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PERIODIC BOUNDARY VALUE PROBLEMS FOR FRACTIONAL IMPLICIT DIFFERENTIAL EQUATIONS INVOLVING HILFER FRACTIONAL DERIVATIVE

Abstract. In this paper, a new class of the periodic boundary value problem for nonlinear implicit fractional differential equations involving Hilfer fractional derivative is considered in the weighted space of functions. We establish sufficient conditions for existence, uniqueness, Ulam-Hyers and Ulam-Hyers-Rassias stability of the given problem. The main results are based upon the technique of the Schaefer fixed point theorem, the Banach fixed point theorem, generalized Gronwall inequality, and with the help of some properties of Mittag-Leffler functions. An example is presented to illustrate our main results.

Key words: *fractional differential equations, fractional derivatives, Ulam stability, fixed point theorem, Mittag-Leffler function*

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1. Introduction. Fractional differential equations (FDEs) have recently confirmed to be significant tools in modeling many phenomena in various fields of engineering and science. Their non-local property is suitable for description memory phenomena, such as non-local elasticity, polymers, propagation in complex media, biological, electrochemistry, porous media, viscoelasticity, electromagnetics, etc. (see [11, 19] and references therein). In the recent years, there has been considerable growth in ordinary and partial differential equations, involving Riemann-Liouville, Caputo, and Hilfer fractional derivatives. For details, we refer the reader to monographs of Kilbas et al. [25], Miller and Ross [26], Samko et al. [30], Hilfer [22], Podlubny [28]. The implicit fractional differential equations

(IFDEs) represent a very important class of FDEs. This article is motivated by the importance of implicit ordinary differential equation (IODE) of the form

$$f(t, y(t), y'(t), \dots, y^{(n-1)}(t)) = 0. \tag{1}$$

under different initial and boundary conditions. This kind of equation is important in many disciplines in different fields, such as engineering, physics, chemistry, aerodynamics, polymer rheology, acoustic control, viscoelasticity, and so on. The pair order (α, β) of a fractional derivative ${}^{H}D_{0^+}^{\alpha,\beta}$ ([22]) grants one to interpolate between the Caputo and the Riemann – Liouville derivatives described in [25, 28, 30]. These parameters produce more types of steady states and provide an additional degree of freedom on the initial and boundary conditions. Systems that rely on these derivatives are considered in [1–5, 7, 9, 16, 17, 20–23, 31, 35] and references cited therein. IFDEs have been studied by many researchers, see [2, 6, 12–15, 32, 33]. The stability analysis is very important and it has many applications, such as numerical analysis, optimization, etc. The Ulam-type stability problems have been considered by a large number of mathematicians, for more details see [5, 8, 10, 24, 27, 29]. Recently, Gao et al., in [18] established existence and uniqueness of solutions to the Hilfer non-local boundary-value problem. He used some properties of Hilfer fractional derivative, Mittag-Leffler functions, and fixed-point methods to obtain the existence and uniqueness results. On the other hand, Vivek et al. [34] investigated existence, uniqueness, and stability results for IFDE

$$\begin{aligned} D_{0^+}^{\alpha,\beta} x(t) &= f(t, x(t), D_{0^+}^{\alpha,\beta} x(t)), \quad t \in J := [0, T] \\ I_{0^+}^{1-\gamma} x(0) &= \sum_{i=1}^{m} c_i x(\tau_i), \end{aligned}$$

where $D_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative of order $0 < \alpha < 1$ and type of $0 < \beta < 1$, $\tau_i \in [0, T]$. The obtained results is based on the fixed-point theorems of Schaefer and Banach, and the Gronwall inequality.

The aim of this paper is to study existence, uniqueness, and different types of stabilities of solutions for the following problem:

$${}^{H}D_{0^+}^{\alpha,\beta} y(t) - \lambda y(t) = f(t, y(t), {}^{H}D_{0^+}^{\alpha,\beta} y(t)), \quad t \in (0, b], \tag{2}$$

$$I_{0^+}^{1-\gamma} y(0) = I_{0^+}^{1-\gamma} y(b), \tag{3}$$

where ${}^{H}D_{0^+}^{\alpha,\beta}$ denotes the Hilfer fractional derivative of order $\alpha \in (0, 1)$ and type $\beta \in [0, 1]$, $I_{0^+}^{1-\gamma}$ is the Reimann-Liouville fractional integral of order $1-\gamma$, $\gamma=\alpha+\beta(1-\alpha)$, $\lambda<0$, and $f:(0,b]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given function that satisfies some assumptions specified later.

