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FINITE INTEGRAL FORMULA INVOLVING ALEPH-FUNCTION AND GENERALIZED MITTAG-LEFFLER FUNCTION

Abstract. The aim of this paper is to establish general definite integrals involving product of the Aleph function and the generalized Mittag-Leffler function with general arguments. This integral yields a number of known results as special cases. For the sake of illustration, several corollaries are also presented as special case of our main results.

Key words: *Aleph-function, I-function, H-function, Generalized Mittag-Leffler function*

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1. Introduction and preliminaries. Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}_0^-$ and $\mathbb{N}$ be sets of complex numbers, real and positive numbers, non-positive and positive integers, respectively, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
Gösta Mittag-Leffler [9] introduced and studied the function $E_\alpha(z)$, defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (\mathrm{Re}(\alpha) > 0). \tag{1}$$

A generalization of this function was given by Wiman [23], as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0). \tag{2}$$

A generalization of the Mittag-Leffler function (M–L function) (2) by a series representation is introduced by Prabhakar [11] as

$$E^{\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n z^n}{\Gamma(\alpha n + \beta)\, n!} \quad (\mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0,\ \mathrm{Re}(\delta) > 0). \tag{3}$$

Further, Shukla and Prajapati [18] (see also Srivastava and Tomovski [19]) studied the function $E_{\alpha,\beta}^{\gamma,q}(z)$, defined by

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{nq} z^n}{\Gamma(\alpha n + \beta) n!}, \tag{4}$$

where $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, q \in (0,1) \cup \mathbb{N}$.

Since the Mittag-Leffler function provides solutions to certain problems formulated in terms of fractional order differential, integral and difference equations, a number of generalizations of the this function have been introduced and studied by many authors.

Salim and Faraj [15] have introduced a new generalization of the Mittag-Leffler function by means of the power series

$$E_{\mu,\rho,p}^{\delta,\zeta,q}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{nq} z^n}{\Gamma(\mu n + \rho) n! (\zeta)_{pn}}, \tag{5}$$

where $\mathrm{Re}(\delta)$, $\mathrm{Re}(\mu)$, $\mathrm{Re}(\rho)$, $\mathrm{Re}(\zeta)$, $p, q > 0$, such that $q \leqslant \mathrm{Re}(\mu) + p$.

Gupta et al. [5] gave certain relations of generalized fractional calculus associated with the generalized Mittag-Leffler function (3) in terms of the generalized Wright function; Kumar et al. [6] give the solution of a general family of fractional kinetic equations associated with the generalized Mittag-Leffler function; to mention a few.

The Aleph ($\aleph$)-function is a very general higher transcendental function, which is a generalization of the I-function [16]. The Aleph-function was introduced by Südland et al. [20], [21], and is defined by means of a Mellin-Barnes type integral in the following manner:

$$\begin{aligned}\aleph[z] = \aleph_{p_i,q_i,c_i;r}^{m,n}\left[z \,\middle|\, \begin{matrix}(a_j,A_j)_{1,n}, [c_j(a_{ji},A_{ji})]_{n+1,p_i;r} \\ (b_j,B_j)_{1,m}, [c_j(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right] := \\ := \frac{1}{2\pi\omega}\int_{\mathcal{L}} \Omega_{p_i,q_i,c_i;r}^{m,n}(s)\, z^{-s}\,\mathrm{d}s,\end{aligned} \tag{6}$$

where $z \in \mathbb{C} \setminus \{0\}$, $\omega = \sqrt{-1}$, and

$$\begin{aligned}&\Omega_{p_i,q_i,c_i;r}^{m,n}(s) = \\ &= \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j - A_j s)}{\sum_{i=1}^{r} c_i \left\{\prod_{j=n+1}^{p_i}\Gamma(a_{ji} + A_{ji}s)\prod_{j=m+1}^{q_i}\Gamma(1 - b_{ji} - B_{ji}s)\right\}}.\end{aligned} \tag{7}$$

