pi}|u(z+re^{i\varphi})-u(z)|d\varphi\leqslant Kr.$$

In applications of approximation theorems, one often needs joint approximation of several subharmonic functions. In the next theorem we consider the joint approximation of two subharmonic functions which Riesz measure satisfying condition (1).

Theorem 2. *Let u_j, $j=1,2$, be two subharmonic functions on the plane, and μ_j, $j=1,2$, their Riesz measures satisfy the conditions*

$$\mu_j(z,t) \leqslant Mt,\ t \in (0;1), \tag{2}$$

and the measure μ_2 satisfies the condition

$$\int_1^\infty \frac{\mu_2(r)dr}{r^2} < \infty. \tag{3}$$

Then there exist entire functions f_j, $j=1,2$, such that all zeros of $f = f_1 f_2$ are simple; for some $\delta > 0$ the disks $B_\delta(\lambda) = B(\lambda, \delta(|\lambda|+1)^{-1})$, $\lambda \in N(f)$, are pairwise disjoint; and for some constants B, C, $C' > 0$ the relations

$$|\ln|f_j(\lambda)| - u_j(\lambda)| \leqslant B\ln(|\lambda|+1) + C, \quad \lambda \notin \bigcup_{z\in N(f_j)} B_\delta(z),$$

$$|\ln|f_j'(\lambda)| - u_j(\lambda)| \leqslant B\ln(|\lambda|+1) + C', \quad \lambda \in N(f_j),$$

hold true. Here the constant $B>0$ is independent on the constant M and the functions u_j, and the constants C, C', δ depend on M, but not on u_j.

2. Proof of Theorem 1.

Lemma 1. *Let u be a subharmonic function on $\mathbb{C}$, $u(0)=0$, and its Riesz measure μ satisfies (1). Then there exists a subharmonic and infinitely differentiable function v on $\mathbb{C}$ satisfying the conditions $u(\lambda) \leqslant v(\lambda) \leqslant u(\lambda) + M$, $\Delta v(\lambda) \leqslant c$, $\lambda \in \mathbb{C}$, where the constant $c > 0$.*

Proof. Let $\alpha(t) \in C^\infty(\mathbb{R})$; $\alpha(t) = 0$, $t \notin (0;1)$; $\alpha(t) > 0$, $t \in (0;1)$; and

$$\int_0^1 t\alpha(t)\,dt = \frac{1}{2\pi}.$$

If $\alpha(\lambda) = \alpha(|\lambda|)$, $\lambda \in \mathbb{C}$, and dm is the planar Lebesgue measure, then

$$v(\lambda) := \int\limits_{\mathbb{C}} \alpha(\lambda - w)u(w)dm(w) = \int\limits_{\mathbb{C}} \alpha(w)u(\lambda - w)dm(w),\ \lambda \in \mathbb{C},$$

is a subharmonic and infinitely differentiable function (see [7, p. 51]). By definition $v(\lambda) - u(\lambda) = \int_{\mathbb{C}}(u(w) - u(\lambda))\alpha(\lambda - w)dm(w)$. Converting to polar coordinates and using Jensen's formula, we obtain

$$v(\lambda) - u(\lambda) = 2\pi \int\limits_0^1 \alpha(t)\Big(\int\limits_0^t \frac{\mu(\lambda,s)}{s}ds\Big)tdt.$$

It is obvious that $v(\lambda) \geqslant u(\lambda)$, $\lambda \in \mathbb{C}$. By condition (1) this implies $v(\lambda) - u(\lambda) \leqslant M \int_{\mathbb{C}} \alpha(\lambda)dm(\lambda) = M$. The first statement of lemma is proved. Let us estimate Δv. Considering u as a distribution, we obtain $\Delta v(\lambda) = \pi \int_{\mathbb{C}} \alpha(\lambda - w)d\mu(w)$. If $\alpha = \max_t \alpha(t)$, then in view of (1), we have $\Delta v(\lambda) \leqslant \pi\alpha\mu(\lambda,1) \leqslant \pi\alpha M$. $\square$

Thus, we can prove Theorem 1 assuming function u to satisfy condition

$$\Delta u(\lambda) \leqslant M,\ \lambda \in \mathbb{C}. \tag{4}$$

Let us show that it is enough to prove Theorem 1 for $M = 1$. In fact, suppose the following theorem is proved.

Theorem 1′. *Let $v \in C^\infty$ be a subharmonic function on the plane, $v(0) = 0$, and $\Delta v(\lambda) \leqslant 1$, $\lambda \in \mathbb{C}$. Then there exists an entire function g with simple zeros w_n such that for some $\delta_0 \in (0;1)$ the disks $B(w_n, \delta_0(|w_n| + 1)^{-1})$ are pairwise disjoint and the function satisfies the relations*

$$|\ln|g(w)| - v(w)| \leqslant A_0 \ln(|w| + 1) + C_0\ \lambda \notin \bigcup_n B(w_n, \delta_0(|w_n| + 1)^{-1}),$$

$$|\ln|g'(\lambda)| - v(\lambda)| \leqslant A_0 \ln(|\lambda| + 1) + C_0',\ \lambda \in N(g).$$

Here the constants $A_0, C_0, C_0' > 0$ are independent on the function v.

