ON THE CONVERGENCE OF MAPPINGS WITH $k$-FINITE DISTORTION.

Abstract. We prove that a locally uniform limit of a sequence of homeomorphisms with finite $k$-distortion is also a mapping with finite $k$-distortion. We obtain also an estimation for the distortion coefficient of the limit mapping.

Key words: mapping with $k$-finite distortion, distortion coefficient, passing to the limit, differential form.

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1. Introduction. It is well known that the limit of a uniformly converging sequence of analytic functions is an analytic function. Reshetnyak generalized this result to mappings with bounded distortion: the limit of a locally uniformly converging sequence of mappings with bounded distortion is a mapping with bounded distortion.

Definition 1. [8] A mapping $f : \Omega \rightarrow \mathbb{R}^n$ is called mapping with bounded distortion if $f$ is continuous, $f \in W^1_{n,loc}(\Omega)$, the Jacobian $J(x, f)$ does not change the sign in the domain $\Omega$ and

$$|Df(x)|^n \leq K|J(x,f)| \quad \text{for almost all } x \in \Omega. \quad (1)$$

The smallest constant in this inequality is called the distortion coefficient of the mapping $f$ and is denoted by the symbol $K(f)$. It is clear that

$$K(f) = \sup \left\{ \frac{|Df(x)|^n}{|J(x,f)|} : x \in \Omega, \quad J(x,f) \neq 0 \right\}.$$ 

Reshetnyak used the weak convergence of Jacobians to prove the following theorem on the limit of a sequence of mappings with bounded distortion.

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Theorem 1. [8] Let \( f_m : \Omega \to \mathbb{R}^n, m = 1, 2, \ldots, \) be an arbitrary sequence of mappings with bounded distortion, locally converging in \( L_n(\Omega) \) to a mapping \( f_0 : \Omega \to \mathbb{R}^n \). Assume that the sequence of distortion coefficients \( K(f_m), m = 1, 2, \ldots, \) is bounded. Then the limit mapping \( f_0 \) is a mapping with bounded distortion and the following inequality holds:

\[
K(f_0) \leq \lim_{k \to \infty} K(f_m).
\]

We briefly outline the proof in the case of non-negative Jacobians. For a test function \( \varphi \in C_0^\infty(\Omega) \) we have

\[
\int_{\Omega} |Df_0(x)|^n \varphi(x) \, dx \leq \lim_{m \to \infty} \int_{\Omega} |Df_m(x)|^n \varphi(x) \, dx \leq K \lim_{m \to \infty} \int_{\Omega} J(x,f_m) \varphi(x) \, dx = K \int_{\Omega} J(x,f_0) \varphi(x) \, dx.
\]

To justify the limit in the last equality, we apply the weak convergence of Jacobians. Consequently, for the limit mapping, the point-wise inequality \( |Df_0(x)|^n \leq KJ(x,f_0) \) holds a.e. in \( \Omega \).

More recently, research has begun on mappings with finite distortion. They are a natural generalization of mappings with bounded distortion.

Definition 2. [7] Let a mapping \( f : \Omega \to \mathbb{R}^n \) belong to the Sobolev class \( W_{n,\text{loc}}^1(\Omega) \) and \( J(x,f) \geq 0 \). We define the pointwise distortion coefficient \( K(x,f) \) of the mapping \( f \) as a value

\[
K(x,f) = \begin{cases} 
\frac{|Df(x)|^n}{|J(x,f)|} & \text{if } J(x,f) > 0, \\
1 & \text{otherwise}.
\end{cases}
\]

The mapping \( f : \Omega \to \mathbb{R}^n \) is called mapping with finite distortion \((f \in FD(\Omega))\) if

\[
|Df(x)|^n \leq K(x,f)J(x,f) \quad \text{where } 1 \leq K(x,f) < \infty \text{ for almost all } x \in \Omega.
\]

Remark. In other words, the condition of finite distortion is that the partial derivatives of the mapping \( f \in W_{n,\text{loc}}^1(\Omega) \) vanish a.e. on the set of zeros of the Jacobian \( J(x,f) \).
For the first time, the essential properties of mappings with finite distortion were investigated in the paper [15] in the study of homeomorphisms inducing a bounded composition operator. The name "mapping with finite distortion" was proposed much later in the paper [7].