Here Γ denotes the Gamma function, the path of integration $\mathcal{L}=\mathcal{L}_{\iota\gamma\infty}$ $(\gamma\in\mathbb{R})$ goes from $\gamma-\iota\infty$ to $\gamma+\iota\infty$ and is a contour of the Mellin-Barnes type integral. The poles of the Gamma functions $\Gamma(1-a_j-A_js)$ $(j,n\in\mathbb{N};1\leqslant j\leqslant n)$ do not coincide with those of $\Gamma(b_j+B_js)$ $(j,m\in\mathbb{N};$ $1\leqslant j\leqslant m)$. The parameters $p_i,q_i\in\mathbb{N}_0$ satisfy the conditions $0\leqslant n\leqslant p_i$, $1\leqslant m\leqslant q_i$, $\tau_i>0$ $(1\leqslant i\leqslant r)$; the parameters A_j,B_j,A_{ji}, $B_{ji}>0$ and $a_j,b_j,a_{ji},b_{ji}\in\mathbb{C}$. The empty product in (7) is interpreted as unity. The existence conditions for the defining integral (6) depend on the following conditions:

$$\varphi_i>0 \quad\text{and}\quad |\arg(z)|<\frac{\pi}{2}\varphi_i, \tag{8}$$

and

$$\varphi_i\geqslant 0,\quad |\arg(z)|<\frac{\pi}{2}\varphi_i \quad\text{and}\quad \mathrm{Re}\{\zeta_i\}+1<0, \tag{9}$$

where

$$\varphi_i=\sum_{j=1}^{n}A_j+\sum_{j=1}^{m}B_j-c_i\left(\sum_{j=n+1}^{p_i}A_{ji}+\sum_{j=m+1}^{q_i}B_{ji}\right) \tag{10}$$

and

$$\zeta_i=\sum_{j=1}^{m}b_j-\sum_{j=1}^{n}a_j+c_i\left(\sum_{j=m+1}^{q_i}b_{ji}-\sum_{j=n+1}^{p_i}a_{ji}\right)+\frac{1}{2}\left(p_i-q_i\right). \tag{11}$$

Here $i=1,\cdots,r$.

Remark 1. *Setting $c_i\to 1(i=1,\ldots,r)$ in (6) gives the I-function due to Saxena [16], given by following relation:*

$$I^{m,n}_{p_i,q_i;r}(z)=\aleph^{m,n}_{p_i,q_i,1;r}(z)=\aleph^{m,n}_{p_i,q_i,1;r}\left(z\,\middle|\,\begin{matrix}(a_j,A_j)_{1,n},(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m},(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right)=$$
$$=\frac{1}{2\pi\omega}\int_{\mathcal{L}}\Omega^{m,n}_{p_i,q_i,1;r}(s)\,z^{-s}\mathrm{d}s, \tag{12}$$

where the kernel $\Omega^{m\,n}_{p_i\,q_i,1;r}(s)$ is given in (7) for $c_i\to 1$. The conditions of existence of the integral in (12) are the same as stated in (8)–(11), with $c_i\to 1$ $(i=1,\ldots,r)$.

Remark 2. *If we, further, take $r=1$ in (12), then it reduces to the H-function, defined by Fox [4] (see also, [8]):*

$$H^{m,n}_{p,q}(z)=\aleph^{m,n}_{p_i,q_i,1;1}(z)=\aleph^{m,n}_{p_i,q_i,1;1}\left(z\,\middle|\,\begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right)=$$

$$= \frac{1}{2\pi\omega} \int\limits_{\mathcal{L}} \Omega^{m,\,n}_{p_i,\,q_i,\,1;1}\left(s\right) z^{-s} \mathrm{d}s, \tag{13}$$

where the kernel $\Omega^{m,\,n}_{p_i,\,q_i,\,1;r}\left(s\right)$ *is given in* (7) *for* $c_i \to 1$ $(i = 1, \ldots, r)$ *and* $r = 1$. *The existence conditions for the integral in* (13) *are the same as given by* (8)–(11) *with* $c_i \to 1$ $(i = 1, \ldots, r)$ *and* $r = 1$.

Recently, Ayant and Kumar [1] have given finite double integrals involving the product of two hypergeometric functions and the Aleph-function. Kumar et al. [7] derived various integral formulas involving the Aleph-function multiplied by algebraic functions and special functions; Suthar et al. [22] established certain integrals involving the product of the Aleph-function with the exponential function and the multi Gauss hypergeometric function. For more details and work on the Aleph-function, the reader can refer the recent work [13], [17].

2. Main results. In order to avoid ambiguities, we use the symbols $\mathbf{n}$ and $\mathbf{r}$ in the main results of this section instead of n and r used in equation (6). We will need the following integral (see, Brychkov [2, eqn. 40, p. 115]) to define main results:

Lemma 1.

$$\int\limits_{-1}^{1} \frac{(1-x^2)^{s-\frac{1}{2}}}{(1+2ax+a^2)^s(1+2bx+b^2)^s}\,\mathrm{d}x = \frac{\sqrt{\pi}\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}\,{}_2F_1\left[s, 2s; s+1; ab\right], \tag{14}$$

where $\mathrm{Re}(s) > -\frac{1}{2}$ *and* ${}_2F_1$ *is the Gauss hypergeometric function* [12].