This paper is organized as follows. In Section 2, we recall the basic definitions and lemmas used throughout this paper. In Section 3, we study existence, uniqueness, and stability results of the Hilfer fractional implicit differential equation by using some fixed-point theorems of Schaefer and Banach and the generalized Gronwall inequality. In the last Section, we give an example to illustrate our results.

2. Preliminaries. Let $C\left([0,b],\mathbb{R}\right)$ be the Banach space of all continuous function on $[0,b]$ into $\mathbb{R}$ with the norm $\|y\|=\max\{|y(t)|:t\in[0,b]\}$. We define the weighted spaces $C_{1-\gamma}\left([0,b],\mathbb{R}\right)$, and $C^n_{1-\gamma}\left([0,b],\mathbb{R}\right)$ by

$$C_{1-\gamma}\left([0,b],\mathbb{R}\right)=\left\{y:[0,b]\to\mathbb{R};t^{1-\gamma}y(t)\in C\left([0,b],\mathbb{R}\right)\right\},$$

and

$$C^n_{1-\gamma}\left([0,b],\mathbb{R}\right)=\left\{y\in C^{n-1}\left([0,b],\mathbb{R}\right):y^{(n)}\in C_{1-\gamma}\left([0,b],\mathbb{R}\right)\right\},$$

Obviously, $C_{1-\gamma}\left([0,b],\mathbb{R}\right)$ and $C^n_{1-\gamma}\left([0,b],\mathbb{R}\right)$ are Banach spaces with the norms

$$\|y\|_{c_{1-\gamma}}=\max_{t\in[0,b]}\left|t^{1-\gamma}y(t)\right|,$$

and

$$\|y\|_{C^n_{1-\gamma}}=\sum_{k=0}^{n-1}\left\|y^{(k)}\right\|_C+\left\|y^{(n)}\right\|_{C_{1-\gamma}},n\in\mathbb{N},$$

respectively. Here we have $C^0_{1-\gamma}\left([0,b],\mathbb{R}\right)=C_{1-\gamma}\left([0,b],\mathbb{R}\right)$. In the forthcoming analysis, we need the following space:

$$C^\gamma_{1-\gamma}\left([0,b],\mathbb{R}\right)=\left\{y\in C_{1-\gamma}\left([0,b],\mathbb{R}\right),\ D^\gamma_{0^+}y\in C_{1-\gamma}\left([0,b],\mathbb{R}\right)\right\},\qquad(4)$$

Definition 1. [25] *The left-sided Riemann-Liouville fractional integral of order $\alpha>0$ with the zero lower limit for a function $y:\mathbb{R}^+\longrightarrow\mathbb{R}$ is defined by*

$$(I^\alpha_{0^+}y)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds,\quad t>0,$$

provided that the right-hand side is pointwise on $\mathbb{R}^+$, where Γ is the gamma function.

Definition 2. [25] *The left-sided Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ with the lower limit zero for a function $y : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is defined by*

$$D^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{\alpha-1}y(s)\,ds \quad t > 0,$$

provided that the right-hand side is pointwise on $\mathbb{R}^+$.

Definition 3. [25] *The left-sided Caputo fractional derivative of order $0 < \alpha < 1$ with the lower limit zero for a differentiable function $y : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is given by*

$${}^{c}D^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{\alpha-1}y'(s)ds.$$

Definition 4. [22] *The left-sided Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 \le \beta \le 1$ with the lower limit zero of a function $y : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is given by*

$$D^{\alpha,\beta}_{0+}y(t) = I^{\beta(1-\alpha)}_{0+}DI^{(1-\beta)(1-\alpha)}_{0+}y(t),$$

where $D = \frac{d}{dt}$. One has

$$D^{\alpha,\beta}_{0+}y(t) = I^{\beta(1-\alpha)}_{0+}D^{\gamma}_{0+}y(t), \tag{5}$$

where

$$D^{\gamma}_{0+}y(t) = DI^{1-\gamma}_{0+}y(t), \ \gamma = \alpha + \beta(1-\alpha).$$