Let u satisfy (4). If $M > 1$, then we consider the function $v(w) = u\Big(\dfrac{w}{M}\Big)$. Since $\mu_v(z,t) = \mu\Big(\dfrac{z}{M}, \dfrac{t}{M}\Big)$, then $\mu_v(z,t) \leqslant t$ for $t < 1$. By Theorem 1′, there exists a function g that satisfies the appropriate estimates. Let us take the function $f(\lambda) = g(M\lambda)$ with simple zeros $\lambda_n = \frac{1}{M}w_n$. Under the mapping $w \longrightarrow \frac{w}{M}$ the pairwise disjoint disks $B(w_n, c\delta_0(|w_n|+1)^{-1})$ are mapped to the disjoint disks $B\big(\lambda_n, \frac{\delta_0}{M}(M|\lambda_n|+ + 1)^{-1}\big)$, and outside these disks the estimate

$$|\ln|f(\lambda)| - u(\lambda)| \leqslant A_0 \ln(|\lambda|+1) + A_0 \ln M + C_0 \tag{5}$$

holds true. Since

$$\frac{\delta_0}{M^2}(|\lambda_n|+1)^{-1} \leqslant r_n := \frac{\delta_0}{M}(M|\lambda_n|+1)^{-1} \leqslant \frac{\delta_0}{M}(|\lambda_n|+1)^{-1},$$

the disks $B\big(\lambda_n, \frac{\delta_0}{M^2}(|\lambda_n|+1)^{-1}\big)$ are also pairwise disjoint. We extend estimate (6) outside these disks. Let H be the minimal harmonic majorant of u on $B(\lambda_n, r_n)$; then, by Green's formula and by condition (4) for $\lambda \in B(\lambda_n, r_n)$, we have

$$0 \leqslant H(\lambda) - u(\lambda) = \int\limits_{B(\lambda_n, r_n)} G(\lambda, z)d\mu(z) =$$

$$= \frac{1}{\pi} \int\limits_{B(\lambda_n, r_n)} G(\lambda, z)\Delta u(z)dm(z) \leqslant \frac{M}{\pi} \int\limits_{B(\lambda_n, r_n)} G(\lambda, z)dm(z),$$

where $G(\lambda, z)$ is Green's function of the Dirichlet problem for $B(\lambda_n, r_n)$. Taking into account that for the function $A(\lambda) = |\lambda - \lambda_n|^2$ the relation $\Delta A(\lambda) \equiv 4$ holds true and the minimal harmonic majorant of A is identically equal to r_n^2, we obtain the estimate

$$0 \leqslant H(\lambda) - u(\lambda) = \frac{M}{4} \max_{z \in B(\lambda_n, r_n)} (r_n^2 - |\lambda_n - z|^2) \leqslant \frac{1}{4M}, \quad \lambda \in B(\lambda_n, r_n). \tag{6}$$

By the maximum principle and by (6) for $\lambda \in B(\lambda_n, r_n)$ we have

$$\Big|H(\lambda) - \Big(\ln|f(\lambda)| - \ln\frac{|\lambda - \lambda_n|}{r_n}\Big)\Big| \leqslant \max_{|z-\lambda_n| = r_n} \big|u(z) - \ln|f(z)|\big| \leqslant$$

$$\leqslant A_0 \ln(|\lambda_n| + r_n + 1) + A_0 \ln M + C_0 \leqslant A_0 \ln(|\lambda_n|+1) + A_0 \ln(2M) + C_0. \tag{7}$$

If
$$\frac{\delta_0}{M^2}\big(|\lambda_n|+1\big)^{-1} \leqslant |\lambda-\lambda_n| \leqslant r_n \leqslant \frac{\delta_0}{M}\big(M|\lambda_n|+1\big)^{-1},$$
then $\Big|\ln\Big|\dfrac{\lambda-\lambda_n}{r_n}\Big|\Big| \leqslant \ln M$ and therefore
$$|H(\lambda)-\ln|f(\lambda)|| \leqslant A_0\ln(|\lambda|+1)+(A_0+1)\ln(2M)+C_0,$$
where $\lambda \in B(\lambda_n, r_n)\setminus B\big(\lambda_n, d\frac{\delta_0}{M^2}(|\lambda_n|+1)^{-1}\big)$. This together with (6) implies the required estimate for the function f
$$|u(\lambda)-\ln|f(\lambda)|| \leqslant A_0\ln(|\lambda|+1)+(A_0+1)\ln(2M)+C_0+\frac{1}{4M},$$
where $\lambda \notin B\Big(\lambda_n, \dfrac{\delta_0}{M^2}\big(|\lambda_n|+1\big)^{-1}\Big)$. Passing to the limit as $\lambda\to\lambda_n$ in (7) and applying (6), we obtain the required estimate for $f'(\lambda_n)$:
$$|u(\lambda_n)-\ln|f'(\lambda_n)|| \leqslant A_0\ln(|\lambda_n|+1)+C_0'+\frac{1}{4M}+\ln M,\ n\in\mathbb{N}.$$

To prove Theorem $1'$, one more lemma is needed.

Lemma 2. *Let u be a smooth subharmonic function, and Δu satisfy (4) with $M=1$. Denote the square centered at the origin with sides of length 3^n and parallel to the axes by Q_n, $n\in\mathbb{N}\cup\{0\}$. Then*
$$Q_{n+1}\setminus Q_n = \bigcup_{j=1}^{8} Q_{n,j},\ \ n\in\mathbb{N}\cup\{0\},$$
where $Q_{n,j}$ are the squares obtained by shifting the square Q_n by the vectors $(\pm 3^n,0)$, $(0,\pm 3^n)$, $(\pm 3^n,\pm 3^n)$ and enumerated counterclockwise, starting from the square which intersects with the positive ray of the real axe. There exists a subharmonic function $\widetilde{u}$ with the Riesz measure $\widetilde{\mu}$ such that

1) *inside the squares $Q_{n,j}$ the function $\widetilde{u}$ is smooth and estimate (4) holds true;*
2) *$\widetilde{\mu}(Q_{n,j})$ is a non-negative integer;*
3) *the estimate $|u(\lambda)-\widetilde{u}(\lambda)| \leqslant 45\ln(|\lambda|+e)+142$, $\lambda\in\mathbb{C}$, holds true.*