In [5] F. Gehring and T. Iwaniec showed that the limit of a weakly converging sequence of mappings with finite distortion is also a mapping with finite distortion, and they obtained also an estimation for the distortion coefficient of the limit mapping.

**Theorem 2.** [5] Let $f_m : \Omega \to \mathbb{R}^n$, $m = 1, 2, \ldots$, be an arbitrary sequence of mappings with finite distortion converging weakly in $W^{1,n}_{\text{loc}}(\Omega)$ to a mapping $f_0 : \Omega \to \mathbb{R}^n$. Assume that
\[
K(x, f_m) \leq M(x) < \infty, \quad m = 1, 2, \ldots,
\]
where $\Omega \ni x \mapsto M(x) \in [1, \infty]$ is a measurable function. Then the limit mapping $f_0$ is a mapping with finite distortion and the inequality
\[
K(x, f_0) \leq M(x)
\]
holds.

More precisely, in this paper existence of a subsequence $f_{m_k}$ such that
\[
K(x, f_0) \leq b \ast \lim_{k \to \infty} K(x, f_{m_k})
\]
was shown. Here the limit is understood in the sense of so-called biting convergence.

**Definition 3.** [2] Let $h$ and $h_k$, $k \in \mathbb{N}$, be Lebesgue measurable functions defined on a set $E \subset \mathbb{R}^n$. The sequence $h_k$ is said to converge in the biting sense on $E$ to $h$ if there exists an increasing sequence $E_\nu$ of measurable subsets of $E$,
\[
\bigcup_\nu E_\nu = E,
\]
such that
\[
\lim_{k \to \infty} \int_{E_\nu} \varphi h_k \, dx = \int_{E_\nu} \varphi h \, dx
\]
for any function $\varphi \in L_\infty(E_\nu)$. 

In the paper [4] the limit of a sequence of homeomorphisms with finite distortion converging weakly in $W^1_1$ was shown to have, also, a finite distortion under the condition that the limit mapping is a homeomorphism.

**Theorem 3.** [4] Let $\Omega, \Omega'$ be bounded domains in $\mathbb{R}^n$, $f_j : \Omega \to \Omega'$, $j \in \mathbb{N}$, $f : \Omega \to \Omega'$, be homeomorphisms belonging to $W^1_1(\Omega)$, and $f_j \to f$ weakly in $W^1_1(\Omega)$. Assume that $|Df_j(x)|^n \leq K(x,f_j)J(x,f_j)$ for almost all $x \in \Omega$, where $K(x,f_j) : \Omega \to [1,\infty)$ are Borel functions for all $j$, and the sequence $K(x,f_j)$ converges in the biting sense to $K(x)$ as $j \to \infty$. Then the limit mapping $f$ is a mapping with finite distortion and the inequality $K(x,f) \leq K(x)$ holds for almost all $x \in \Omega$.

In the paper [1] mappings with bounded $(q,p)$-distortion ($n - 1 < q \leq p < \infty$) were defined and investigated; these mappings coincide with the class of mappings with bounded distortion if $q = p = n$.

In the paper [17] a locally uniform limit of a sequence of mappings with bounded $(\theta,1)$-weighted $(q,p)$-distortion was shown to be, also, a mapping with a bounded $(\theta,1)$-weighted $(q,p)$-distortion and an estimation similar to (2) was established. The proofs of the theorems like those in the article [17] and in this work are based on the method developed in [11] for extending Reshetnyak’s result to Carnot groups.