Lemma 1. [17, Lemma 20] *Let $\alpha > 0$, $\beta > 0$, and $\gamma = \alpha + \beta - \alpha\beta$. If $y \in C^{\gamma}_{1-\gamma}([0, b], \mathbb{R})$, then*

$$I^{\gamma}_{0+}D^{\gamma}_{0+}y = I^{\alpha}_{0+}D^{\alpha,\beta}_{0+}y, \qquad D^{\gamma}_{0+}I^{\alpha}_{0+}y = D^{\beta(1-\alpha)}_{0+}y, \qquad I^{\alpha}_{0+}I^{\beta}_{0+}y(t) = I^{\alpha+\beta}_{0+}y(t).$$

Theorem 1. [25] *Let $y \in C_{1-\gamma}([0, b], \mathbb{R})$, $\alpha > 0$, and $0 \le \beta \le 1$. Then, we have*

$${}^{H}D^{\alpha,\beta}_{0+}I^{\alpha}_{0+}y(t) = y(t).$$

Lemma 2. [25, Property 2.1, p. 74] *Let $\alpha, \sigma > 0$. Then we have*

$$I_{0^+}^{\alpha} t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\alpha+\sigma)} t^{\alpha+\sigma-1}, \quad D_{0^+}^{\alpha} t^{\alpha-1} = 0, \quad \alpha \in (0,1).$$

Lemma 3. [25] *Let* $0 < \alpha < 1$, $0 \le \gamma < 1$, *and* $y \in C_{1-\gamma}([0,b],\mathbb{R})$, $I_{0^+}^{1-\alpha} y \in C_{1-\gamma}^{1}([0,b],\mathbb{R})$. *Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} y(t) = y(t) - \frac{I_{0^+}^{1-\alpha} y(0)}{\Gamma(\alpha)} t^{\alpha-1}.$$

Lemma 4. [16, Lemma 13] *Let* $y \in C_{1-\gamma}([0,b],\mathbb{R})$, $0 < \alpha < 1$. *Then*

$$I_{0^+}^{\alpha}\, y(0) = \lim_{t \longrightarrow 0^+} I_{0^+}^{\alpha} y(t) = 0, \ 0 \le \gamma < \alpha$$

Lemma 5. [25] *Let* $\alpha > 0$, $\beta > 0$, $\gamma > 0$, *and* $\lambda \in \mathbb{R}$. *Then*

$$I_{0^+}^{\alpha} t^{\beta-1} E_{\gamma,\beta}(\lambda t^{\gamma}) = t^{\alpha+\beta-1} E_{\gamma,\alpha+\beta}(\lambda t^{\gamma}).$$

Lemma 6. [36, Lemma 2] *Let* $\alpha \in (0,2]$, *and* $\beta > 0$ *be arbitrary. The function* $E_{\alpha}(\cdot)$, $E_{\alpha,\alpha}(\cdot)$, *and* $E_{\alpha,\beta}(\cdot)$ *are non-negative, and for all* $z < 0$

$$E_{\alpha}(z) := E_{\alpha,1}(z) \le 1, \quad E_{\alpha,\alpha}(z) \le \frac{1}{\Gamma(\alpha)}, \quad E_{\alpha,\beta}(z) \le \frac{1}{\Gamma(\beta)}.$$

Moreover, for any $c < 0$ *and* $t_1, t_2 \in [0,1]$,

$$E_{\alpha,\alpha+\beta}(ct_2^{\alpha}) \longrightarrow E_{\alpha,\alpha+\beta}(ct_1^{\alpha}) \ \text{ as } t_1 \longrightarrow t_2. \tag{6}$$

Lemma 7. [18] *Let* $\alpha > 0$, $\beta > 0$, $k > 0$, $\lambda \in \mathbb{R}$, $z \in \mathbb{R}$ *and* $f \in C_{1-\gamma}([0,1],\mathbb{R})$, *then*

$$I_{0^+}^{k} \int_0^z (z-t)^{\alpha-1} E_{\alpha,\alpha}(\lambda (z-t)^{\alpha}) f(t) dt =$$

$$= \int_0^z (z-t)^{\alpha+k-1} E_{\alpha,\alpha+k}(\lambda (z-t)^{\alpha}) f(t) dt.$$