Proof. Let $\mu(Q_{n,j}) := m_{n,j} + q_{n,j}$, $j = 1, 2, \dots, 8$, $n \in \mathbb{N}$, where $q_{n,j} = \{\mu(Q_{n,j})\} \in [0;1)$ is the fractional part of $\mu(Q_{n,j})$. Let $q_n^+ = \sum_j q_{n,j} \in [0;8)$, $q_n^- = \sum_j (q_{n,j} - 1) \in [-8;0)$. Define the sequence q_n as follows: let $q_0 = \{\mu(Q_0)\}$; if q_j are defined for $j \leqslant k-1$, then, as $\sum_{j \leqslant k-1} q_j \geqslant 0$, let $q_k := q_k^-$; $q_k := q_k^+$ otherwise. Thus, $\sum_{k=0}^{n} q_k \in (-8;8)$, $n \in \mathbb{N}$. Then define the sequence of natural numbers $N_0, N_{n,j}$, $j = 1, \dots, 8$, $n \in \mathbb{N}$. Let $N_0 = [\mu(Q_0)]$; if $q_n = q_n^-$, then $N_{n,j} = \mu(Q_{n,j}) - (q_{n,j} - 1)$; and if $q_n = q_n^+$, then $N_{n,j} = \mu(Q_{n,j}) - q_{n,j}$. Thus, either $N_{n,j} = m_{n,j} + 1$ or $N_{n,j} = m_{n,j}$. The restriction of the measure μ to the square $Q_{n,j}$ is denoted by $\mu_{n,j}$, $\mu_0 = \mu\,|_{Q_0}$ Let $\widetilde{\mu}_0 = \dfrac{N_0}{\mu(Q_0)}\mu_0$, if $\mu(Q_0) \neq 0$, and

$$\widetilde{\mu}_{n,j} = \frac{N_{n,j}}{\mu(Q_{n,j})}\mu_{n,j},\ j = 1, \dots, 8,\ n \in \mathbb{N},$$

if $\mu(Q_{n,j}) \neq 0$. If $\mu(Q_0) = 0$, then $\widetilde{\mu}_0 = 0$. And if $\mu(Q_{n,j}) = 0$, then $\widetilde{\mu}_{n,j} = 0$. Then $\widetilde{\mu}_{n,j}(\mathbb{C}) = N_{n,j}$ are non-negative integers. Let $\nu_0 = \mu_0 - \widetilde{\mu}_0$, $\nu_{n,j} = \mu_{n,j} - \widetilde{\mu}_{n,j}$; then

$$\nu_{n,j}(\mathbb{C}) \in (-1;1), \quad \Big(\sum_{j=1}^{8} \nu_{n,j}\Big)(\mathbb{C}) \in (-8;8). \tag{8}$$

Let

$$\nu = \nu_0 + \sum_{n=1}^{\infty}\sum_{j=1}^{8} \nu_{n,j},\ \nu^+ = \nu_0 + \sum_{q_n = q_n^+}\sum_{j=1}^{8} \nu_{n,j},\ \nu^- = -\sum_{q_n = q_n^-}\sum_{j=1}^{8} \nu_{n,j},$$

then $\nu^{\pm}$ are non-negative measures and $\nu = \nu^+ - \nu^-$. At that,

$$\nu^{\pm}\Big(\bigcup_{j=1}^{8} Q_{n,j}\Big) = q_n^{\pm} \in (-8;8).$$

Let us prove that

$$\pi(\lambda) := \int_{\mathbb{C}} \ln\Big|1 - \frac{\lambda}{w}\Big| d\nu(w) \leqslant 45\ln(|\lambda| + e) + 142,\ \lambda \in \mathbb{C}. \tag{9}$$

Then Lemma 2 holds true for the function $\widetilde{u}(\lambda) = u(\lambda) - \pi(\lambda)$.

Choose $\lambda \in Q_{n+1} \setminus Q_n$. If $w \in Q_{m+1} \setminus Q_m$, then $\frac{3^m}{2} \leqslant |w| \leqslant \frac{1}{\sqrt{2}} 3^{m+1}$ and $|\ln(|\zeta|+1)| \leqslant 2|\zeta|$ for $|\zeta| \leqslant \frac{1}{2}$; therefore

$$\Bigl| \int\limits_{\mathbb{C}\setminus Q_{n+2}} \ln\Bigl|1-\frac{\lambda}{w}\Bigr| d\nu(w)\Bigr| \leqslant \sum_{m=2}^{\infty} \Bigl| \int\limits_{Q_{n+m+1}\setminus Q_{n+m}} \ln\Bigl|1-\frac{\lambda}{w}\Bigr| d\nu(w)\Bigr| \leqslant$$

$$\leqslant \frac{32|\lambda|}{3^n} \sum_{m=2}^{\infty} \frac{1}{3^m} \leqslant 8\sqrt{2}. \tag{10}$$

Similarly, we have

$$\Bigl| \int\limits_{Q_{n-1}} \ln\Bigl|1-\frac{w}{\lambda}\Bigr| \, d\nu(w)\Bigr| \leqslant \frac{2}{|\lambda|}\Bigl(8\sum_{m=1}^{n-1} \frac{3^{n-m}}{\sqrt{2}} + \frac{\sqrt{2}}{2}\Bigr) \leqslant 10\sqrt{2}. \tag{11}$$

Let us prove that

$$|\nu(t)| = |\nu(B(0,t))| \leqslant 17,\ t \geqslant 0. \tag{12}$$

Indeed, if $t < \frac{3}{\sqrt{2}}$, then $B(0,t) \subset Q_2$ therefore

$$|\nu(t)| \leqslant |\nu(Q_1)| + \sum_{j=1}^{8} |\nu_{1,j}(\mathbb{C})| \leqslant 9.$$

For $t \geqslant \dfrac{3}{\sqrt{2}}$ we denote the maximal natural number, for which $\dfrac{3^n}{\sqrt{2}} \leqslant t$ by n. Then $Q_n \subset B(0,t)$ and

$$\frac{3^{n+2}}{2} = \frac{3}{\sqrt{2}} \frac{3^{n+1}}{\sqrt{2}} \geqslant \frac{3}{\sqrt{2}} t > t.$$