In this paper we extend the above-mentioned assertions to the class of mappings with $k$-finite distortion, which arise naturally in the problem of operating with differential forms of degree $k$ (see [12]).

**2. Preliminaries.** Let $U$ be a domain in $\mathbb{R}^n$. We consider the Banach space $\mathcal{L}_p(U,\Lambda^k)$ of differential forms $\omega$ of degree $k$, $k = 1, \ldots, n$, with measurable coefficients, which have the following finite norm: $\|\omega\|_p = (\int_U |\omega|^p \, dx)^{1/p}$.

A mapping $f : U \to \mathbb{R}^n$ is said to be approximate differentiable at a point $x \in U$ [3], if there exists a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\lim_{r \to 0} \frac{\left| \{ y \in B(x,r) : |f(y) - f(x) - L(y-x)| > \varepsilon \} \right|}{r^n} = 0$$

for any $\varepsilon > 0$. Here the symbol $| \cdot |$ denotes the Lebesgue measure. It is well known that the approximate differential is unique [3] if $x$ is a density point. In what follows, it is denoted by the symbol $ap \, Df(x)$. 
In our paper we consider the mappings belonging to some Sobolev class. Such mappings are unconditionally approximately differentiable.

Let \( \omega = \sum \omega_\beta \, dy^\beta \) be any \( k \)-form, \( k = 1, \ldots, n \), in \( W \) with continuous coefficients \( \omega_\beta : W \to \mathbb{R} \), where the summation is over all \( k \)-dimensional ordered multi-indices \( \beta = (\beta_1, \ldots, \beta_k) \), \( 1 \leq \beta_1 < \cdots < \beta_k \leq n \), and \( dy^\beta = dy_{\beta_1} \wedge dy_{\beta_2} \wedge \cdots \wedge dy_{\beta_k} \). Let a mapping \( f = (f_1, \ldots, f_n) : U \to W \) of Euclidean domains \( U, W \subset \mathbb{R}^n \) be approximately differentiable almost everywhere in \( U \). We write the pull-back of the \( k \)-form \( \omega \) in the following way:

\[
f^* \omega(x) = \sum_\beta \omega_\beta(f(x)) \, df_{\beta_1} \wedge df_{\beta_2} \wedge \cdots \wedge df_{\beta_k} = \sum_\alpha \sum_\beta \omega_\beta(f(x)) M_{\alpha \beta}(x) \, dx^\alpha.
\]

In other words, it is a \( k \)-form with measurable coefficients, which are defined for almost all \( x \in U \) (here \( df_{\beta_k} = \sum_{i=1}^n \frac{\partial f_{\beta_k}}{\partial x_i} \, dx_i \) and the partial derivatives are understood in the approximate sense, \( M_{\alpha \beta}(x) \) are \((k \times k)\)-minors of the matrix \( ap Df(x) = (\frac{\partial f_j}{\partial x_i}) \), \( i, j = 1, \ldots, n \), with ordered lines and columns).

We recall that the approximate differential \( ap Df(x) : T_x U \to T_{f(x)} W \) is defined a.e. in \( U \). It generates canonically the linear mapping \( \Lambda_k f(x) : \Lambda_k T_x U \to \Lambda_k T_{f(x)} W \) of the spaces of \( k \)-vectors, and the pullback operation \( f^* \) of \( k \)-forms. We denote the norm of the last linear mapping by the symbol \( |\Lambda_k f(x)| \).

The minimal analytic and geometric properties of the mapping \( f \) were obtained in [12] for generating a bounded pullback operator

\[
f^* : \mathcal{L}_p(W, \Lambda^k) \to \mathcal{L}_q(U, \Lambda^k), \quad 1 \leq q \leq p \leq \infty,
\]

of differential forms of degree \( k = 1, \ldots, n \).