Lemma 8. [37] **The generalized Gronwall inequality.** *Let* $v, w : [0,b] \to [0,+\infty)$ *be continuous functions. If* w *is non-decreasing and there are constants* $k > 0$ *and* $0 < \alpha < 1$, *such that*

$$v(t) \le w(t) + k \int_0^t (t-s)^{\alpha-1} v(s) ds, \quad t \in [0,b],$$

then

$$v(t) \leq w(t) + \int\limits_0^t \left(\sum_{n=1}^{\infty} \frac{(k\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} w(s) \right) ds, \quad t \in [0, b].$$

Remark 1. *In particular, if $w(t)$ is a non-decreasing function on $[0, b]$, then*

$$v(t) \leq w(t) E_\alpha(k\Gamma(\alpha)(t)^\alpha).$$

3. Main Results. Here we present the existence, uniqueness, and stability theorems for solutions to Hilfer equation (2) with the periodic condition (3).

The following lemma establishes existence of a solution to the problem (2) – (3).

Lemma 9. *Let $\alpha \in (0,1)$, $\beta \in [0,1]$ and $g : (0, b] \to \mathbb{R}$ be a continuous function. Then the problem*

$$\begin{aligned} {}^H D_{0+}^{\alpha,\beta} y(t) - \lambda y(t) = g(t), & \qquad t \in (0, b], \\ I_{0+}^{1-\gamma} y(0) = I_{0+}^{1-\gamma} y(b), & \qquad \alpha \leq \gamma = \alpha + \beta - \alpha\beta \end{aligned} \tag{7}$$

is equivalent to the integral equation

$$y(t) = \frac{t^{\gamma-1} E_{\alpha,\gamma}(\lambda t^\alpha)}{1 - E_{\alpha,1}(\lambda b^\alpha)} \int\limits_0^b (b-s)^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}(\lambda (b-s)^\alpha) g(s) ds +$$
$$+ \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^\alpha) g(s) ds,$$

where $E_{\alpha,1}(\lambda b^\alpha) \neq 1$.

Proof. By [23], the solution of the following problem

$$\begin{aligned} {}^H D_{0+}^{\alpha,\beta} y(t) - \lambda y(t) = g(t), & \quad t \in (0, b], \\ I_{0+}^{1-\gamma} y(0) = y_0, & \quad \alpha \leq \gamma = \alpha + \beta - \alpha\beta < 1 \end{aligned}$$

is given by

$$y(t) = t^{\gamma-1} E_{\alpha,\gamma}(\lambda t^\alpha) I_{0+}^{1-\gamma} y(0) + \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^\alpha) g(s) ds. \tag{8}$$

Next, by multiplying both sides of (8) by the operator $I_{0^+}^{1-\gamma}$ and using Lemmas 5, 7, we get

$$I_{0^+}^{1-\gamma}y(t)=E_{\alpha,1}(\lambda t^{\alpha})I_{0^+}^{1-\gamma}y(0)+\int_0^t(t-s)^{\alpha-\gamma}E_{\alpha,\alpha-\gamma+1}(\lambda(t-s)^{\alpha})g(s)ds. \quad (9)$$

Taking the limit as $t\longrightarrow b$ in both sides of (9), we get

$$I_{0^+}^{1-\gamma}y(0)=\frac{I_{0^+}^{1-\gamma}y(b)}{E_{\alpha,1}(\lambda b^{\alpha})}-\frac{1}{E_{\alpha,1}(\lambda b^{\alpha})}\int_0^b(b-s)^{\alpha-\gamma}E_{\alpha,\alpha-\gamma+1}(\lambda(b-s)^{\alpha})g(s)ds.$$

Since $I_{0^+}^{1-\gamma}y(0)=I_{0^+}^{1-\gamma}y(b)$, we obtain

$$I_{0^+}^{1-\gamma}y(0)=\frac{1}{1-E_{\alpha,1}(\lambda b^{\alpha})}\int_0^b(b-s)^{\alpha-1}E_{\alpha,\alpha-\gamma+1}(\lambda\,(b-s)^{\alpha})\,g(s)ds. \quad (10)$$

From (8) and (10), it follows that

$$y(t)=\frac{t^{\gamma-1}E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\int_0^b(b-s)^{\alpha-\gamma}E_{\alpha,\alpha-\gamma+1}(\lambda\,(b-s)^{\alpha})\,g(s)ds+$$
$$+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda\,(t-s)^{\alpha})\,g(s)ds. \quad (11)$$

