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INEQUALITIES FOR THE NORM AND NUMERICAL RADIUS FOR HILBERT C^*-MODULE OPERATORS

Abstract. In this paper, we introduce some inequalities between the operator norm and the numerical radius of adjointable operators on Hilbert C^*-module spaces. Moreover, we establish some new refinements of numerical radius inequalities for Hilbert space operators. More precisely, we prove that if $T \in B(H)$ and

$$\min\Big(\frac{\|T+T^*\|^2}{2}, \frac{\|T-T^*\|^2}{2}\Big) \leq \max\Big(\inf_{\|x\|=1}\|Tx\|^2, \inf_{\|x\|=1}\|T^*x\|^2\Big),$$

then

$$\|T\|\leq \sqrt{2}\omega(T);$$

this is a considerable improvement of the classical inequality

$$\|T\|\leq 2\omega(T).$$

Key words: *Bounded linear operator, Hilbert space, Norm inequality, Numerical radius*

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1. Introduction and preliminaries. Let $B(H)$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$. If $\dim H = n$, we identify $B(H)$ with the space M_n of all $n \times n$ matrices with entries in the complex field. For $T \in B(H)$, let $\|T\|$ denote the usual operator norm and

$$\omega(T) = \sup\{|\langle Tx, x\rangle|: \|x\|= 1\}$$

denote the numerical radius of T. It is well known that $\omega(\cdot)$ is a norm on $B(H)$ and that

$$\frac{\|T\|}{2} \leq \omega(T) \leq \|T\| \tag{1}$$

for all $T \in B(H)$. The first inequality becomes an equality if $T^2 = 0$ (use the first Kittaneh inequality below). The second inequality becomes an equality if T is normal. Recently, the authors of [13] tried to show that $\|T\| \leq \sqrt{2}\omega(T)$ holds, whenever T is an invertible operator. However, Cain [1] constructed some counterexamples. Several numerical radius inequalities improving those in (1) have been recently given in [2–5], [11], and [14]. For instance, Dragomir proved that

$$\omega^2(T) \leq \frac{1}{2}(\omega(T^2) + \|T\|^2)$$

for any $T \in B(H)$. And Kittaneh proved that, for any $T \in B(H)$,

$$\omega(T) \leq \frac{1}{2}(\|T\| + \|T^2\|^{\frac{1}{2}})$$

and

$$\frac{\|TT^* + T^*T\|}{4} \leq \omega^2(T) \leq \frac{\|TT^* + T^*T\|}{2}.$$

Theese inequalities can be found in [9], [10], respectively. Furthermore, Holbrook in [7] showed that, for any $R, S \in B(H)$,

$$\omega(RS) \leq 4\omega(R)\omega(S), \tag{2}$$

and

$$\omega(RS) \leq 2\omega(R)\omega(S), \tag{3}$$

when $RS = SR$.

See [6] for other results and historical comments on the numerical radius. Now, here is a reminder of the definition of a Hilbert module, according to [12].

Let $\mathcal{A}$ be a C^*-algebra (not necessarily unital or commutative). An inner-product $\mathcal{A}$-module is a linear space E, which is a right $\mathcal{A}$-module (with a compatible scalar multiplication: $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for all $x \in E$, $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$), together with a map $\langle \cdot, \cdot \rangle : E \times E \longrightarrow \mathcal{A}$, such that

(i) $\langle x, x \rangle \geqq 0$, meaning it is one of the positive operators in $\mathcal{A}$; $\langle x, x \rangle = 0$ iff $x = 0$,

(ii) $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$,

(iii) $\langle x, ya \rangle = \langle x, y \rangle a$,

(iv) $\langle x, y\rangle = \langle y, x\rangle^*$,

for all $x, y, z \in E, a \in \mathcal{A}, \lambda \in \mathbb{C}$.

For $x \in E$, we write $\|x\|= \|\langle x, x\rangle\|^{\frac{1}{2}}$. An inner-product $\mathcal{A}$-module that is complete with respect to its norm is called a Hilbert $\mathcal{A}$-module, or a Hilbert C^*-module, over the C^*-algebra $\mathcal{A}$. We denote, by $L(E)$, the C^*-algebra of all adjointable operators on E (i. e., of all maps $T : E \longrightarrow E$, such that there exists a $T^* : E \longrightarrow E$ with the property $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$, for all $x, y \in E$) and let $L^{-1}(E)$ denote the set of all invertible operators in $L(E)$.