We say that an approximately differentiable mapping \( f : U \to W \) has \( k \)-finite distortion, \( 1 \leq k \leq n \), (shortly \( f \in CD^k(U; W) \)) if rank \( ap Df(x) < k \) almost everywhere on a set \( Z \). (Hereinafter \( Z = \{ x \in U : \det ap Df(x) = 0 \} \).) For \( k = 1 \) \( (k = n - 1) \) and \( f \in W^1_{1, \text{loc}}(U) \) \( (f \in W^1_{n-1, \text{loc}}(U)) \), this notion is well-known in literature: it is just the class of Sobolev mappings with finite distortion (codistortion), which is characterized by the property: \( ap Df(x) = 0 \) \( (\text{adj } Df(x) = 0) \) almost everywhere on \( Z \) (see [10], [14] for the second notion).

Besides of the property of \( k \)-finite distortion we consider mappings with a certain behavior of some characteristics of the distortion containing in
itself the ratio \( \frac{|\Lambda^k f(x)|^q}{|J(x,f)|} \), where \( J(x,f) = \det ap Df(x) \) \cite{12}: operator (4) is bounded if and only if the mapping \( f \in CD^k(U;W) \) and the distortion function \( W \ni y \mapsto H_{k,q}(y) = \)

\[
\left\{ \begin{array}{ll}
\left( \sum_{x \in f^{-1}(y) \setminus (\Sigma \cup Z)} \frac{|\Lambda^k f(x)|^q}{|J(x,f)|^{1/p}} \right)^{1/q} & \text{if } f^{-1}(y) \setminus (\Sigma \cup Z) \neq \emptyset, \\
0 & \text{otherwise},
\end{array} \right.
\]

belongs to \( L_\varkappa(W) \) where \( \frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p} \) if \( q < p \), and \( \varkappa = \infty \) \( (\varkappa = q) \) if \( q = p \) \( (p = \infty) \). Moreover, the norm of the operator \( f^* \) is comparable with the value \( \|H_{k,q}(\cdot) | L_\varkappa(\Omega)\| : \alpha_{q,p} \|H_{k,q}(\cdot) | L_\varkappa(\Omega)\| \leq \|f^*\| \leq \|H_{k,q}(\cdot) | L_\varkappa(\Omega)\| \)

where \( \alpha_{q,p} \) is some constant.

Hereinafter \( \Sigma \subset U \) is a set of measure zero outside of which the mapping \( f \) has the Luzin property \( \mathcal{N} \).

For homeomorphic mappings one can use a simpler characteristic.

**Corollary 1.** \cite{12} Let \( f : U \to W \) be an approximately differentiable homeomorphism. The operator \( f^* : L_p(W, \Lambda^k) \to L_q(U, \Lambda^k), 1 \leq q \leq p \leq \infty, \]

\[ k = 1, \ldots, n, \]

is bounded if and only if the following conditions are satisfied:

1) \( f : U \to W \) has the \( k \)-finite distortion;

2) the function \( K_{k,p}(x,f) = \)

\[ \left\{ \begin{array}{ll}
\frac{|\Lambda^k f(x)|}{|J(x,f)|^{1/p}} & \text{if } J(x,f) \neq 0, \\
0 & \text{otherwise},
\end{array} \right. \]

belongs to \( L_\varkappa(U) \), where \( \frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p} \) if \( q < p \), and \( \varkappa = \infty \) \( (\varkappa = q) \) if \( q = p \) \( (p = \infty) \).

In this case, the norm of the operator \( f^* \) is comparable with

\[ \|K_{k,p}(\cdot,f) | L_\varkappa(\Omega)\| : \alpha_{q,p} \|K_{k,p}(\cdot,f) | L_\varkappa(\Omega)\| \leq \|f^*\| \leq \|K_{k,p}(\cdot,f) | L_\varkappa(\Omega)\| , \]

where \( \alpha_{q,p} \) is some constant.