Theorem 1. *If* $A_1 = (\frac{1}{2}-s-c\mathbf{n}; e), (1-s-\mathbf{r}-c\mathbf{n}; e), (1-\mathbf{r}-2(s+c\mathbf{n}); 2e)$; $B_1 = (-s-\mathbf{r}-c\mathbf{n}; e), (1-s-c\mathbf{n}; e), (1-2(s+c\mathbf{n}); 2e)$; *and*

$$X = \frac{(1-x^2)}{(1+2ax+a^2)\,(1+2bx+b^2)}, \tag{15}$$

then we have

$$\int\limits_{-1}^{1} \frac{\left(1-x^2\right)^{s-\frac{1}{2}}}{\left(1+2ax+a^2\right)^s\left(1+2bx+b^2\right)^s}\, E^{\delta,\,\zeta,\,q}_{\mu,\,\rho,\,p}\left(yX^c\right) \aleph\left(zX^e\right) \mathrm{d}x =$$

$$= \sqrt{\pi} \sum_{\mathbf{n},\,\mathbf{r}=0}^{\infty} \frac{\left(\delta\right)_{\mathbf{n}q}\, y^{\mathbf{n}}}{\Gamma\left(\mu\mathbf{n}+\rho\right)\mathbf{n}!\,\left(\zeta\right)_{p\mathbf{n}}} \frac{\left(ab\right)^{\mathbf{r}}}{\mathbf{r}!} \times$$

$$\times \ \aleph^{m,\,n+3}_{p_i+3,\,q_i+3,\,c_i;r}\left(z \,\middle|\, \begin{array}{l} A_1, (a_j, A_j)_{1,n}, [c_i\,(a_{ji}, A_{ji})]_{n+1,\,p_i;r} \\ \\ B_1, (b_j, B_j)_{1,m}, [c_i\,(b_{ji}, B_{ji})]_{m+1,\,q_i;r} \end{array} \right), \tag{16}$$

provided that

a) $\mathrm{Re}(\delta), \mathrm{Re}(\mu), \mathrm{Re}(\rho), \mathrm{Re}(\zeta), p, q > 0$, *such that* $q \leqslant \mathrm{Re}(\mu) + p$, $\min\{a, b, c, e\} > 0$;

b) $\mathrm{Re}(t + \mathbf{n}c) + e \min\limits_{1 \leqslant j \leqslant m} \mathrm{Re}\left[\left(\dfrac{b_j^{(i)}}{B_j^{(i)}}\right)\right] > -\dfrac{1}{2}$;

c) $|\arg(zX^e)| < \dfrac{1}{2}\pi\varphi_i \ (i = 1, \ldots, r)$, *where* φ_i *is defined by* (10).

Proof. Invoke (5) and (6) and change the order of integration and summation (this can be done due to absolute convergence of the integral) to transform the LHS of (16) to

$$\mathcal{I} = \sum_{\mathbf{n}=0}^{\infty} \frac{(\delta)_{\mathbf{n}q}\, y^{\mathbf{n}}}{\Gamma(\mu\mathbf{n} + \rho)\mathbf{n}!\,(\zeta)_{p\mathbf{n}}} \int_{-1}^{1} \frac{(1 - x^2)^{s + c\mathbf{n} - \frac{1}{2}}}{(1 + 2ax + a^2)^{s + c\mathbf{n}}(1 + 2bx + b^2)^{s + c\mathbf{n}}} \times$$
$$\times \left[\frac{1}{2\pi\omega} \int_{\mathcal{L}} \Omega^{m,n}_{p_i,q_i,c_i;r}(u)\, z^{-u} \frac{(1 - x^2)^{-eu}}{(1 + 2ax + a^2)^{-eu}(1 + 2bx + b^2)^{-eu}}\, \mathrm{d}u\right] \mathrm{d}x. \tag{17}$$

Interchanging the order of the x-integral and the u-integral, which is permissible under the given conditions, we have

$$\mathcal{I} = \sum_{\mathbf{n}=0}^{\infty} \frac{(\delta)_{\mathbf{n}q}\, y^{\mathbf{n}}}{\Gamma(\mu\mathbf{n} + \rho)\,\mathbf{n}!\,(\zeta)_{p\mathbf{n}}} \frac{1}{2\pi\omega} \int_{\mathcal{L}} \Omega^{m,n}_{p_i,q_i,c_i;r}(u)\ z^{-u} \times$$
$$\times \ \Big[\int_{-1}^{1} \frac{(1 - x^2)^{s + c\mathbf{n} - eu - \frac{1}{2}}}{(1 + 2ax + a^2)^{s + c\mathbf{n} - eu}(1 + 2bx + b^2)^{s + c\mathbf{n} - eu}}\, \mathrm{d}u\Big] \mathrm{d}x. \tag{18}$$