Hence, $Q_{n+2} \supset B(0,t)$. Thus, in view of (8), we obtain

$$|\nu(t)| \leqslant |\nu(Q_n)| + \sum_{i=n}^{n+1} \Bigl| \sum_{j=1}^{8} \nu_{i,j}(\mathbb{C})\Bigr| \leqslant 17.$$

Let $\widetilde{\nu}_n$ be the restriction of the measure μ to the square Q_n. Then

$$\Bigl| \int\limits_{Q_{n-1}} \ln\frac{|\lambda|}{|w|} d\nu(w)\Bigr| \leqslant \ln\frac{|\lambda|}{3^n} |\nu(Q_{n-1})| + \Bigl| \int\limits_{0}^{3^n} \frac{\widetilde{\nu}_{n-1}(t)}{t} dt\Bigr|.$$

By condition (4) with $M = 1$, in view of (12), we have

$$\Big|\int\limits_0^{3^n} \frac{\widetilde{\nu}_{n-1}(t)}{t} dt\Big| \leqslant \int_0^1 \frac{\mu(t)}{t} dt + \int\limits_1^{3^n} \frac{17dt}{t} \leqslant 1 + 17\ln(|\lambda| + e).$$

Hence,

$$\Big|\int\limits_{Q_{n-1}} \ln \frac{|\lambda|}{|w|}\, d\nu(w)\Big| \leqslant 18 + 17\ln(|\lambda| + e),$$

and using (11), we obtain

$$\Big|\int\limits_{Q_{n-1}} \ln \Big|1 - \frac{\lambda}{w}\Big|\, d\nu(w)\Big| \leqslant 17\ln(|\lambda| + e) + 33. \tag{13}$$

If $w \in Q_{n+2} \setminus Q_{n-1}$, then $|w| \in [\frac{3^{n-1}}{2}; \frac{1}{\sqrt{2}}3^{n+2}]$ and $|\lambda| \in [\frac{3^n}{2}; \frac{1}{\sqrt{2}}3^{n+1}]$; Therefore, for $w \notin B(\lambda,1)$ the estimate

$$\frac{\sqrt{2}}{9} \cdot 3^{-n} \leqslant \Big|\frac{\lambda - w}{w}\Big| \leqslant 51,$$

holds true. Hence, $\big|\ln|1 - \frac{\lambda}{w}|\big| \leqslant \ln(|\lambda| + e) + 4$. Consequently,

$$\Big|\int\limits_{(Q_{n+2}\setminus Q_{n-1})\setminus B(\lambda,1)} \ln\Big|1 - \frac{\lambda}{w}\Big| d\nu(w)\Big| \leqslant 24\ln(|\lambda| + e) + 96. \tag{14}$$

It remains to estimate the integral over the disk $B(\lambda,1)$. The measure ν is a part of the measure μ by construction, so it satisfies (1) with $M = 1$. Taking this into account and integrating by parts, we obtain

$$\Big|\int\limits_{B(\lambda,1)} \ln|\lambda - w| d\nu(w)\Big| \leqslant 1, \quad \Big|\int\limits_{B(\lambda,1)} \ln|w| d\nu(w)\Big| \leqslant \pi\ln(|\lambda| + e).$$

Hence,

$$\int\limits_{B(\lambda,1)} \Big|\ln\Big|1 - \frac{\lambda}{w}\Big|\Big| d\nu(w) \leqslant \pi\ln(|\lambda| + e) + 1.$$

This, together with (10), (13) and (14), implies (9). $\square$

Lemma 3. *There exist measures μ_n, $\mu_n(\mathbb{C}) = 1$, and rectangles P_n, $n \in \mathbb{N}$, such that*

1) $\sum_n \mu_n = \widetilde{\mu}$;
2) *interiors of convex hulls of supports of the measures* μ_n *are pairwise disjoint;*
3) *support of the measure* μ_n *is located in* P_n, $n \in \mathbb{N}$;
4) *sides of the rectangles* P_n *are parallel to the axes and the ratio of length of the sides for rectangles* P_n *lies in the interval* $[3^{-1}; 3]$;
5) *each point of the plane belongs to at most four rectangles* P_n;
6) *if* F_n *is convex hull of support of the measure* μ_n, *then*

$$\textit{diam}\ F_n \leqslant 2\sqrt{2} \min_{\lambda \in F_n} |\lambda| + \sqrt{2}.$$

Proof. Let us apply theorem 1 from [8] to restrictions of the measure $\widetilde{\mu}$ on the squares $Q_{n,j}$. After renumbering, we obtain a set of unit measures satisfying properties $1-5$ of Lemma 3. Property 6 follows from the corresponding property of the squares $Q_{n,j}$. □

Let us continue the proof of Theorem $1'$. Centres of mass of the unit measures μ_n constructed in Lemma 3 are denoted by λ_n:

$$\int\limits_{\mathbb{C}} w\, d\mu_n(w) = \lambda_n,\ \ n \in \mathbb{N}.$$

By $\widetilde{\mu}_n$ we denote the restriction of the measure $\widetilde{\mu}$ to the square Q_n, and let π_n be the potential of this measure

$$\pi_n(\lambda) = \int_{\mathbb{C}} \ln \left|1 - \frac{\lambda}{w}\right| d\widetilde{\mu}_n(\lambda).$$

Then the measure $\widetilde{\mu}_n$ satisfies the conditions of theorem 3 from [8]. In the terminology of this paper, each point $\lambda \in \mathbb{C}$ for each $s = s(\lambda) \in (0; 1]$ is (π, s)-normal with respect to the measure $\widetilde{\mu}$, by (4). Hence, this theorem shows that for the polynomial

$$P_n(\lambda) = \prod_{\lambda_k \in Q_n} \left(1 - \frac{\lambda}{\lambda_k}\right)$$

the relation $|\pi_n(\lambda) - \ln|P_n(\lambda)|| \leqslant A\ln(|\lambda| + 1) + B\ln(s(\lambda) + 1) + C$ holds true outside the set of discs $B_k(s) = B(\lambda_k, s(\lambda_k))$, $\lambda_k \in Q_n$. Here constants A, B, C are independent on μ and n. Thanks to the latter fact, we can justify the passage to the limit in the usual way. As a result, we see that there exists an entire function f with simple zeros at the points λ_n satisfying the condition

$$|\widetilde{u}(\lambda)-\ln|f(\lambda)||\leqslant A\ln(|\lambda|+1)+B\ln(s(\lambda)+1)+C,\ \lambda\notin\bigcup_n B_n(s). \quad (15)$$