**3. Main results.**

**Definition 4.** \cite{12} An approximately differentiable homeomorphism \( f : U \to W \) belongs to the class \( CD^k_{q,p}(U;W) \) if the following conditions hold

1) \( f \in CD^k(U;W); \)
Theorem 4. Let \( f_m \in CD_{q,p}^k(U;W), m \in \mathbb{N}, \) be a sequence of homeomorphisms of the Sobolev class \( W^1_{l,\text{loc}}(U) \) with \( k < l, q \leq l/k, 1 < q \leq p < \infty. \) Suppose that the sequence \( f_m \) is locally bounded in \( W^1_l(U), \) and locally uniformly converges to a homeomorphism \( f: U \to W \) as \( m \to \infty. \) Assume also that there exists a sequence of functions \( U \ni x \mapsto M_m(x), \) belonging to \( L_\kappa(U), \) that is bounded in \( L_\kappa(U), \) for which the inequality

\[
K_{k,p}(x,f_m) \leq M_m(x) \text{ for almost all } x \in U
\]

is true. Then there exists a function \( U \ni x \mapsto M(x) \) of \( L_\kappa(U) \) such that some subsequence

(i) in the case \( 1 < q < p < \infty: \) of functions \( \{M_m(x)^{\kappa}\}_{m \in \mathbb{N}} \) converges in the biting sense to \( M(x)^\kappa; \)

(ii) in the case \( 1 < q = p < \infty: \) of numbers \( \{\|M_m| L_\infty(U)\|\}_{m \in \mathbb{N}} \) converges to

\[
M = \lim_{m \to \infty} \|M_m| L_\infty(U)\|
\]

the limit mapping \( f \) belongs to \( CD^k_{q,p}(U;W) \) and \( K_{k,p}(\cdot,f) \in L_\kappa(U), \)

where \( \frac{1}{\kappa} = \frac{1}{q} - \frac{1}{p}. \)

Moreover, the inequalities

\[
\begin{cases}
K_{k,p}(x,f) \leq M(x) & \text{in the case } q < p,
K_{k,p}(x,f) \leq M & \text{in the case } q = p,
\end{cases}
\]

hold for almost all \( x \in U. \)

In the proof we use some arguments from the paper [17], where Theorem 4 is proved for \( k = 1. \)

Proof. It follows from the conditions of the theorem that \( f \in W^1_{l,\text{loc}}(U). \) First, we show that the limit mapping \( f \) belongs to \( CD^k(U;W). \) For doing this, we show that the mapping \( f \) induces a bounded operator \( f^* : \mathcal{L}_p(W, \Lambda^k) \to \mathcal{L}_q(U, \Lambda^k), 1 < q \leq p < \infty. \) Since every mapping \( f_m \in CD_{q,p}^k(U;W), \) it follows from Corollary 1 that the homeomorphism \( f_m : U \to W \) induces the bounded operator \( f_m^* : \mathcal{L}_p(W, \Lambda^k) \to \mathcal{L}_q(U, \Lambda^k), 1 < q \leq p < \infty, m \in \mathbb{N}. \) Moreover, the norms of the operators \( f_m^* \) are totally bounded

\[
\|f_m^*\| \leq \|K_{k,p}(\cdot,f_m)| L_\kappa(U)\| \leq \|M_m(\cdot)| L_\kappa(U)\| \leq \tilde{M} < \infty.
\]
Take a $k$-form $\omega \in \mathcal{L}_p(W,\Lambda^k) \cap \mathcal{C}(W,\Lambda^k)$ and set $\sigma_m = f_m^*(\omega)$. Since $\|f_m^*\| \leq \tilde{M}$, the sequence of forms $\sigma_m$ is bounded in $\mathcal{L}_q(U,\Lambda^k)$. Therefore, we can extract a weakly converging subsequence. We assume that the sequence $\sigma_m$ converges weakly in $\mathcal{L}_q(U,\Lambda^k)$ to a form $\sigma_0$. The weak convergence of forms means that coefficients of the forms $\sigma_m$ converge weakly in $L_q(U)$ to the corresponding coefficients of the form $\sigma_0$. Since the sequence $\sigma_m$ converges weakly in $\mathcal{L}_q(U,\Lambda^k)$ to $\sigma_0$ as $m \to \infty$, we have

