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A LOWER BOUND FOR THE $L_2[-1,1]$-NORM OF THE LOGARITHMIC DERIVATIVE OF POLYNOMIALS WITH ZEROS ON THE UNIT CIRCLE

Abstract. Let C be the unit circle $\{z : |z| = 1\}$ and $Q_n(z)$ be an arbitrary C-polynomial (i. e., all its zeros $z_1, \dots, z_n \in C$). We prove that the norm of the logarithmic derivative Q_n'/Q_n in the complex space $L_2[-1,1]$ is greater than $1/8$.

Key words: *logarithmic derivative, C-polynomial, simplest fraction, norm, unit circle*

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1. The main result. The problem of whether the logarithmic derivatives of C-polynomials (see the Abstract), i. e., the rational functions of the form

$$\sum_{k=1}^{n} \frac{1}{z - z_k} \qquad (|z_1| = \dots = |z_n| = 1),$$

are dense in the complex space $L_2[-1,1]$, was raised by Nasyrov in 2014 during the talk of Borodin at the conference "Complex Analysis and its Applications" in Petrozavodsk (see [1, §4]). (Sums $\sum_1^n 1/(z - z_k)$ are also known as the *simplest fractions* or *simple partial fractions*.)

We find that the Nasyrov's question has the negative answer, namely,

$$\left(\int_{-1}^{1} \left| \sum_{k=1}^{n} \frac{1}{x - z_k} \right|^2 dx \right)^{1/2} > \frac{1}{8} \tag{1}$$

for any $z_1, \dots, z_n$ on the unit circle.

The related result for the area integral was obtained by Newman [2]:

$$\iint\limits_{|z|<1} \left| \sum_{k=1}^{n} \frac{1}{z - z_k} \right| dxdy \geqslant \frac{\pi}{18} \qquad (|z_1| = \dots = |z_n| = 1), \tag{2}$$

where $x+iy=z$. Using the techniques in [2], Chui and Shen [3, §5] derived the order of approximation by sums of the form $\sum_1^n 1/(z-z_k)$, $z_k\in C$, in the Bers spaces $A_q(D)$, $q>2$ (D is the interior of C).

To prove (1), we use some ideas in [2], too.

2. Proof of (1). Let $|z_1|=\cdots=|z_n|=1$. Obviously,

$$\sum_{k=1}^n \frac{1}{z-z_k}=-\frac{1}{2z}\left(\sum_{k=1}^n \frac{z_k+z}{z_k-z}-n\right), \tag{3}$$

$$P(z_k;z):=\operatorname{Re}\frac{z_k+z}{z_k-z}=\frac{1-|z|^2}{1-2\operatorname{Re}(\overline{z}z_k)+|z|^2}\geqslant 0, \qquad |z|\leqslant 1,$$

and for any $h>0$ and every $k=1,\dots,n$ the set of points z for which $P(z_k;z)\geqslant h$, i. e.,

$$(h+1)|z|^2-2h\operatorname{Re}(\overline{z}z_k)+h-1\leqslant 0,$$

is the disk $|z-z_kh/(h+1)|\leqslant 1/(h+1)$ in the closed unit disk. Put

$$c_n=1-6(5n)^{-2}.$$

Case 1: *There is a pole $z^*\in\{z_1,\dots,z_n\}$ such that* $|\operatorname{Re} z^*|\geqslant c_n$. We can assume that $c_n\leqslant \operatorname{Re} z^*\leqslant 1$. Then there is the segment $[\alpha_n,\beta_n]$,

$$0<\alpha_n<\beta_n\leqslant 1, \qquad \beta_n-\alpha_n>\frac{4}{5n+4}, \qquad \alpha_n\beta_n=\frac{5n-4}{5n+4},$$

such that

$$P(z^*;x)\geqslant 5n/4 \qquad \text{for} \qquad \alpha_n\leqslant x\leqslant\beta_n.$$

Indeed, the intersection of the disk $\{z: P(z^*;z)\geqslant 5n/4\}$ with the real axis is the segment $[x_1,x_2]\subset(0,1]$, where x_1 and $x_2>x_1$ are the (real) roots of the equation

$$(5n+4)x^2-2(5n)x\operatorname{Re} z^*+5n-4=0$$

with the discriminant

$$d\geqslant 4(5n)^2\{1-12(5n)^{-2}+36(5n)^{-4}\}-4\{(5n)^2-16\}>16.$$

It is clear that $x_1x_2=(5n-4)/(5n+4)$ and $x_2-x_1=\sqrt{d}/(5n+4)$.

Now, using the identity (3) and putting $Z_n = \{z_1, \dots, z_n\}$, we get

$$I(Z_n) := \int_{-1}^{1} \left| \sum_{k=1}^{n} \frac{1}{x - z_k} \right|^2 dx > \int_{-1}^{1} \left(\sum_{k=1}^{n} P(z_k; x) - n \right)^2 \frac{dx}{4x^2} >$$

$$> \int_{\alpha_n}^{\beta_n} (P(z^*; x) - n)^2 \frac{dx}{4x^2} > \frac{n^2}{4^3} \frac{\beta_n - \alpha_n}{\alpha_n \beta_n} > \frac{n^2}{4^3} \frac{4}{5n-4} \geqslant \frac{1}{24} > \frac{1}{64}.$$

Case 2: $|\operatorname{Re} z_k| < c_n$ *for any* $k = 1, \dots, n$. We have

$$P(z_k; x) < 1/3 \qquad \text{for} \quad x \in \Delta_n := [-1, -\gamma_n] \cup [\gamma_n, 1], \qquad k = 1, \dots, n,$$