We need to show that for sufficiently small $\delta>0$, the discs $B_n=B_n(\lambda_n, \delta(|\lambda_n|+1)^{-1})$ are pairwise disjoint. Let us estimate the distance d_n from the point λ_n to the boundary of convex hull F_n of support of the measure μ_n. Let w_n be a point where this distance is attained:

$$|\lambda_n-w_n|=\inf\{|\lambda_n-w|,\ w\notin F_n\}.$$

Let $w_n-\lambda_n=e^{i\varphi_n}|\lambda_n-w_n|$ and $z=Tw=e^{-\varphi_n}(\lambda_n-w)$. Under such transformation, the image F^* of the hull F_n is located in the half-plane $\{\mathrm{Re}\ z\leqslant d_n\}$ and the image of the measure $d\mu^*(z)=d\mu_n(\lambda_n-e^{i\varphi_n}z)$ satisfies the conditions

$$\int\limits_{\mathbb{C}} d\mu^*(z)=1,\quad \int\limits_{\mathbb{C}} zd\mu^*(z)=0,$$

$$d\mu^*(z)=\frac{1}{\pi}\chi_n(z)\Delta\widetilde{u}(\lambda_n-e^{i\varphi_n}z)dm(z),$$

where $\chi_n(z)$ is the characteristic function of the set F^*. Let

$$\delta(x)=\frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\chi_n(x+iy)\Delta\widetilde{u}(\lambda_n-e^{i\varphi_n}(x+iy))dy.$$

Then $\delta(x)$ is a compactly supported function on the segment $[a;d_n]$ and by statements 6 in Lemma 3, $0\leqslant\delta(x)\leqslant 3\pi(|\lambda_n|+1):=M_n$. Moreover, it follows from the properties of μ^* that

$$\int\limits_a^{d_n}\delta(x)dx=1,\quad \int\limits_a^{d_n}x\delta(x)dx=0.$$

The next lemma is proved in [2] (see [2, Proposition 2]).

Lemma 4. [2] *Let $\delta(x)$ be a non-negative continuous compactly supported function satisfying the conditions*

$$1)\ \operatorname{conv}\operatorname{supp}\delta=[a;d],\quad 2)\sup_x\delta(x)\leqslant M_0<\infty,$$

$$3)\int\limits_a^d\delta(x)\,dx=1,\qquad 4)\int\limits_a^d x\delta(x)\,dx=0.$$

Then $d\geqslant\dfrac{1}{6M_0}$.

Lemma 4 implies that $d_n\geqslant\frac{1}{18\pi}(1+|\lambda_n|)^{-1}$, $n\in\mathbb{N}$. By the property 2 in Lemma 3, the disks $B_n=B(\lambda_n,\delta(1+|\lambda_n|)^{-1})$, $\delta<\frac{1}{18\pi}$, are pairwise disjoint. In particular, each point λ outside these disks is $(\pi,(1+|\lambda_n|)^{-1})$-normal with respect to the measures $\widetilde{\mu}$ and $\nu=\sum_k\delta(\lambda_k)$, here $\delta(w)$ is a unit mass concentrated at the point w. By relation (15), the estimate $|\widetilde{u}(\lambda)-\ln|f(\lambda)||\leqslant A\ln(|\lambda|+1)+C$, $\lambda\notin\bigcup_n B_n$, holds true outside the disks B_n. By usual tricks and by Cauchy's formula $\frac{1}{f'(\lambda_n)}=\frac{1}{2\pi i}\int_{\partial B_n}\frac{dz}{f(z)}$, one can obtain the necessary estimates for the derivatives at the points λ_n.

3. Proof of Theorem 2.

By Theorem 1, for each of the functions u_j there exists an entire function f_j satisfying the estimates

$$|\ln|f_j(\lambda)|-u_j(\lambda)|\leqslant A\ln(|\lambda|+1)+C,\ \lambda\notin\bigcup_{\lambda\in N(f_j)}B(\lambda,\delta(|\lambda|+1)^{-1}),\quad(16)$$

$$|\ln|f_j'(\lambda)|-u_j(\lambda)|\leqslant A\ln(|\lambda|+1)+C',\ \lambda\in N(f_j).$$

At that, the disks $B_\delta(\lambda)=B(\lambda,\delta(|\lambda|+1)^{-1})$, $\lambda\in N(f_j)$, are pairwise disjoint for each of $j=1,2$ separately. Let ν_2 be the Riesz measure of $\ln|f_2|$. Let us formulate the properties of the measures μ_2 and ν_2 in the next lemma.