Next, we evaluate the inner integral of (18) with the help of Lemma 1:

$$\int_{-1}^{1} \frac{(1 - x^2)^{s + c\mathbf{n} - eu - \frac{1}{2}}}{(1 + 2ax + a^2)^{s + c\mathbf{n} - eu}(1 + 2bx + b^2)^{s + c\mathbf{n} - eu}}\, \mathrm{d}x =$$

$$
= \frac{\sqrt{\pi}\Gamma(s + c\mathbf{n} - eu + \frac{1}{2})}{\Gamma\left(s + c\mathbf{n} - eu + 1\right)} {}_2F_1\left(\begin{matrix} s + c\mathbf{n} - eu, 2\left(s + c\mathbf{n} - eu\right) \\ \\ s + c\mathbf{n} - eu + 1 \end{matrix} \middle| ab\right). \tag{19}
$$

Now, by substituting (19) in (18), we arrive at

$$
\mathcal{I} = \sum_{\mathbf{n}=0}^{\infty} \frac{(\delta)_{\mathbf{n}q}\, y^{\mathbf{n}}}{\Gamma\left(\mu\mathbf{n} + \rho\right)\mathbf{n}!\,(\zeta)_{p\mathbf{n}}} \frac{1}{2\pi\omega} \int_{\mathcal{L}} \Omega^{m,\,n}_{p_i,\,q_i,\,c_i;r}\left(u\right)\, z^{-u} \times
$$

$$
\times \frac{\sqrt{\pi}\Gamma\left(s + c\mathbf{n} - eu + \frac{1}{2}\right)}{\Gamma\left(s + c\mathbf{n} - eu + 1\right)} {}_2F_1\left(\begin{matrix} s + c\mathbf{n} - eu, 2\left(s + c\mathbf{n} - eu\right) \\ \\ s + c\mathbf{n} - eu + 1 \end{matrix} \middle| ab\right) \mathrm{d}u. \tag{20}
$$

$$
\mathcal{I} = \sum_{\mathbf{n}=0}^{\infty} \frac{(\delta)_{\mathbf{n}q}\, y^{\mathbf{n}}}{\Gamma\left(\mu\mathbf{n} + \rho\right)\mathbf{n}!\,(\zeta)_{p\mathbf{n}}} \frac{1}{2\pi\omega} \int_{\mathcal{L}} \Omega^{m,\,n}_{p_i,\,q_i,\,c_i;r}\left(u\right)\, z^{-u} \times
$$

$$
\times \frac{\sqrt{\pi}\Gamma\left(s + c\mathbf{n} - eu + \frac{1}{2}\right)}{\Gamma\left(s + c\mathbf{n} - eu + 1\right)} \sum_{\mathbf{r}=0}^{\infty} \frac{\left(s + c\mathbf{n} - eu\right)_{\mathbf{r}} \left(2(s + c\mathbf{n} - eu)\right)_{\mathbf{r}} (ab)^{\mathbf{r}}}{\left(s + c\mathbf{n} - eu + 1\right)_{\mathbf{r}}\ \mathbf{r}!} \mathrm{d}u. \tag{21}
$$

We change the order of integration and summation, which is justified due to the absolute convergence of the u-integral and the $\mathbf{n}$-series involved in the process.

$$
\mathcal{I} = \sqrt{\pi} \sum_{\mathbf{n},\,\mathbf{r}=0}^{\infty} \frac{(\delta)_{\mathbf{n}q}\, y^{\mathbf{n}}}{\Gamma\left(\mu\mathbf{n} + \rho\right)\mathbf{n}!\,(\zeta)_{p\mathbf{n}}} \frac{(ab)^{\mathbf{r}}}{\mathbf{r}!} \left[\frac{1}{2\pi\omega} \int_{\mathcal{L}} \Omega^{m,\,n}_{p_i,\,q_i,\,c_i;r}\left(u\right)\, z^{-u} \times \right.
$$

$$
\left. \times \frac{\Gamma\left(s + c\mathbf{n} - eu + \frac{1}{2}\right)}{\Gamma\left(s + c\mathbf{n} - eu + 1\right)} \frac{\left(s + c\mathbf{n} - eu\right)_{\mathbf{r}} \left(2\left(s + c\mathbf{n} - eu\right)\right)_{\mathbf{r}}}{\left(s + c\mathbf{n} - eu + 1\right)_{\mathbf{r}}} \mathrm{d}u\right]. \tag{22}
$$