Definition 1. *For $T \in L(E)$, let*

$$\delta(T) = \sup\{\|\langle Tx, x\rangle\| : \|x\|= 1\},$$
$$\|T\|= \sup\{\|Tx\| : \|x\|= 1\},$$

respectively, denote the numerical radius and operator norm of T.

Recently, in [15], we have shown that

$$\|T\|\leq 2\delta(T), \tag{4}$$

and

$$\delta(TS) \leq 4\delta(T)\delta(S). \tag{5}$$

We are able to improve the inequalities (4) and (5). The results in this paper considerably improve inequalities (1) and (2).

2. Main results. Let $T \in L(E)$. For the sake of convenience, we prepare the following notation:

$$m(T) = \min\left(\frac{\|T - T^*\|^2}{2}, \frac{\|T + T^*\|^2}{2}\right)$$

and

$$M(T) = \max\left(\inf_{\|x\|=1}\|Tx\|^2, \inf_{\|x\|=1}\|T^*x\|^2\right).$$

In order to derive our main results, we need the following lemmas.

Lemma 1. *If $T \in L(E)$ is self-adjoint, then*

$$\delta(T) = \|T\|. \tag{6}$$

Proof. First, we show that the result holds for positive operators.

Let $G \in L(E)$ be positive. Since L(E) is a C^*-algebra, we know that $\|G^*G\|= \|G\|^2$. Then,

$$\|G^*G\|= \sup_{\|x\|=1} \|\langle G^*Gx, x\rangle\|.$$

Replacing G by $\sqrt{G}$ gives

$$\|G\|= \sup_{\|x\|=1} \|\langle Gx, x\rangle\|. \tag{7}$$

Now, let $T \in L(E)$ be just self-adjoint. By Proposition 1.1 in [12],

$$\delta(T) = \sup_{\|x\|=1} \|\langle Tx, x\rangle\| \leq \|T\|. \tag{8}$$

On the other hand, being self-adjoint, T can be decomposed: $T = T_+ - T_-$, such that T_+ and T_- are both positive and $T_+T_- = T_-T_+ = 0$, and also $\|T\|= \max(\|T_+\|, \|T_-\|)$. Note that

$$\sup_{\|x\|=1} \|\langle T_+^3 x, x\rangle\|= \|T_+^3\|, \qquad \text{(by (7))};$$

then there exist a sequence $\{x_n\}$ of unit vectors in E, such that

$$\|T_+^3\|= \lim_{n\to\infty} \|\langle T_+^3 x_n, x_n\rangle\|.$$

Therefore,

$$\begin{aligned}\sup_{\|x\|=1} \|\langle Tx, x\rangle\| &\geq \Big\|\Big\langle T\Big(\frac{T_+x_n}{\|T_+x_n\|}\Big), \ \frac{T_+x_n}{\|T_+x_n\|}\Big\rangle\Big\| = \\ &= \Big\|\Big\langle (T_+ - T_-)\Big(\frac{T_+x_n}{\|T_+x_n\|}\Big), \ \frac{T_+x_n}{\|T_+x_n\|}\Big\rangle\Big\| = \\ &= \frac{\|\langle T_+^3 x_n, \ x_n\rangle\|}{\|T_+x_n\|^2} \geq \frac{\|\langle T_+^3 x_n, x_n\rangle\|}{\|T\|^2}\end{aligned}$$

and so:

$$\sup_{\|x\|=1} \|\langle Tx, x\rangle\| \geq \frac{\|T_+\|^3}{\|T\|^2} = \lim_{n\to\infty} \frac{\|\langle T_+^3 x_n, x_n\rangle\|}{\|T\|^2}. \tag{9}$$

Similarly,

$$\sup_{\|x\|=1} \|\langle Tx, x\rangle\| \geq \frac{\|T_-\|^3}{\|T\|^2}. \tag{10}$$

By (9) and (10),

$$\delta(T) = \sup_{\|x\|=1} \|\langle Tx, x\rangle\| \geq \max\left(\frac{\|T_+\|^3}{\|T\|^2}, \frac{\|T_-\|^3}{\|T\|^2}\right) = \|T\|. \tag{11}$$

The result follows from inequalities (8) and (11). □

Lemma 2. *If $T \in L(E)$, then*

(a) $m(T) \leq 2\delta^2(T)$.