$$\|\sigma_0 \mid L_q(U)\| \leq \lim_{m \to \infty} \|\sigma_m \mid L_q(U)\| = \lim_{m \to \infty} \|f_m^* \omega \mid L_q(U)\| \leq \lim_{k \to \infty} \|f_m^*\| \cdot \|\omega \mid L_p(U)\| \leq \tilde{M} \cdot \|\omega \mid L_p(U)\|. \quad (6)$$

The following lemma is proved in the book [8, Chapter 2, §4].

**Lemma 1.** [8] Suppose $U$ is an open subset in $\mathbb{R}^n$, and suppose that $\varphi_m = (\varphi_{m1}, \varphi_{m2}, \ldots, \varphi_{mk})$, $1 \leq k \leq n$, $m = 1, 2, \ldots$, is a sequence of vector-functions of $W^1_{1,\text{loc}}(U)$, $k \leq l$, locally bounded in $W^1_l(U)$. Assume that, as $m \to \infty$, the functions $\varphi_m$ converge in $L_{1,\text{loc}}$ to a vector function $\varphi_0 = (\varphi_{01}, \varphi_{02}, \ldots, \varphi_{0k})$, and set $\omega_m = d\varphi_m \wedge d\varphi_{m2} \wedge \ldots \wedge d\varphi_{mk}$. Then the sequence of forms $\omega_m$ weakly converges in $L_{l/k,\text{loc}}(U)$ to a form $\omega_0$.

Since the homeomorphisms $f_m$ locally uniformly converge to $f$ and the form $\omega$ has continuous coefficients, the functions $\omega_\beta(f_m(x))$ converge locally uniformly to $\omega_\beta(f(x))$ as $m \to \infty$. Lemma 1 implies that the minors of the matrices $Df_m$ converge weakly in $L_{l/k,\text{loc}}(U)$ to minors of the matrix $Df$. Therefore, the forms $\sigma_m$ converge weakly in $L_{l/k,\text{loc}}(U)$ to $f^*(\omega)$.

It is not hard to see that both limits $\sigma_0$ and $f^*(\omega)$ coincide: $\sigma_0 = f^*(\omega)$. In view of (6) the mapping $f$ induces a bounded operator $f^* : \mathcal{L}_p(W,\Lambda^k) \to \mathcal{L}_q(U,\Lambda^k)$, $1 < q < p < \infty$. By Corollary 1, $f \in \mathcal{CD}^k(U;W)$.

First, we consider the case $q < p$. The following lemma is valid.

**Lemma 2.** [2] Every sequence of mappings $h_m$, $m = 1, 2, \ldots$, that is bounded in $L_1(U)$, contains a subsequence, converging in the biting sense to some function $h \in L_1(U)$.

This lemma implies existence of a function $U \ni x \mapsto M(x)$ of $L_\infty(U)$ such that some subsequence of the function $h_m = M_m(x)x$ converges

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1It means that the sequence of forms $\omega_m$ converges weakly in $L_{l/k}(D)$ to a form $\omega_0$ on every subdomain $D \subseteq U$. 
in the biting sense to \( h = M(x) \). We assume that the given sequence \( M_m(x) \) converges in the biting sense to the function \( M(x) \) (the set of the Definition 3 is denoted by \( E_\nu \)).

Now we estimate the distortion coefficient of the limit mapping \( f \). For this, we consider estimates on a set \( E_\nu \). Let \( Z_m \) be the set of zeros of the Jacobian of the mapping \( f_m \). Since the rank of the matrix \( Df_m \) on the set \( Z_m \) is less then \( k \), it follows that all \( k \)-th-order minors are equal to zero on the set \( Z_m \).