By using relation $(a)_r = \dfrac{\Gamma\left(a + r\right)}{\Gamma\left(a\right)}$ for $(a \neq 0, -1, -2, \ldots)$, we get

$$
\mathcal{I} = \sqrt{\pi} \sum_{\mathbf{n},\,\mathbf{r}=0}^{\infty} \frac{(\delta)_{\mathbf{n}q}\, y^{\mathbf{n}}}{\Gamma\left(\mu\mathbf{n} + \rho\right)\mathbf{n}!\,(\zeta)_{p\mathbf{n}}} \frac{(ab)^{\mathbf{r}}}{\mathbf{r}!} \left[\frac{1}{2\pi\omega} \int_{\mathcal{L}} \Omega^{m,\,n}_{p_i,\,q_i,\,c_i;r}\left(u\right)\, z^{-u} \times \right.
$$

$$\left. \times \frac{\Gamma(s+c\mathbf{n}-eu+\frac{1}{2})}{\Gamma(s+c\mathbf{n}+\mathbf{r}-eu+1)} \frac{\Gamma(s+c\mathbf{n}+\mathbf{r}-eu)\Gamma(2s+c\mathbf{n}-eu)+\mathbf{r})}{\Gamma(s+c\mathbf{n}-eu)\Gamma(2(s+c\mathbf{n}-eu))} \,\mathrm{d}u \right]. \quad (23)$$

Finally, interpreting the result (23), with the Mellin-Barnes counter integral, we arrive at the desired formula (16). □

Theorem 2. *Assume that* $A_2 = (\frac{1}{2} - s + c\mathbf{n}; e)$, $(1 - s - \mathbf{r} + c\mathbf{n}; e)$, $(1 - \mathbf{r} - 2(s - c\mathbf{n}); 2e)$; $B_2 = (-s - \mathbf{r} + c\mathbf{n}; e)$, $(1 - s + c\mathbf{n}; e)$, $(1 - 2(s - c\mathbf{n}); 2e)$; *and* $X \neq 0$, *where* X *is given by* (15). *Then we have*

$$\int_{-1}^{1} \frac{(1-x^2)^{s-\frac{1}{2}}}{(1+2ax+a^2)^s (1+2bx+b^2)^s} E^{\delta,\zeta,q}_{\mu,\rho,p}\left(yX^{-c}\right) \aleph\left(zX^e\right) \mathrm{d}x =$$

$$= \sqrt{\pi} \sum_{\mathbf{n},\mathbf{r}=0}^{\infty} \frac{(\delta)_{\mathbf{n}q}\, y^{\mathbf{n}}}{\Gamma(\mu\mathbf{n}+\rho)\,\mathbf{n}!\,(\zeta)_{p\mathbf{n}}} \frac{(ab)^{\mathbf{r}}}{\mathbf{r}!} \times$$

$$\times\ \aleph^{m,n+3}_{p_i+3,q_i+3,c_i;r}\left(z \,\middle|\, \begin{array}{l} A_2, (a_j, A_j)_{1,n}, [c_i(a_{ji}, A_{ji})]_{n+1,p_i;r} \\ \\ B_2, (b_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{array}\right), \quad (24)$$

provided that

a) $\mathrm{Re}(\delta), \mathrm{Re}(\mu), \mathrm{Re}(\rho), \mathrm{Re}(\zeta), p, q > 0$, *such that* $q \leqslant \mathrm{Re}(\mu) + p$, $\min\{a, b, c, e\} > 0$.

b) $\mathrm{Re}(t - \mathbf{n}c) + e \min\limits_{1\leqslant j\leqslant m} \mathrm{Re}\left[\left(\dfrac{b_j^{(i)}}{B_j^{(i)}}\right)\right] > -\frac{1}{2}$.

c) $|\arg(zX^e)| < \frac{1}{2}\pi\varphi_i$ $(i = 1\ldots, r)$, *where* φ_i *is defined by* (10).

Proof. We can derive result (24) by following Theorem 1 and replacing c by $-c$. □

3. Special cases.

In this section, we give some special cases of Theorem 1 and Theorem 2.