$\gamma_n := 1 - (9/5)(5n)^{-2}$, since the product of the roots of every polynomial

$$q(x) = q(z_k; x) = 4x^2 - 2x \operatorname{Re} z_k - 2$$

is negative $(= -1/2)$, while the values of q at the points ± 1 and $\pm\gamma_n$ are positive, so that $q(x) > 0$ for $x \in \Delta_n$. Thus

$$I(Z_n) > \int_{\Delta_n} \left(n - \sum_{k=1}^{n} P(z_k; x) \right)^2 \frac{dx}{4x^2} > 2 \int_{\gamma_n}^{1} \frac{n^2}{9} \, dx = \frac{2}{125} > \frac{1}{64}$$

and the proof is complete. Note that $I(Z_n) > n/80$ in Case 1.

3. Refinements of (1). It is of interest to find an order of $\inf\{I(Z_n) : Z_n \subset C\}$. We find this order in two cases. Let C^+ (C_+) be the intersection of the unit circle with the upper (right) closed half-plane.

Proposition. *There is an absolute constant* $0 < c \leqslant \pi/2$ *such that*

$$I(Z_n) \geqslant cn^2 \tag{4}$$

for all n *in* $\mathbb{N}$ *and any* Z_n *in* C^+ *or* C_+. *This bound is sharp in order* n.

Proof. First let $Z_n \subset C^+$. By the theorem of Govorov and Lapenko [4, Theorem 2] with $r = 1$ and $\delta = (2/3)(35e)^{-1}$, we have

$$\left| \sum_{k=1}^{n} \frac{1}{x - z_k} \right| > n \cdot \frac{2}{3} \cdot \frac{1}{35e} \quad \text{for} \quad x \in \Delta \subset [-1,1], \qquad \operatorname{mes} \Delta > \frac{2}{3};$$

therefore $I(Z_n) > c'n^2, \quad c' := (2/3)^3 (35e)^{-2}$.

If $Z_n \subset C_+$, i.e., all $\operatorname{Re} z_k \geqslant 0$, then we have $q(z_k; x) \geqslant 0$ for any $x \in [-1, -1/\sqrt{2}]$ and every k (see Case 2, § 2), so that

$$I(Z_n) > \int_{-1}^{-1/\sqrt{2}} (n - n/3)^2 \frac{dx}{4x^2} = c''n^2, \qquad c'' := \frac{\sqrt{2}-1}{9},$$

$c'' > c'$. Thus, (4) holds with $c = c'$.

Finally, the sharpness of (4) in order n as well as the estimate $c \leqslant \pi/2$ follow from the example of the C-polynomial $(z-i)^n$:

$$\int_{-1}^{1} |n/(x-i)|^2 dx = 2n^2 \int_0^1 dx/(x^2+1) = \pi n^2/2.$$

□

It seems that in the general case, $Z_n \subset C$, the following result is true.

CONJECTURE. *There is an absolute constant* $0 < c \leqslant \ln 2$ *such that*

$$I(Z_n) \geqslant cn \tag{5}$$

for all n *in* $\mathbb{N}$ *and any* Z_n *in* C. *This bound is sharp in order* n.

To prove the sharpness of (5) we use the C-polynomial $z^n - i$:

$$\int_{-1}^{1} \left| \frac{nx^{n-1}}{x^n - i} \right|^2 dx = 2n \int_0^1 \frac{t^{1-(1/n)}}{t^2+1} dt = n \ln 2 + \int_0^1 t^{-1-(1/n)} \ln(t^2+1) dt$$

(the last integral is less than 1, because $\ln(t^2+1) < t^2$ for $0 < t \leqslant 1$).

4. A lower bound for the L_2-norm in the case of the unit disk. It follows from (2) and the Schwarz inequality that

$$\left(\iint_{|z|<1} \left| \sum_{k=1}^{n} \frac{1}{z - z_k} \right|^2 dxdy \right)^{1/2} \geqslant \frac{\sqrt{\pi}}{18} \qquad (|z_1| = \cdots = |z_n| = 1).$$

Using the techniques in the proof of (1), we can derive a more sharp bound. Indeed, the consideration of two cases (cf. §2):

1) there is a pole $z^* \in \{z_1, \ldots, z_n\}$ such that $c_n \leqslant \operatorname{Re} z^* \leqslant 1$,

2) $-1 \leqslant \operatorname{Re} z_k < c_n$ for all $k = 1, \ldots, n$

shows that for any $Z_n = \{z_1, \ldots, z_n\} \subset C$

$$\int_0^1 \left| \sum_{k=1}^n \frac{1}{x - z_k} \right|^2 x dx > \frac{1}{2 \cdot 8^2}.$$

But all points $z_k e^{-i\varphi}$, $\varphi \in \mathbb{R}$, also belong to C, so that

$$I(Z_n; \varphi) := \int_0^1 \left| \sum_{k=1}^n \frac{1}{re^{i\varphi} - z_k} \right|^2 r dr > \frac{1}{2 \cdot 8^2} \qquad \text{for any } \varphi \in \mathbb{R}.$$

Finally, for any $z_1, \ldots, z_n$ on the unit circle, we have

$$\left(\iint_{|z|<1} \left| \sum_{k=1}^n \frac{1}{z - z_k} \right|^2 dxdy \right)^{1/2} = \left(\int_0^{2\pi} I(Z_n; \varphi)\, d\varphi \right)^{1/2} > \frac{\sqrt{\pi}}{8}.$$

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