Conversely, applying $I_{0^+}^{1-\gamma}$ to both sides of (9), using Lemmas 5 and 7, we have

$$\begin{aligned} I_{0^+}^{1-\gamma}y(t) &= \frac{E_{\alpha,1}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\int_0^b(b-s)^{\alpha-\gamma}E_{\alpha,\alpha-\gamma+1}(\lambda\,(b-s)^{\alpha})\,g(s)ds+\\ &+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha-\gamma+1}(\lambda\,(t-s)^{\alpha})\,g(s)ds. \end{aligned} \quad (12)$$

By Lemma 4, and passing to the limit as $t\to 0$,

$$I_{0^+}^{1-\gamma}y(0)=\frac{1}{1-E_{\alpha,1}(\lambda b^{\alpha})}\int_0^b(b-s)^{\alpha-\gamma}E_{\alpha,\alpha-\gamma+1}(\lambda\,(b-s)^{\alpha})\,g(s)ds. \quad (13)$$

Similarly, passing to the limit as $t \to b$ of (12), we have

$$I_{0^+}^{1-\gamma} y(b) = \frac{E_{\alpha,1}(\lambda b^\alpha)}{1 - E_{\alpha,1}(\lambda b^\alpha)} \int_0^b (b-s)^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}(\lambda (b-s)^\alpha)\, g(s) ds +$$
$$+ \int_0^b (b-s)^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}(\lambda (b-s)^\alpha)\, g(s) ds =$$
$$= \frac{1}{1 - E_{\alpha,1}(\lambda b^\alpha)} \int_0^b (b-s)^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}(\lambda (b-s)^\alpha)\, g(s) ds. \quad (14)$$

From (13) and (14) the relation $I_{0^+}^{1-\gamma} y(0) = I_{0^+}^{1-\gamma} y(b)$ follows.

On the other hand, apply $D_{0^+}^{\gamma}$ to both sides of (11), use Lemma 1 and Theorem 1, then apply $I_{0^+}^{\beta(1-\alpha)}$ on the result to get

$${}^H D_{0^+}^{\alpha,\beta} y(t) - \lambda y(t) = g(t)$$

from Lemma 3 and equation (5). $\square$

For our analysis, the following assumptions must hold.

(H_1) Let $f : (0, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function and let there exist positive constants $M > 0$ and $0 < L < 1$, such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le M |u_1 - u_2| + L |v_1 - v_2|,$$

for any $u_i, v_i \in \mathbb{R}, i = 1, 2$ and $t \in (0, b]$.

(H_2) There exist $m, q, p \in C([0, b], \mathbb{R})$ such that

$$|f(t, u, v)| \le m(t) + q(t) |u| + p(t) |v|,$$

with $p^* = \sup_{t\in[0,b]} p(t) < 1$, $q^* = \sup_{t\in[0,b]} q(t)$, and $m^* = \sup_{t\in[0,b]} m(t)$, for all $t \in (0, b]$, and for each $u, v \in \mathbb{R}$.

(H_3) The following inequality holds:

$$\Theta := \frac{\lambda + M}{1 - L} \left(\frac{E_{\alpha,\gamma}(\lambda b^\alpha)}{1 - E_{\alpha,1}(\lambda b^\alpha)} \frac{\Gamma(\gamma) b^\alpha}{\Gamma(\alpha+1)} + \frac{B(\alpha, \gamma) b^{1-\gamma+\alpha}}{\Gamma(\alpha)} \right) < 1.$$

3.1. Existence Result Via Schaefer's Fixed Point Theorem. We begin with an existence result via Schaefer's fixed point theorem:

Theorem 2. *Assume that $f:(0,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous, and the condition (H_2) holds. If*

$$\varrho := \left(\frac{E_{\alpha,\gamma}(\lambda b^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\frac{\Gamma(\gamma)}{\Gamma(\alpha+1)}+\frac{B(\alpha,\gamma)}{\Gamma(\alpha)}\right)\frac{(\lambda+q^{*})\,b^{\alpha}}{(1-p^{*})}<1, \tag{15}$$

then the Hilfer problem (2) –(3) has at least one solution in $C_{1-\gamma}([0,b]\,,\mathbb{R})$.