Corollary 1. *By taking* $c_i \to 1$ *in Theorem 1, the Aleph-function reduces to the I-function* [16], *and we obtain*

$$\int_{-1}^{1} \frac{(1-x^2)^{s-\frac{1}{2}}}{(1+2ax+a^2)^s (1+2bx+b^2)^s} E^{\delta,\zeta,q}_{\mu,\rho,p}\left(yX^{c}\right) I\left(zX^e\right) \mathrm{d}x =$$

$$= \sqrt{\pi} \sum_{\mathbf{n},\,\mathbf{r}=0}^{\infty} \frac{(\delta)_{\mathbf{n}q}\, y^{\mathbf{n}}}{\Gamma(\mu\mathbf{n}+\rho)\,\mathbf{n}!\,(\zeta)_{p\mathbf{n}}} \frac{(ab)^{\mathbf{r}}}{\mathbf{r}!} \times$$

$$\times\ I^{m,n+3}_{p_i+3,q_i+3;r}\left(z \,\middle|\, \begin{matrix} A_1, (a_j, A_j)_{1,n}, (a_{ji}, A_{ji})_{n+1,p_i} \\ B_1, (b_j, B_j)_{1,m}, (b_{ji}, B_{ji})_{m+1,q_i} \end{matrix}\right), \quad (25)$$

under the same notation and conditions a) – b) of (16). *Also,* $|\arg(zX^e)| < \frac{1}{2}\pi\varphi_i'$, $(i = 1\ldots, r)$, *where*

$$\varphi_i' = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q_i} B_{ji} - \sum_{j=n+1}^{p_i} A_{ji}.$$

Corollary 2. *If we set* $r = 1$ *in the above corollary, then the* I*-function reduces to Fox's* H*-function* [4]*:*

$$\int_{-1}^{1} \frac{(1-x^2)^{s-\frac{1}{2}}}{(1+2ax+a^2)^s (1+2bx+b^2)^s} E^{\delta,\zeta,q}_{\mu,\rho,p}(yX^c)\ H(zX^e)\,\mathrm{d}x =$$

$$= \sqrt{\pi} \sum_{\mathbf{n},\,\mathbf{r}=0}^{\infty} \frac{(\delta)_{\mathbf{n}q}\, y^{\mathbf{n}}}{\Gamma(\mu\mathbf{n}+\rho)\,\mathbf{n}!\,(\zeta)_{p\mathbf{n}}} \frac{(ab)^{\mathbf{r}}}{\mathbf{r}!}\ H^{m,\,n+3}_{p+3,\,q+3}\left(z \,\middle|\, \begin{matrix} A_1, (a_j, A_j)_{1,\,p} \\ B_1, (b_j, B_j)_{1,\,q} \end{matrix}\right), \quad (26)$$

under the same conditions a) – b), stated in (16), *and* $|\arg(zX^e)| < \frac{1}{2}\varphi$, *where* $\varphi = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j - \sum_{j=n+1}^{p} A_j$.

Remark 3. *By using similar methods, we can obtain the relation analogous to the* I*-function of one variable defined by Rathie* [14]*, the* $\overline{H}$*-function defined by Buschman and Srivastava* [3].

If we take $A_j\ (j = 1, \cdots, p) = B_j\ (j = 1, \ldots, q) = 1$, then Fox's H-function reduces to Meijer's G-function [8]. We have

Corollary 3.

$$\int_{-1}^{1} \frac{(1-x^2)^{s-\frac{1}{2}}}{(1+2ax+a^2)^s (1+2bx+b^2)^s} E^{\delta,\,\zeta,\,q}_{\mu,\,\rho,\,p}(yX^c)\ G(zX^e)\ \mathrm{d}x =$$

$$= \sqrt{\pi} \sum_{\mathbf{n},\,\mathbf{r}=0}^{\infty} \frac{(\delta)_{\mathbf{n}q}\, y^{\mathbf{n}}}{\Gamma(\mu\mathbf{n}+\rho)\,\mathbf{n}!\,(\zeta)_{p\mathbf{n}}} \frac{(ab)^{\mathbf{r}}}{\mathbf{r}!}\ G^{m,\,n+3}_{p'+3,\,q'+3}\left(z \,\middle|\, \begin{matrix} A_1, (a_j)_{1,\,p'} \\ (b_j)_{1,\,q'}, B_1 \end{matrix}\right), \quad (27)$$

where A_1, B_1 and X are the same as given in Theorem 1. Formula (27) holds under the following assumptions [8]:

a) $0 \leqslant m \leqslant q'$ *and* $0 \leqslant n \leqslant p'$, *where* m,n,p' *and* q' *are integers.*

b) $a_k - b_j \neq 1, 2, 3, \ldots$ $(k = 1, \ldots, n;\ j = 1, \ldots, m)$, *which implies that no pole of any* $\Gamma\left(b_j - s\right)$ $(j = 1, \ldots, m)$ *coincides with any pole of* $\Gamma\left(1 - b_k + s\right)$ $(k = 1, \ldots, n)$.

c) $z \neq 0$, δ, ζ, μ, $\rho \in \mathbb{C}$, $\mathrm{Re}(\delta)$, $\mathrm{Re}(\mu)$, $\mathrm{Re}(\rho)$, $\mathrm{Re}(\zeta)$, $p, q > 0$, *such that* $q \leqslant \mathrm{Re}(\mu) + p$, $\min\{a, b, c, e\} > 0$.