Lemma 5.

$$1)\ \mu_2(t)=o(t),\quad \nu_2(t)=o(t),\ t\longrightarrow\infty,\qquad(17)$$

$$2)\ \textit{if } \lambda\notin\bigcup_{z\in N(f_2)}B_\delta(z),\ \textit{then}\ \int\limits_0^{\frac12}\frac{\nu_2(\lambda,\tau)d\tau}{\tau}\leqslant 2A\ln(1+|\lambda|)+C''.$$

Proof. The first assertion of the lemma is an immediate consequence of condition (3). To prove the second assertion, we use Jensen's formula for the function $v(z) = u_2(z) - \ln|f_2(z)|$ with the Riesz measure $\nu = \mu_2 - \nu_2$:

$$\frac{1}{2\pi}\int_0^{2\pi} v(\lambda + re^{i\varphi})d\varphi = v(\lambda) + \int_0^r \frac{\nu(\lambda,\tau)d\tau}{\tau}.$$

By (17), there exists $R > 2$ such that $\mu_2(t) \leqslant \frac{t}{4}$ for every $t \geqslant R$. We take $|\lambda| \geqslant R$ and we let $r(w) = \delta(1 + |w|)^{-1}$. The disk $B(w_k, r(w_k))$ can intersect with the circles $C(\lambda, r)$, $r \in [\frac{1}{2}; 1)$, only if the center w_k lies in $B(\lambda, 1 + \delta)$. For these disks $r(w_k) \leqslant 2r(\lambda)$, therefore,

$$\sum_{w_k \in B(\lambda, 1+\delta)} r(w_k) \leqslant 2r(\lambda)\mu_2(1 + \delta + |\lambda|) < \frac{1}{2}.$$

Hence, there exists a number $r \in [\frac{1}{2}; 1)$ such that the circle $C(\lambda, r)$ does not intersect with the exceptional set. By construction, the inequality $|v(z)| = |u_2(z) - \ln|f_2(z)|| \leqslant A\ln(1 + |z|) + C$ holds true on this circle. The same inequality holds true at the point λ. By Jensen's formula, we obtain

$$\Big|\int_0^r \frac{\mu_2(t) - \nu_2(t)}{t}\,dt\Big| \leqslant 2A\ln(1 + |z|) + C_1.$$

This, together with (2), implies the second assertion of Lemma (5). □

Next we prove a lemma on exceptional sets.

Lemma 6. *If estimate (5) holds true for u_2 and δ, then estimate (5) holds true for u_2 and each $\delta' \in (0; \delta)$ outside the discs $B(\lambda, \delta'(|\lambda| + 1)^{-1})$, $\lambda \in N(f_2)$, possibly with another constant C depending on δ'.*

Proof. In fact, let us take an arbitrary $\lambda \notin E' := \bigcup_{z \in N(f_2)} B(z, \delta'(|z| + 1)^{-1})$. If, in addition, $\lambda \notin E := \bigcup_{z \in N(f_2)} B(z, \delta(|z| + 1)^{-1})$, then the estimates hold true by the hypothesis of the lemma. If $\lambda \in E$, then there exists a number n such that $r' := \delta'(1 + |\lambda_n|)^{-1} \leqslant |\lambda - \lambda_n| < r := \delta(1 + |\lambda_n|)^{-1}$. Let $G(\lambda, w)$ be Green's function for the disk $B(\lambda_n, r)$, then there are representations

$$u_2(\lambda) = h(\lambda) - \int_{B(\lambda_n, r)} G(\lambda, w)d\mu_2(w), \ \ln|f_2(\lambda)| = h_0(\lambda) - G(\lambda, \lambda_n),$$

where the functions h, h_0 are the harmonic majorants of u_2 and $\ln|f_2|$ respectively on the disk $B(\lambda_n, r)$. The difference $|h(\lambda) - h_0(\lambda)|$ is estimated by the maximum principle for harmonic functions. For some constant C_1

$$|h_0(\lambda) - h(\lambda)| \leqslant A\ln(1+|\lambda|) + C_1.$$

We estimate the potential of the measure μ_2 using (2). Let h_1 be the harmonic majorant of u_2 on the disk $B(\lambda, 2r)$. Then for $\delta < \frac{1}{2}$

$$\int\limits_{B(\lambda_n, r)} G(\lambda, w)\, d\mu_2(w) = h(\lambda) - u_2(\lambda) \leqslant$$

$$\leqslant h_1(\lambda) - u_2(\lambda) = \int\limits_0^{2r} \frac{\mu_2(\lambda, t)\, dt}{t} \leqslant 2Mr \leqslant M.$$

And finally $G(\lambda, \lambda_n) = \ln \frac{r}{|\lambda_n - \lambda|} \leqslant \ln \frac{r}{r'} = \ln\left(\frac{\delta}{\delta'}\right), \lambda \notin B(\lambda_n, r')$. Thus, $|u_2(\lambda) - \ln|f_2(\lambda)|| \leqslant A\ln(1+|\lambda|) + C_1 + M + \ln\left(\frac{\delta}{\delta'}\right)$ for $\lambda \notin B(\lambda_n, r')$. $\Box$

Let $z \in N(f_1)$. Then there are no other zeros of the function f_1 in the disk $B_{\frac{\delta}{2}}(z)$ and there can be only one zero of the function f_2 in this disk. If $w \in N(f_2) \bigcap B_{\frac{\delta}{2}}(z)$, then we move this point to the nearest point w' on the boundary of the disk $B_{\frac{\delta}{4}}(z)$. Let $N(f_2) = \{w_k\}_{k=1}^{\infty}$. We denote by $\widetilde{w}_k$ the point w_k if w_k does not belong to the union of the disks $B_{\frac{\delta}{2}}(\lambda)$, $\lambda \in N(f_1)$, and the point w'_k if the point w_k belongs to one of these disks. Also, we denote by $\widetilde{N}$ the set of points $\widetilde{w}_k$. Obviously, the disks $B_{\frac{\delta}{8}}(\lambda)$, $\lambda \in N(f_1) \bigcup \widetilde{N}$, are pairwise disjoint.

Lemma 7. *Let*

$$\pi_2(\lambda) = \sum_{k=1}^{\infty} \ln\left|1 - \frac{\lambda}{w_k}\right|, \ \widetilde{\pi}(\lambda) = \sum_{k=1}^{\infty} \ln\left|1 - \frac{\lambda}{\widetilde{w}_k}\right|, \ \lambda \in \mathbb{C}. \qquad (18)$$

Then outside the set of pairwise disjoint disks $E := \bigcup_{w \in N(f_2) \bigcup \widetilde{N}} B_{\frac{\delta}{8}}(w)$ *for each* $\varepsilon > 0$ *and for some constant* C *the estimate*

$$|\pi_2(\lambda) - \widetilde{\pi}(\lambda)| \leqslant \varepsilon \ln(1+|\lambda|) + C$$

holds true.