Proof. According to Lemma 9, the solution of the Hilfer problem (2) –(3) can be expressed by the integral equation

$$\begin{aligned} y(t) &= \frac{t^{\gamma-1}E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\int_0^b (b-s)^{\alpha-\gamma}E_{\alpha,\alpha-\gamma+1}(\lambda\,(b-s)^{\alpha})\,K_y(s)ds+ \\ &\quad+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda\,(t-s)^{\alpha})\,K_y(s)ds, \end{aligned}$$

where K_y is the solution of the functional integral equation

$$\begin{aligned} K_y(t) = \lambda\bigg(&\frac{t^{\gamma-1}E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\int_0^b (b-s)^{\alpha-\gamma}E_{\alpha,\alpha-\gamma+1}(\lambda\,(b-s)^{\alpha})\,K_y(s)ds+ \\ &+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda\,(t-s)^{\alpha})\,K_y(s)ds, K_y(t)\bigg)+ \\ +f\bigg(t,&\frac{t^{\gamma-1}E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\int_0^b (b-s)^{\alpha-\gamma}E_{\alpha,\alpha-\gamma+1}(\lambda\,(b-s)^{\alpha})\,K_y(s)ds+ \\ &+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda\,(t-s)^{\alpha})\,K_y(s)ds, K_y(t)\bigg). \end{aligned} \tag{16}$$

Here $K_y(t) := \lambda y(t) + f(t, y(t), K_y(t))$.

Consider the operator $\Lambda : C_{1-\gamma}\,[0,b] \longrightarrow C_{1-\gamma}\,[0,b]$

$$y(t) \longrightarrow \Lambda y(t) =$$

$$= \frac{t^{\gamma-1}E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\int_0^b (b-s)^{\alpha-\gamma}E_{\alpha,\alpha-\gamma+1}(\lambda(b-s)^{\alpha})K_y(s)ds +$$

$$+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})K_y(s)ds. \tag{17}$$

It is obvious that the operator Λ is well defined. Define a bounded closed convex set $B_r = \{y \in C_{1-\gamma}[0,b] : \|y\|_{C_{1-\gamma}} \leq r\} \subset C_{1-\gamma}[0,b]$ with $r \geq \frac{\omega}{1-\varrho}$, $\varrho < 1$ and

$$\omega := \left(\frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\frac{1}{\Gamma(\alpha-\gamma+2)} + \frac{1}{\Gamma(\alpha+1)}\right)\frac{m^{*}b^{\alpha-\gamma+1}}{(1-p^{*})}.$$

Claim(1). The operator Λ is continuous. Consider a sequence $\{y_n\}_{n=1}^{\infty}$, such that $y_n \longrightarrow y$ in B_r. In view of Lemmas 6, 7, and for $t \in (0,b]$ it follows that

$$\left|t^{1-\gamma}[\Lambda y_n(t) - \Lambda y(t)]\right| \leq$$

$$\leq \frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\int_0^b (b-s)^{\alpha-\gamma}E_{\alpha,\alpha-\gamma+1}(\lambda(b-s)^{\alpha})\left|K_{y_n}(s)-K_y(s)\right|ds+$$

$$+ t^{1-\gamma}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})\left|K_{y_n}(s)-K_y(s)\right|ds \leq$$

$$\leq \frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\frac{1}{\Gamma(\alpha-\gamma+1)}\int_0^b (b-s)^{\alpha-\gamma}\left|K_{y_n}(s)-K_y(s)\right|ds+$$

$$+ \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|K_{y_n}(s)-K_y(s)\right|ds = I_1 + I_2, \tag{18}$$

where

$$I_1 = \frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\frac{1}{\Gamma(\alpha-\gamma+1)}\int_0^b (b-s)^{\alpha-\gamma}\left|K_{y_n}(s)-K_y(s)\right|ds \leq$$

$$\leq \frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\frac{1}{\Gamma(\alpha-\gamma+1)}\times$$

$$\times\int_0^b (b-s)^{\alpha-\gamma}[\lambda\,|y_n(s)-y(s)|+|f(s,y_n(s),K_{y_n}(s))-f(s,y(s),K_y(s))|]\,ds \leq$$

$$\leq \frac{E_{\alpha,\gamma}(\lambda b^\alpha)}{1-E_{\alpha,1}(\lambda b^\alpha)}\Big(\frac{\Gamma(\gamma)\lambda b^\alpha}{\Gamma(\alpha+1)}\,\|y_n-y\|_{C_{1-\gamma}}+$$
$$+\frac{b^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}\,\|f(\cdot,\,y_n(\cdot),K_{y_n}(\cdot))-f(\cdot,y(\cdot),K_y(\cdot))\|_C\Big), \quad (19)$$