The generalized M-L function introduced by Salim and Faraj [15] reduces to the function given by Shukla and Prajapati [18]; then we have

Corollary 4.

$$\int\limits_{-1}^{1} \frac{\left(1-x^2\right)^{s-\frac{1}{2}}}{\left(1+2ax+a^2\right)^s\left(1+2bx+b^2\right)^s}\, E_{\alpha,\beta}^{\delta,q}\left(yX^c\right) \aleph\left(zX^e\right) \mathrm{d}x =$$

$$= \sqrt{\pi} \sum_{\mathbf{n},\mathbf{r}=0}^{\infty} \frac{(\delta)_{\mathbf{n}q}\, y^{\mathbf{n}}}{\Gamma\left(\alpha\mathbf{n}+\beta\right)\mathbf{n}!} \frac{(ab)^{\mathbf{r}}}{\mathbf{r}!} \times$$

$$\times\ \aleph_{p_i+3,\, q_i+3,\, c_i; r}^{\,m, n+3}\left(z \,\middle|\, \begin{matrix} A_1, \left(a_j, A_j\right)_{1,\mathbf{n}}, \left[c_i\left(a_{ji}, A_{ji}\right)\right]_{\mathbf{n}+1,\, p_i; r}, \\ \\ \left(b_j, B_j\right)_{1,m}, \left[c_i\left(b_{ji}, B_{ji}\right)\right]_{m+1,\, q_i; r}, B_1 \end{matrix}\right), \tag{28}$$

where quantities A_1, B_1 and X are the same as given in Theorem 1. Also, provided that

a) $\mathrm{Re}\left(\alpha\right), \mathrm{Re}\left(\beta\right), \mathrm{Re}\left(\delta\right) > 0$, $q \in (0, 1) \cup \mathbb{N}$.

b) $\mathrm{Re}\left(t + \mathbf{n}c\right) + e \min\limits_{1 \leqslant j \leqslant m} \mathrm{Re}\left[\left(\dfrac{b_j^{(i)}}{B_j^{(i)}}\right)\right] > -\frac{1}{2}$.

c) $\left|\arg\left(zX^e\right)\right| < \frac{1}{2}\varphi_i\pi$, *where* φ_i *is defined by the equation* (10).

The M-L function, defined by Shukla and Prajapati [18], reduces to a generalization of Mittag-Leffler function (3) introduced by Prabhakar [11], and we obtain the following formula:

Corollary 5.

$$\int\limits_{-1}^{1} \frac{\left(1-x^2\right)^{s-\frac{1}{2}}}{\left(1+2ax+a^2\right)^s\left(1+2bx+b^2\right)^s}\, E_{\alpha,\beta}^{\delta}\left(yX^c\right) \times$$

$$\times\ \aleph^{m,\,n}_{p_i,\,q_i,\,c_i;r}\left(zX^e\,\middle|\,\begin{array}{l}(a_j,A_j)_{1,\mathbf{n}}\,,\left[c_i\left(a_{ji},\,A_{ji}\right)\right]_{\mathbf{n}+1,\,p_i;r}\\ (b_j,\,B_j)_{1\,,m}\,,\left[c_i\left(b_{ji},\,B_{ji}\right)\right]_{m+1,\,q_i;r}\end{array}\right)\mathrm{d}x=$$

$$=\sqrt{\pi}\sum_{\mathbf{n},\mathbf{r}=0}^{\infty}\frac{(\delta)_{\mathbf{n}}\,y^{\mathbf{n}}}{\Gamma\left(\alpha\mathbf{n}+\beta\right)\mathbf{n}!}\frac{(ab)^{\mathbf{r}}}{\mathbf{r}!}\times$$

$$\times\ \aleph^{m,\,n+3}_{p_i+3,\,q_i+3,\,c_i;r}\left(z\,\middle|\,\begin{array}{l}A_1,(a_j,A_j)_{1,\mathbf{n}}\,,\left[c_i\left(a_{ji},\,A_{ji}\right)\right]_{\mathbf{n}+1,\,p_i;r}\,,\\ (b_j,\,B_j)_{1,\,m}\,,\left[c_i\left(b_{ji},\,B_{ji}\right)\right]_{m+1,\,q_i;r}\,,B_1\end{array}\right),\tag{29}$$

where, $\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\delta)>0.$ *Also the existence condition b) and c) are satisfied, as stated in above corollary.*