Applying the Hölder inequality, and taking into account that \( \frac{q}{p} + \frac{q}{p} = 1 \) on each intersection \( E_\nu \cap B(x_0,r) \), where \( x_0 \in E_\nu, B(x_0,r) \subseteq U \), in view of (5) we have

\[
\int_{E_\nu \cap B(x_0,r)} |\Lambda^k f_m(x)|^q \, dx = \int_{(E_\nu \cap B(x_0,r)) \setminus Z_m} |\Lambda^k f_m(x)|^q \, dx \leq \left( \int_{(E_\nu \cap B(x_0,r)) \setminus Z_m} \left| \frac{\Lambda^k f_m(x)}{|J(x,f_m)|} \right|^{\frac{q}{p}} \, dx \right)^{\frac{q}{p}} \leq \left( \int_{E_\nu \cap B(x_0,r)} (K_{k,p}(f_m))^{\alpha(x)} \, dx \right)^{\frac{q}{p}} \left( \int_{E_\nu \cap B(x_0,r)} |J(x,f_m)| \, dx \right)^{\frac{q}{p}} \leq \left( \int_{E_\nu \cap B(x_0,r)} M_m^\alpha(x) \, dx \right)^{\frac{q}{p}} \left( \int_{B(x_0,r)} |J(x,f_m)| \, dx \right)^{\frac{q}{p}}. \tag{7}
\]

Elements of the matrix \( \Lambda^k(f_m)(x) \) are the \( k \)-th-order minors of ap \( D_m f(x) \). In view of Lemma 1, they converge weakly in \( L_{\ell/k, loc}(U) \) to elements of the matrix \( \Lambda^k(f)(x) \). Since \( q < \ell/k \), \( \Lambda^k(f_m)(x) \) converges weakly in \( L_{q, loc}(U) \) to \( \Lambda^k(f)(x) \). Since the norm is semicontinuous in the Banach space \( L_q \), the left-hand side of the inequality can be estimated as

\[
\int_{E_\nu \cap B(x_0,r)} |\Lambda^k f(x)|^q \, dx \leq \lim_{m \to \infty} \int_{E_\nu \cap B(x_0,r)} |\Lambda^k f_m(x)|^q \, dx.
\]

We have also \( \int_{B(x_0,r)} |J(x,f_m)| \, dx \leq |f_m(B(x_0,r))| \).

Since \( |f(B(x_0,r))| < \infty \) and the mapping \( f \) is a homeomorphism, the images \( f(S(x_0,r)) \) of the spheres \( S(x_0,r) \) do not intersect under different
r. It follows that the n-measure of the image of any sphere is zero for almost all \( r \): \(|f(S(x_0, r))| = 0\). We fix \( r \) so that

\[ |f(S(x_0, r))| = 0 \]

and surround the image of the sphere \( f(S(x_0, r)) \) by an \( \varepsilon \)-neighborhood \( U_\varepsilon \). Since the mappings \( f_m \) converge locally uniformly to the mapping \( f \), it follows that, starting from a number \( m_0 \), the images of the spheres \( f_m(S(x_0, r)) \), \( m \geq m_0 \), are contained in this \( \varepsilon \)-neighborhood. It is clear that \( |U_\varepsilon| \to 0 \) as \( \varepsilon \to 0 \), and hence \(|f_m(B(x_0, r))| \to |f(B(x_0, r))|\) as \( m \to \infty \).