and

$$I_2=\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,|K_{y_n}(s)-K_y(s)|\,ds \leq$$
$$\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\lambda|y_n(s)-y(s)|+$$
$$+|f(s,y_n(s),K_{y_n}(s))-f(s,y(s),K_y(s))|]ds \leq$$
$$\leq \frac{\Gamma(\gamma)\lambda b^\alpha}{\Gamma(\alpha+\gamma)}\,\|y_n-y\|_{C_{1-\gamma}}+$$
$$+\frac{b^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}\,\|f(\cdot,y_n(\cdot),K_{y_n}(\cdot))-f(\cdot,y(\cdot),K_y(\cdot))\|_C. \quad (20)$$

In (19) and (20), the function f is continuous and $y_n \longrightarrow y$ as $n \longrightarrow \infty$; it follows that $I_1 \to 0$ and $I_2 \to 0$, as $n \to \infty$. Hence,

$$\|\Lambda y-\Lambda y_n\|_{C_{1-\gamma}} \to 0 \text{ as } n \to \infty.$$

Thus, the operator Λ is continuous.

Claim(2). Λ maps bounded sets into bounded sets in $C_{1-\gamma}([0,b]\,,\mathbb{R})$.

By using Lemma 6, and for $t \in (0,b]$, we get

$$\left|t^{1-\gamma}\Lambda y(t)\right| \leq \frac{E_{\alpha,\gamma}(\lambda t^\alpha)}{1-E_{\alpha,1}(\lambda b^\alpha)}\int_0^b (b-s)^{\alpha-\gamma}E_{\alpha,\alpha-\gamma+1}(\lambda\,(b-s)^\alpha)|K_y(s)|\,ds+$$
$$+\,t^{1-\gamma}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda\,(t-s)^\alpha)\,|K_y(s)|\,ds \leq$$

$$\leq \frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\frac{1}{\Gamma(\alpha-\gamma+1)}\int_0^b (b-s)^{\alpha-\gamma}\left|K_y(s)\right| ds+$$

$$+\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|K_y(s)\right| ds. \quad (21)$$

In view of (H_2), we have

$$\begin{aligned}\left|K_y(t)\right| &\leq \lambda\left|y(t)\right| + \left|f(t,y(t),K_y(t))\right| \leq \\ &\leq \lambda\left|y(t)\right| + m(t) + q(t)\left|y(t)\right| + p(t)\left|K_y(t)\right| \leq \\ &\leq \lambda\left|y(t)\right| + m^* + q^*\left|y(t)\right| + p^*\left|K_y(t)\right|.\end{aligned}$$

Since $p^* < 1$, it follows that

$$\left|K_y(t)\right| \leq \frac{m^* + (\lambda+q^*)\left|y(t)\right|}{(1-p^*)}. \quad (22)$$

Relations (22) and (21) together give

$$\begin{aligned}\left|t^{1-\gamma}\Lambda y(t)\right| \leq{}& \frac{m^*}{(1-p^*)}\frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\frac{b^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}+ \\ &+\frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\frac{(\lambda+q^*)}{(1-p^*)}\frac{\Gamma(\gamma)b^{\alpha}}{\Gamma(\alpha+1)}\left\|y\right\|_{C_{1-\gamma}}+ \\ &+\frac{m^* t^{\alpha-\gamma+1}}{(1-p^*)\Gamma(\alpha+1)}+\frac{(\lambda+q^*)}{(1-p^*)}\frac{B(\alpha,\gamma)t^{\alpha}}{\Gamma(\alpha)}\left\|y\right\|_{C_{1-\gamma}}. \quad (23)\end{aligned}$$

For any $y \in B_r$, the last inequality leads to

$$\begin{aligned}\left|t^{1-\gamma}\Lambda y(t)\right| \leq{}& \left(\frac{E_{\alpha,\gamma}(\lambda b^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\frac{1}{\Gamma(\alpha-\gamma+2)}+\frac{1}{\Gamma(\alpha+1)}\right)\frac{m^* b^{\alpha-\gamma+1}}{(1-p^*)}+ \\ &+\left(\frac{E_{\alpha,\gamma}(\lambda b^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\frac{\Gamma(\gamma)}{\Gamma(\alpha+1)}+\frac{B(\alpha,\gamma)}{\Gamma(\alpha)}\right)\frac{(\lambda+q^*)b^{\alpha}}{(1-p^*)}r \leq \omega + \varrho r,\end{aligned}$$

which implies

$$\left\|\Lambda y\right\|_{C_{1-\gamma}} \leq r.$$

Thus, $\Lambda : B_r \longrightarrow B_r$, that is ΛB_r is uniformly bounded.