Next, by taking $c_i\to 1$ in Theorem 2, the Aleph-function reduces to the I-function, and we obtain

Corollary 6.

$$\int_{-1}^{1}\frac{\left(1-x^2\right)^{s-\frac{1}{2}}}{\left(1+2ax+a^2\right)^s\left(1+2bx+b^2\right)^s}\,E^{\delta,\,\zeta,\,q}_{\mu,\,\rho,\,p}\left(yX^{-c}\right)\,I\left(zX^e\right)\mathrm{d}x=$$

$$=\sqrt{\pi}\sum_{\mathbf{n},\,\mathbf{r}=0}^{\infty}\frac{(\delta)_{\mathbf{n}q}\,y^{\mathbf{n}}}{\Gamma\left(\mu\mathbf{n}+\rho\right)\mathbf{n}!\,(\zeta)_{p\mathbf{n}}}\frac{(ab)^{\mathbf{r}}}{\mathbf{r}!}\times$$

$$\times\ I^{m,\,n+3}_{p_i+3,\,q_i+3;r}\left(z\,\middle|\,\begin{array}{l}A_2,(a_j,A_j)_{1,\,\mathbf{n}}\,,(a_{ji},A_{ji})_{\mathbf{n}+1,\,p_i}\\ (b_j,B_j)_{1,\,m}\,,(b_{ji},B_{ji})_{m+1,\,q_i}\,,B_2\end{array}\right),\tag{30}$$

under the same conditions that a) and b) that (24)*, and* $|\arg(zX^e)|<$ $<\frac{1}{2}\varphi_i'\pi$*, where* $\varphi_i'=\sum_{j=1}^{M}B_j+\sum_{j=1}^{N}A_j-\left(\sum_{j=M+1}^{Q_i}B_{ji}+\sum_{j=N+1}^{P_i}A_{ji}\right)>0$ $(i=1,\ldots,r)$.

If we, further, take $r=1$ in the above corollary 6, then the I-function of one variable reduces to Fox's H-function, and we have

Corollary 7.

$$\int_{-1}^{1}\frac{\left(1-x^2\right)^{s-\frac{1}{2}}}{\left(1+2ax+a^2\right)^s\left(1+2bx+b^2\right)^s}\,E^{\delta,\,\zeta,\,q}_{\mu,\,\rho,\,p}\left(yX^{-c}\right)\,H\left(zX^e\right)\mathrm{d}x=$$

$$= \sqrt{\pi} \sum_{\mathbf{n},\,\mathbf{r}=0}^{\infty} \frac{(\delta)_{\mathbf{n}q}\, y^{\mathbf{n}}}{\Gamma\left(\mu\mathbf{n}+\rho\right) \mathbf{n}!\, (\zeta)_{p\mathbf{n}}} \frac{(ab)^{\mathbf{r}}}{\mathbf{r}!} H^{m,\,n+3}_{p+3,\,q+3}\left(z \,\middle|\, \begin{matrix} A_2, (a_j, A_j)_{1,\,p} \\ (b_j, B_j)_{1,\,q}, B_2 \end{matrix} \right), \tag{31}$$

under the same condition a) that stated in (24). *Also the following conditions are satisfied:*
$\operatorname{Re}(t - \mathbf{n}c) + e \min_{1\leqslant j\leqslant m} \operatorname{Re}\left[\left(\frac{b_j}{B_j}\right)\right] > -\frac{1}{2}$, *and* $|\arg(zX^e)| < \frac{1}{2}\varphi_i''\pi$, *where*
$\varphi_i'' = \sum_{j=1}^{M} B_j + \sum_{j=1}^{N} A_j - \left(\sum_{j=M+1}^{Q} B_j + \sum_{j=N+1}^{P} A_j\right) > 0.$

4. Concluding Remarks. The Aleph-function is a transcendental function of very general character and may be specialized to include most of special functions of one variable occurring in mathematical physics and chemistry. We conclude our present study by remarking that several further consequences of our results can easily be derived by using some known and new relationships between Aleph-functions, which is an elegant unification of various special functions, such as the H-function and the I-functions (see, [4], [8], [16]), after some suitable parametric replacements. Secondly, specializing the parameters of the generalized Mittag-Leffler function, we can obtain others special functions. The integral obtained in this paper is of the general nature, we can get other known and new integrals by specializing different parameters. One can obtain new definite integrals involving more generalized special functions: this may have many applications in physics and engineering science.

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