(b) $M(T) = \dfrac{1}{\|T^{-1}\|^2}$, *if T is invertible.*

Proof. (a) Since $T + T^*$ is self-adjoint, from Lemma 1 we have:

$$\|T + T^*\| = \delta(T + T^*)$$

So,

$$\frac{\|T + T^*\|^2}{2} = \frac{(\delta(T + T^*))^2}{2} \leq \frac{(\delta(T) + \delta(T^*))^2}{2} = 2\delta^2(T).$$

Consequently,

$$\frac{\|T + T^*\|^2}{2} \leq 2\delta^2(T). \tag{12}$$

Since $m(T) \leq \dfrac{\|T + T^*\|^2}{2}$, the result follows from (12).

(b) See [8, p. 41]. □

Lemma 3. *Let E be a Hilbert C^*- module. Then*

$$\|\langle a, a\rangle + \langle b, b\rangle\| \leq \frac{1}{2}(\|a + b\|^2 + \|a - b\|^2), \tag{13}$$

for any $a, b \in E$.

Proof. Suppose that $a, b \in E$; then

$$\langle a + b, a + b\rangle = \langle a, a\rangle + \langle a, b\rangle + \langle b, a\rangle + \langle b, b\rangle,$$

$$\langle a - b, a - b\rangle = \langle a, a\rangle - \langle a, b\rangle - \langle b, a\rangle + \langle b, b\rangle.$$

Thus,

$$\langle a + b, a + b\rangle + \langle a - b, a - b\rangle = 2(\langle a, a\rangle + \langle b, b\rangle).$$

Therefore,

$$2\|\langle a, a\rangle + \langle b, b\rangle\| \leq \|a + b\|^2 + \|a - b\|^2.$$

This completes the proof. □

Theorem 1. *If $T \in L(E)$ be such that*

$$\inf_{\|x\|=1} \|Tx\|^2 + \|T^*x\|^2 \leq \|\langle Tx, Tx\rangle + \langle T^*x, T^*x\rangle\|$$

and

$$\inf_{\|x\|=1} \|T^*x\|^2 + \|Tx\|^2 \leq \|\langle Tx, Tx\rangle + \langle T^*x, T^*x\rangle\|$$

for all $x \in E$ with $\|x\| = 1$; then

$$\|T\|^2 + M(T) - m(T) \leq 2\delta^2(T). \tag{14}$$

Proof. Suppose that $u \in E$ with $\|u\| = 1$. Choose $a = Tu, b = T^*u$ in (13) to give

$$\|\langle Tu, Tu\rangle + \langle T^*u, T^*u\rangle\| \leq \frac{1}{2}(\|Tu + T^*u\|^2 + \|Tu - T^*u\|^2). \tag{15}$$

By the assumption, $\inf_{\|x\|=1} \|Tx\|^2 + \|T^*u\|^2 \leq \|\langle Tu, Tu\rangle + \langle T^*u, T^*u\rangle\|$ gives

$$\inf_{\|x\|=1} \|Tx\|^2 + \|T^*u\|^2 \leq \frac{1}{2}\Big(\|Tu - T^*u\|^2 + \|Tu + T^*u\|^2\Big). \qquad \text{(by (15))}$$

Taking the supremum over $u \in E$ with $\|u\| = 1$ gives

$$\inf_{\|x\|=1} \|Tx\|^2 + \|T\|^2 \leq \frac{1}{2}\Big(\|T - T^*\|^2 + \|T + T^*\|^2\Big). \qquad \text{(since } \|T\| = \|T^*\|)$$

Since $(T + T^*)$ is self-adjoint, (6) yields

$$\|T + T^*\| \leq 2\delta(T).$$

Therefore,

$$\inf_{\|x\|=1} \|Tx\|^2 + \|T\|^2 \leq 2\delta^2(T) + \frac{\|T - T^*\|^2}{2}. \tag{16}$$

Similarly, by the assumption,

$$\inf_{\|x\|=1} \|T^*x\|^2 + \|Tx\|^2 \leq \|\langle Tx, Tx\rangle + \langle T^*x, T^*x\rangle\|,$$

gives

$$\inf_{\|x\|=1} \|T^*x\|^2 + \|T\|^2 \leq 2\delta^2(T) + \frac{\|T - T^*\|^2}{2}$$

and, so,

$$\|T\|^2+M(T)-\frac{\|T-T^*\|^2}{2}\leq 2\delta^2(T). \qquad \text{(by (16))}.$$

Replacing T by iT in the last inequality gives

$$\|T\|^2+M(T)-\frac{\|T+T^*\|^2}{2}\leq 2\delta^2(T).$$

Thus,

$$\|T\|^2+M(T)-\min\left(\frac{\|T-T^*\|^2}{2},\frac{\|T+T^*\|^2}{2}\right)\leq 2\delta^2(T),$$

which is exactly the desired result. □

The following particular case is of interest.