Proof. The convergence of the series in (18) follows from (3). Since $\sup_{\lambda\notin E}\sum_{k\leqslant n}\left|\ln\left|1-\frac{\lambda}{w_k}\right|-\ln\left|1-\frac{\lambda}{\widetilde{w}_k}\right|\right|\leqslant C(n)$, for each $n\in\mathbb{N}$, then without loss of generality one can consider only w_k with sufficiently large absolute value. Thus, let us assume that $|w_k|\geqslant 25$. Since for $|z-w|\leqslant 1$ and $|z|\geqslant 25$

$$\frac{23}{24}\leqslant\frac{1+|z|}{1+|w|}\leqslant\frac{25}{24}, \tag{19}$$

then we can assume that $|w_k-\widetilde{w}_k|\leqslant\frac{\delta}{3}(1+|w_k|)^{-1}$ and, so, $\left|\frac{w_k-\widetilde{w}_k}{w_k}\right|\leqslant\frac{1}{2}$. Then, taking into account the simple inequality $|\ln|1-\zeta||\leqslant 2|\zeta|$ for $|\zeta|\leqslant\frac{1}{2}$, we have

$$\left|\ln\left|1-\frac{w_k-\widetilde{w}_k}{w_k}\right|\right|\leqslant\frac{2|w_k-\widetilde{w}_k|}{|w_k|}\leqslant\frac{\delta}{|w_k|(1+|w_k|)}.$$

By (17), we obtain

$$\sum_{k=1}^{\infty}\left|\ln\left|\frac{\widetilde{w}_k}{w_k}\right|\right|=\sum_{k=1}^{\infty}\left|\ln\left|1-\frac{\widetilde{w}_k-w_k}{w_k}\right|\right|\leqslant\sum_{k=1}^{\infty}\frac{\delta}{|w_k|(1+|w_k|)}=$$

$$=\int_1^{\infty}\frac{\delta\,d\nu_2(w)}{|w|(1+|w|)}\leqslant\int_1^{\infty}\frac{\delta\,d\nu_2(t)}{t^2}=2\delta\int_1^{\infty}\frac{\nu_2(t)dt}{t^3}:=C_1. \tag{20}$$

Fix the point λ. Let $I_1(\lambda)$ be the set of indexes k such that $|w_k|\geqslant 2|\lambda|$, let $I_2(\lambda)$ be the set of indexes k such that $|w_k|\leqslant\frac{|\lambda|}{2}$, and let $I_3(\lambda)$ be the set of indexes k such that $\frac{|\lambda|}{2}<|w_k|<2|\lambda|$. We denote by $J_1(\lambda)$ the set of indexes $k\in I_3(\lambda)$ such that $|\lambda-w_k|\geqslant\frac{1}{2}$. Let for some $p>1$ $J_2(\lambda)$ be the set of indexes $k\in I_3(\lambda)$ such that $\frac{1}{2}>|\lambda-w_k|\geqslant p\delta(1+|\lambda|)^{-1}$, and let $J_3(\lambda)$ be all other indexes $k\in I_3(\lambda)$.

Let $k\in I_1(\lambda)$, then $|\lambda|\leqslant|w_k|/2$ and $|\lambda-w_k|\geqslant|w_k|/2$; therefore $\left|\frac{w_k-\widetilde{w}_k}{\lambda-w_k}\right|\leqslant\frac{1}{2}$. Hence,

$$\left|\ln\left|\frac{\lambda-\widetilde{w}_k}{\lambda-w_k}\right|\right|=\left|\ln\left|1-\frac{\widetilde{w}_k-w_k}{\lambda-w_k}\right|\right|\leqslant\frac{2\delta}{|w_k|(1+|w_k|)},$$

and

$$\left|\sum_{k\in I_1}\ln\left|\frac{\lambda-\widetilde{w}_k}{\lambda-w_k}\right|\right|\leqslant\int_{2|\lambda|}^{\infty}\frac{2\delta d\nu_2(w)}{|w|(1+|w|)}\leqslant\int_{2|\lambda|}^{\infty}\frac{2\delta d\nu_2(t)}{t(1+t)}\leqslant C_2. \tag{21}$$

Let $k \in I_2(\lambda)$, then $|\lambda - w_k| \geqslant |\lambda|/2$; therefore $\left|\frac{w_k - \widetilde{w}_k}{\lambda - w_k}\right| \leqslant \frac{1}{2}$ for $|\lambda| \geqslant 1$. Hence,

$$\Big|\ln\Big|\frac{\lambda - \widetilde{w}_k}{\lambda - w_k}\Big|\Big| = \Big|\ln\Big|1 - \frac{\widetilde{w}_k - w_k}{\lambda - w_k}\Big|\Big| \leqslant \frac{2\delta}{|\lambda|(1 + |w_k|)},$$

and

$$\left|\sum_{k \in I_2} \ln\left|\frac{\lambda - \widetilde{w}_k}{\lambda - w_k}\right|\right| \leqslant \int_1^{|\lambda|/2} \frac{2\delta d\nu_2(w)}{|\lambda|(1 + |w|)} \leqslant \int_1^{|\lambda|/2} \frac{2\delta d\nu_2(t)}{|\lambda|(1 + t)} \leqslant C_3. \tag{22}$$

Let $k \in J_1(\lambda)$. Then

$$\left|\frac{w_k - \widetilde{w}_k}{\lambda - w_k}\right| \leqslant \frac{2\delta}{3(1 + |w_k|)} \leqslant \frac{1}{2},$$

and by (17), we have

$$\left|\sum_{k \in J_1} \ln\left|\frac{\lambda - \widetilde{w}_k}{\lambda - w_k}\right|\right| \leqslant \int_{|\lambda|/2}^{2|\lambda|} \frac{4\delta d\nu_2(t)}{3(1 + t)} \leqslant \int_{|\lambda|/2}^{2|\lambda|} \frac{8\delta d\nu_2(t)}{3(|\lambda| + 2)} \leqslant C_4. \tag{23}$$

