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L. Matejíčka

PROOF OF A CONJECTURE ON NIELSEN'S β-FUNCTION

Abstract. In this paper, an inequality for Nielsen's β-function is proved. The inequality was posed by Kwara Nantomah as a conjecture in 2019.

Key words: *Nielsen's β-function, inequality, harmonic mean*

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1. Introduction. It is well known that there are several ways to define Nielsen's β-function (e. g., [5], [6]). We use the following definition:

$$\begin{aligned}\beta(x) &= \int_0^\infty \frac{e^{-xt}}{1+e^{-t}}dt = \int_0^1 \frac{t^{x-1}}{1+t}dt = \sum_{m=0}^\infty \frac{(-1)^m}{m+x} = \\ &= \frac{1}{2}\left(\psi\left(\frac{1+x}{2}\right) - \psi\left(\frac{x}{2}\right)\right) \text{ for } x>0,\end{aligned}$$

where $\psi(x) = d\ln\Gamma(x)/dx$ is the digamma function, $\Gamma(x)$ is the Euler Gamma function [8].

It is also known [8] that the special function $\beta(x)$ is related to the Euler beta function $B(x,y)$ and to the Gauss hypergeometric function ${}_2F_1(a,b;c;d)$ by

$$\begin{aligned}\beta(x) &= -\frac{d}{dx}\left(\ln B\left(\frac{x}{2},\frac{1}{2}\right)\right)\\ \beta(x) &= \frac{1}{x}\left({}_2F_1(1,x;1+x;-1)\right) \text{ for } x>0.\end{aligned}$$

In the recent years, Nielsen's β-function has been very intensively studied. Nantomah and other researchers [5]–[9] introduced and studied some of its properties. A lot of interesting inequalities for Nielsen's β-function have been discovered and proved. For example, in [5] it was shown that

$$\begin{aligned} &\beta(x) + x\beta'(x) < 0, \qquad (1)\\ &2\beta'(x) + x\beta''(x) > 0, \end{aligned}$$

for $x > 0$.

The result follows from the fact that the function $x|\beta^{(m)}(x)|$, $x > 0$, $n \in N_0$ is completely monotonic [7]. It was also shown [5], that

$$\beta(x) + \beta(1-x) = \frac{\pi}{\sin(\pi x)},\ 0 < x < 1,$$

$$\beta(x) = \frac{1}{x} - \beta(x+1),\ x > 0, \qquad (2)$$

$$\beta(1) = \ln(2),\ \beta'(1) = \frac{-\pi^2}{12}. \qquad (3)$$

We recall, that Gautschi [2] proved an interesting inequality involving the Euler gamma function $\Gamma(x)$:

$$\frac{2\Gamma(x)\Gamma(1/x)}{\Gamma(x) + \Gamma(1/x)} \geqslant 1,\ \ x > 0. \qquad (4)$$

Similarly, Alzer and Jameson [1] proved that

$$\frac{2\psi(x)\psi(1/x)}{\psi(x) + \psi(1/x)} \geqslant -\gamma,\ \ x > 0, \qquad (5)$$

where $\gamma = 0.577...$ is the Euler–Mascheroni constant.

In view of the harmonic mean inequalities (4), (5), Nantomah [5] posed the following conjecture:

Conjecture 1. *For $x \in (0, \infty)$, the inequality*

$$\frac{2\beta(x)\beta(1/x)}{\beta(x) + \beta(1/x)} \leqslant \ln 2, \qquad (6)$$

holds, turning into equality at $x = 1$.

For more detailed information on Nielsen's β-function, refer to [1]–[9] and the related references therein.

The aim of this short paper is to prove the Conjecture 1 on Nielsen's β-function.

2. Main Results.

Lemma 1. *The inequality*

$$2\beta'^2(x) - \beta''(x)\beta(x) > 0, \tag{7}$$

holds for $x > 0$.

Proof.

In [5] it was shown

$$\beta(x) + x\beta'(x) < 0, \tag{8}$$

for $x > 0$.

This implies that

$$\beta^2(x) + 2x\beta(x)\beta'(x) + x^2\beta'(x)^2 > 0, \tag{9}$$

is valid for $x > 0$.

So,

$$\beta'^2(x) > -\frac{1}{x^2}\beta^2(x) - \frac{2}{x}\beta(x)\beta'(x). \tag{10}$$

To prove the inequality (7), we need to establish

$$2\left(-\frac{1}{x^2}\beta^2(x) - \frac{2}{x}\beta(x)\beta'(x)\right) - \beta(x)\beta''(x) > 0. \tag{11}$$

Because of

$$\beta(x) = \int_0^\infty \frac{e^{-xt}}{1+e^{-t}} > 0,$$

for $x > 0$ (see [5]), it is sufficient to prove

$$-\frac{2}{x^2}\beta(x) - \frac{4}{x}\beta'(x) - \beta''(x) > 0. \tag{12}$$

The inequality (12) is equivalent to

$$2\beta(x) + 4x\beta'(x) + x^2\beta''(x) < 0, \tag{13}$$

for $x > 0$.