Corollary 1. *Let T be as in Theorem 1. If, in addition, $T\in L^{-1}(E)$, then*

$$\|T\|^2+\frac{1}{\|T^{-1}\|^2}-m(T)\leq 2\delta^2(T). \tag{17}$$

Proof. Result follows immediately from Theorem 1 and Lemma 2(b), since T is invertible. □

Our next corollary includes a refinement of the inequality (5).

Corollary. *Let R, S be as in Theorem 1. Then*

$$\delta(RS)\leq\sqrt{\left(2\delta^2(R)-M(R)+m(R)\right)\left(2\delta^2(S)-M(S)+m(S)\right)}\leq$$
$$\leq 4\delta(R)\delta(S).$$

Proof. By Lemma 2(a),

$$m(R)\leq 2\delta^2(R)$$

and so

$$\|R\|\leq\sqrt{2\delta^2(R)-M(R)+m(R)}\leq 2\delta(R). \qquad \text{(by (14))}$$

Similarly,

$$\|S\|\leq\sqrt{2\delta^2(S)-M(S)+m(S)}\leq 2\delta(S).$$

Therefore,

$$\begin{aligned}\delta(RS) &\leq \|R\|\|S\|\leq \\ &\leq \sqrt{\left(2\delta^2(R)-M(R)+m(R)\right)\left(2\delta^2(S)-M(S)+m(S)\right)} \leq \\ &\leq 4\delta(R)\delta(S).\end{aligned}$$

□

The following applications of Theorem 1 improve inequality (4) for some invertible operators.

Corollary. *Let $R, S \in L^{-1}(E)$ and satisfy the condition of Theorem 1. If $m(R) \leq \|R^{-1}\|^{-2}$ and $m(S) \leq \|S^{-1}\|^{-2}$, then*

$$\|R\|\leq \sqrt{2}\delta(R), \tag{18}$$

$$\delta(RS) \leq 2\delta(R)\delta(S). \tag{19}$$

Proof. Inequality (18) follows from Lemma 2(b) and corollary 1. Similarly,

$$\|S\|\leq \sqrt{2}\delta(S). \tag{20}$$

For inequality (19), observe, using $\delta(RS) \leq \|RS\|$ in the first inequality and (18) and (20) in the third, that

$$\delta(RS) \leq \|RS\|\leq \|R\|\|S\|\leq 2\delta(R)\delta(S).$$

This completes the proof. □

3. New inequalities for Hilbert operators. Since a Hilbert space is a Hilbert $\mathbb{C}^*$-module, the results in section 2 of this paper hold in $B(H)$.

Theorem 2. *If $T \in B(H)$ and $0 < \|T\|^2 + M(T) - m(T)$, then*

$$\|T\|\leq \sqrt{\frac{2}{1+M\left(\frac{T}{\|T\|}\right)-m\left(\frac{T}{\|T\|}\right)}}\ \omega(T). \tag{21}$$

Proof. According to Definition 1, we have $\delta(T) = \omega(T)$.

Replacing T by $\frac{T}{\|T\|}$ in (14) gives

$$\|T\|^2\left(1+M\left(\frac{T}{\|T\|}\right)-m\left(\frac{T}{\|T\|}\right)\right) \leq 2\omega^2(T).$$

Since $\|T\|^2 + M(T) - m(T) > 0$,

$$\|T\|^2 \leq \frac{2}{1 + M(\frac{T}{\|T\|}) - m(\frac{T}{\|T\|})}\,\omega^2(T),$$

which is exactly the desired result. □

In the next result, we provide some conditions for the inequality $\|T\| \leq \sqrt{2}\,\omega(T)$ to be true.

Corollary. *If $T \in B(H)$ and $M(T) \geq m(T)$, then*

$$\|T\| \leq \sqrt{2}\,\omega(T).$$

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