Taking into account that \( M_m(x)^\kappa \) converge in the biting sense to \( M(x)^\kappa \), we pass to the lower limit in (7) as \( m \to \infty \). We get

\[
\int_{B(x_0, r)} |\Lambda^k f(x)|^q \chi_{E^\nu}(x) \, dx \leq \left( \int_{B(x_0, r)} M^\kappa(x) \chi_{E^\nu}(x) \, dx \right)^\frac{q}{\kappa} |f(B(x_0, r)|^\frac{q}{p}.
\]

Dividing both sides of this inequality by the measure of the ball \( B(x_0, r) \), we obtain the following inequality

\[
\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |\Lambda^k f(x)|^q \chi_{E^\nu}(x) \, dx \leq \left( \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} M^\kappa(x) \chi_{E^\nu}(x) \, dx \right)^\frac{q}{\kappa} \left( \frac{|f(B(x_0, r))|}{|B(x_0, r)|} \right)^\frac{q}{p}. \tag{8}
\]

Since the homeomorphism \( f \) is Sobolev differentiable, then by [13, Section 2.3, formula (2.5)] we have

\[
\frac{|f(B(x_0, r))|}{|B(x_0, r)|} \to |J(x_0, f)| \quad \text{as} \quad r \to 0 \quad \text{for almost all} \quad x_0 \in E^\nu.
\]

Hence, by the Lebesgue differentiability theorem, letting \( r \) go to 0 we obtain that

\[
|\Lambda^k f(x)|^q \leq M^q(x) |J(x, f)|^\frac{q}{p} \quad \text{for almost all} \quad x \in E^\nu. \tag{9}
\]

As \( U = \bigcup_{\nu} E^\nu \), the point-wise inequality (9) holds in \( U \) almost everywhere.

In the case \( 1 < q = p < \infty \) we can assume that a sequence of numbers \( \{\|M_m \mid L_\infty(U)\|\}_{m \in \mathbb{N}} \) converges to \( M = \lim_{m \to \infty} \|M_m \mid L_\infty(U)\| \in \mathbb{R} \). In
this case, instead of (7), for any \( \varepsilon > 0 \) there exists \( m_1 \) such that for all \( m \geq m_1 \) we have
\[
\int_{B(x_0, r)} \left| \Lambda^k f_m(x) \right|^q \, dx \leq (M + \varepsilon) \left( \int_{B(x_0, r)} \left| J(x, f_m) \right| \, dx \right).
\]

Further, proceeding as in the case \( q < p \), we obtain the estimation
\[
K_{k,p}(x, f) \leq (M + \varepsilon) \text{ for almost all } x \in U.
\]
Since \( \varepsilon > 0 \) is an arbitrary number, we get the desired estimation.

By the Corollary 1, we have proved that the limit mapping \( f \) belongs to \( C\mathcal{D}^{k}_{q,p}(U; W) \).

□

As a straightforward consequence of Theorem 4 we get the following

**Corollary.** [16] Let \( f_m \in C\mathcal{D}^{k}_{q,p}(U; W) \), \( m \in \mathbb{N} \), be a sequence of homeomorphisms of the Sobolev class \( W^{1,1}_{l,\text{loc}}(U) \) with \( k < l \), \( q \leq l/k \), \( 1 < q \leq \leq p < \infty \). Suppose that the sequence \( f_m \) is locally bounded in \( W^1_l(U) \), and converge locally uniformly to a homeomorphism \( f : U \to W \) as \( m \to \infty \). Assume also that there exists a function \( M(x) \in L^{1/q}(U) \), \( \frac{1}{q} = \frac{1}{q} - \frac{1}{p} \), such that
\[
K_{k,p}(\cdot, f_m)(x) \leq M(x) \text{ for all } m \in \mathbb{N}
\]
in \( U \) almost everywhere. Then the limit mapping \( f \) belongs to \( C\mathcal{D}^{k}_{q,p}(U; W) \) and \( K_{k,p}(\cdot, f) \in L^{1/q}(U) \), where \( \frac{1}{q} = \frac{1}{q} - \frac{1}{p} \).

Moreover, the inequality \( K_{k,p}(x, f) \leq M(x) \) holds almost everywhere.

**Remark.** There exists a misprint in the paper [16]: in the statement of the main result the condition \( q \leq l/k \) is missing.

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