The well-known formulas (see [5])

$$\beta(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m+x},\ \beta'(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m+x)^2},\ \beta''(x) = 2\sum_{m=0}^{\infty} \frac{(-1)^m}{(m+x)^3},$$

give

$$\begin{aligned}&2\beta(x)+4x\beta'(x)+x^2\beta''(x)=\\&=\sum_{m=0}^{\infty}\frac{2(-1)^m}{m+x}+\sum_{m=0}^{\infty}\frac{4x(-1)^{m+1}}{(m+x)^2}+\sum_{m=0}^{\infty}\frac{2x^2(-1)^m}{(m+x)^3}=\\&=\Psi(x)=2\sum_{m=0}^{\infty}\frac{(-1)^m m^2}{(m+x)^3}.\end{aligned}$$

The following classical Laplace formula

$$\frac{2}{(k+x)^3}=\int_0^{\infty}t^2e^{-kt}e^{-xt}dt, \tag{14}$$

for $x>0$, $k\in N_0$ can be found in a table of Laplace transforms (see [3]). So, $\Psi(x)$ can be rewritten as

$$\begin{aligned}\Psi(x)&=\sum_{k=0}^{\infty}(-1)^k k^2\int_0^{\infty}t^2e^{-kt}e^{-xt}dt=\\&=\int_0^{\infty}t^2e^{-xt}\left\{\sum_{k=1}^{\infty}(-1)^k k^2e^{-kt}\right\}dt.\end{aligned}$$

Because of

$$\left|(-1)^k k^2 e^{-kt}\right|\leqslant\frac{k^2}{e^{ka}},$$

where $a>0$ and $t\geqslant a$, the Weierstrass theorem for functional series shows that the series

$$g(t)=\sum_{k=1}^{\infty}(-1)^k k^2e^{-kt},$$

converges uniformly on each (a,∞), where $a>0$. We show that $g(t)<0$ for $t>0$. Put

$$\alpha(t)=\sum_{k=1}^{\infty}(-1)^k e^{-kt}=\frac{-e^{-t}}{1+e^{-t}}=-\frac{1}{1+e^t}.$$

It implies

$$\alpha''(t)=\sum_{k=1}^{\infty}(-1)^k k^2e^{-kt}=-\left(\frac{1}{1+e^t}\right)''=\frac{e^t\left(1-e^t\right)}{\left(1+e^t\right)^3}<0.$$

The proof of Lemma 1 is complete. □

Theorem 1. *Let $x \in (0, \infty)$. Then the inequality*

$$\frac{2\beta(x)\beta(1/x)}{\beta(x)+\beta(1/x)} \leqslant \ln 2, \tag{15}$$

holds, turning to equality if $x = 1$.

Proof. The equality is obvious. Because of $\beta(1) = \ln 2$, the inequality (15) can be rewritten as

$$\beta(x)\left(\beta(1/x)-\beta(1)\right)+\beta(1/x)\left(\beta(x)-\beta(1)\right) \leqslant 0. \tag{16}$$

Some computations show that inequality (16) is equivalent to

$$\frac{\beta(1)}{\beta(x)} - 1 + \frac{\beta(1)}{\beta(1/x)} - 1 \geqslant 0.$$

Put

$$F(t) = \frac{\beta(1)}{\beta(t)} - 1$$

for $t > 0$.

First, we show that $F(t)$ is a convex function on $(0, \infty)$.

By repeated differentiation, we obtain

$$F'(t) = -\frac{\beta(1)\beta'(t)}{\beta^2(t)}$$

and

$$F''(t) = \frac{\beta(1)}{\beta^4(t)}\left(2\beta'^2(t) - \beta(t)\beta''(t)\right)\beta(t).$$

So, it is sufficient to show that

$$2\beta'^2(t) - \beta(t)\beta''(t) \geqslant 0.$$

But this follows from Lemma (1).

Using the Jensen inequality for $F(x)$, we obtain

$$F\left(\frac{x+\frac{1}{x}}{2}\right) \leqslant \frac{1}{2}\left(F(x)+F\left(\frac{1}{x}\right)\right),$$

which is, in our case,

$$\frac{1}{2}\left(\frac{\beta(1)}{\beta(x)} - 1 + \frac{\beta(1)}{\beta\left(\frac{1}{x}\right)} - 1\right) \geqslant \frac{\beta(1)}{\beta\left(\frac{x+\frac{1}{x}}{2}\right)} - 1 = \frac{\beta(1)}{\beta\left(\frac{x^2+1}{2x}\right)} - 1.$$

The proof of the Theorem 1 will be complete, if we show that

$$\frac{\beta(1)}{\beta\left(\frac{x^2+1}{2x}\right)} - 1 \geqslant 0.$$

But, this follows from $(x^2+1)/(2x) \geqslant 1$ and $\beta'(t) < 0$ for $t > 0$ and thus completes the proof of Theorem 1. □

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Ladislav Matejíčka
Faculty of Industrial Technologies in Púchov
Trenčín University of Alexander Dubček in Trenčín
I. Krasku 491/30, 02001 Púchov, Slovakia
E-mail: ladislav.matejicka@tnuni.sk