It remains to estimate the difference of potentials for k, such that $|\lambda - w_k| \leqslant \frac{1}{2}$. For $k \in J_2(\lambda)$, due to the choice of p, we still have

$$\left|\frac{w_k - \widetilde{w}_k}{\lambda - w_k}\right| \leqslant \frac{1}{2}.$$

Therefore, by (19), for $r = p\delta(1 + |\lambda|)^{-1}$ we have

$$\left|\sum_{k \in J_2} \ln\left|\frac{\lambda - \widetilde{w}_k}{\lambda - w_k}\right|\right| \leqslant \frac{2\delta}{3} \sum_{k \in J_2} \left|\frac{1}{(\lambda - w_k)(1 + |w_k|)}\right| \leqslant \frac{\delta}{1 + |\lambda|} \int_r^{\frac{1}{2}} \frac{d\nu_2(\lambda, t)}{t}.$$

Hence,

$$\Big|\sum_{k \in J_2} \ln\Big|\frac{\lambda - \widetilde{w}_k}{\lambda - w_k}\Big|\Big| \leqslant \frac{2\delta\nu_2(\lambda, \frac{1}{2})}{1 + |\lambda|} + \frac{\delta}{1 + |\lambda|} \int_r^{\frac{1}{2}} \frac{\nu_2(\lambda, t)dt}{t^2}. \tag{24}$$

By assertion 2 of Lemma 5, we have $\nu_2(\lambda,t) \leqslant 2A\ln(1+|\lambda|)+C$, hence, $\dfrac{\delta\nu_2(\lambda,\frac{1}{2})}{1+|\lambda|} \leqslant C_5$ and

$$\frac{\delta}{1+|\lambda|}\int\limits_r^{\frac{1}{2}}\frac{\nu_2(\lambda,t)dt}{t^2} \leqslant \frac{\delta}{(1+|\lambda|)r}\int\limits_r^{\frac{1}{2}}\frac{\nu_2(\lambda,t)dt}{t} \leqslant \frac{2A}{p}\ln(1+|\lambda|)+C_6.$$

By the last two estimates and by (8), we obtain

$$\left|\sum_{k\in J_2}\ln\left|\frac{\lambda-\widetilde{w}_k}{\lambda-w_k}\right|\right| \leqslant \frac{2A}{p}\ln(1+|\lambda|)+C_7. \tag{25}$$

The number of indexes $k\in J_3(\lambda)$ is finite and bounded by some absolute constant N. Thus, $\left|\ln\left|\frac{\lambda-\widetilde{w}_k}{\lambda-w_k}\right|\right| \leqslant \text{Const}$, $\lambda\notin E$, $k\in J_3(\lambda)$. Hence,

$$\left|\sum_{k\in J_3}\ln\left|\frac{\lambda-\widetilde{w}_k}{\lambda-w_k}\right|\right| \leqslant C_8. \tag{26}$$

Since

$$|\pi_2(\lambda)-\widetilde{\pi}(\lambda)| \leqslant \sum_k\left|\ln\left|\frac{w_k}{\widetilde{w}_k}\right|\right| + \sum_k\left|\ln\left|1-\frac{\lambda-\widetilde{w}_k}{\lambda-w_k}\right|\right|,$$

then by (20)–(7), (5)–(9), the assertion of Lemma 7 follows. $\square$

Let us now complete the proof of Theorem 2. By condition (3), we have the representation $f_2(\lambda)=e^{g(\lambda)}\prod_k\left(1-\frac{\lambda}{w_k}\right)$, $\lambda\in\mathbb{C}$, where g is an entire function. Let $\widetilde{f}(\lambda)=e^{g(\lambda)}\prod_k\left(1-\dfrac{\lambda}{\widetilde{w}_k}\right)$, $\lambda\in\mathbb{C}$. Then by Lemma 7 for each $\varepsilon>0$, we have

$$\begin{aligned}|\ln|f_2(\lambda)|-\ln|\widetilde{f}(\lambda)|| = |\pi_2(\lambda)-\widetilde{\pi}(\lambda)| \leqslant \\ \leqslant \varepsilon\ln(1+|\lambda|)+C,\ \lambda\notin E=\bigcup_{w\in N(f_2)\bigcup\widetilde{N}} B_{\frac{\delta}{8}}(w).\end{aligned}$$

By Lemma 6, outside the set E, we have an estimate with some constant C and an arbitrary $\varepsilon>0$

$$|u_2(\lambda) - \ln|\widetilde{f}(\lambda)|| \leqslant |u_2(\lambda) - \ln|f_2(\lambda)|| + |\pi_2(\lambda) - \widetilde{\pi}(\lambda)| \leqslant$$
$$\leqslant (A+\varepsilon)\ln(1+|\lambda|) + C,\ \lambda \notin E = \bigcup_{w \in N(f_2) \bigcup \widetilde{N}} B_{\frac{\delta}{8}}(w). \quad (27)$$

We extend this estimate to the union of sets $\bigcup_k (B_{\frac{\delta}{8}}(w_k) \setminus B_{\frac{\delta}{8}}(\widetilde{w}_k))$. The union of $B_{\frac{\delta}{8}}(w_k) \bigcup B_{\frac{\delta}{8}}(\widetilde{w}_k)$ is a subset of the disk $B_{\frac{11\delta}{24}}(\widetilde{w}_k) \subset B_{\frac{14\delta}{24}}(w_k) \subset$ $\subset B_\delta(w_k)$. Thus, the estimate (27) is satisfied outside the pairwise disjoint discs $B_{\frac{11\delta}{24}}(\widetilde{w}_k)$. By Lemma 6, it is also satisfied outside the discs $B_{\frac{\delta}{8}}(\widetilde{w}_k)$.

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