Claim(3). Λ maps bounded sets into equicontinuous sets of $C_{1-\gamma}([0,b],\mathbb{R})$.

Choose any $y \in B_r$ and $t_1, t_2 \in (0, b]$, such that $t_1 \leq t_2$. Using Lemmas 6, 7, we have

$$
\begin{aligned}
&\left|t_2^{1-\gamma}\Lambda y(t_2) - t_1^{1-\gamma}\Lambda y(t_1)\right| = \\
&= \left|\frac{E_{\alpha,\gamma}(\lambda t_2^{\alpha}) - E_{\alpha,\gamma}(\lambda t_1^{\alpha})}{1 - E_{\alpha,1}(\lambda b^{\alpha})}\int_0^b (b-s)^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}(\lambda (b-s)^{\alpha}) K_y(s)ds + \right. \\
&\qquad + t_2^{1-\gamma}\int_0^{t_2} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t_2-s)^{\alpha}) K_y(s)ds - \\
&\qquad \left. - t_1^{1-\gamma}\int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t_1-s)^{\alpha}) K_y(s)ds\right| \leq \\
&\qquad \leq \left|\frac{E_{\alpha,\gamma}(\lambda t_2^{\alpha}) - E_{\alpha,\gamma}(\lambda t_1^{\alpha})}{1 - E_{\alpha,1}(\lambda b^{\alpha})}\frac{1}{\Gamma(\alpha-\gamma+1)} \times \right. \\
&\qquad \times \int_0^b (b-s)^{\alpha-\gamma}\left[s^{\gamma-1}\frac{(\lambda+q^*)}{(1-p^*)}\|y\|_{C_{1-\gamma}} + \frac{m^*}{(1-p^*)}\right]ds + \\
&\qquad + \frac{t_2^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_2} (t_2-s)^{\alpha-1}\left[s^{\gamma-1}\frac{(\lambda+q^*)}{(1-p^*)}\|y\|_{C_{1-\gamma}} + \frac{m^*}{(1-p^*)}\right]ds - \\
&\qquad \left. - \frac{t_1^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}\left[s^{\gamma-1}\frac{(\lambda+q^*)}{(1-p^*)}\|y\|_{C_{1-\gamma}} + \frac{m^*}{(1-p^*)}\right]ds\right| \leq \\
&\leq \left|\frac{E_{\alpha,\gamma}(\lambda t_2^{\alpha}) - E_{\alpha,\gamma}(\lambda t_1^{\alpha})}{1 - E_{\alpha,1}(\lambda b^{\alpha})}\left[\frac{\Gamma(\gamma)b^{\alpha}}{\Gamma(\alpha+1)}\frac{(\lambda+q^*)}{(1-p^*)}r + \frac{b^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}\frac{m^*}{(1-p^*)}\right]\right| + \\
&\qquad + \left|\left[\frac{\Gamma(\gamma)t_2^{\alpha}}{\Gamma(\alpha+\gamma)}\frac{(\lambda+q^*)}{(1-p^*)}r + \frac{t_2^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}\frac{m^*}{(1-p^*)}\right] - \right. \\
&\qquad \left. - \left[\frac{\Gamma(\gamma)t_1^{\alpha}}{\Gamma(\alpha+\gamma)}\frac{(\lambda+q^*)}{(1-p^*)}r + \frac{t_1^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}\frac{m^*}{(1-p^*)}\right]\right| \leq \\
&\leq \left|\frac{E_{\alpha,\gamma}(\lambda t_2^{\alpha}) - E_{\alpha,\gamma}(\lambda t_1^{\alpha})}{1 - E_{\alpha,1}(\lambda b^{\alpha})}\left[\frac{\Gamma(\gamma)b^{\alpha}}{\Gamma(\alpha+1)}\frac{(\lambda+q^*)}{(1-p^*)}r + \frac{b^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}\frac{m^*}{(1-p^*)}\right]\right| + \\
&\quad + \left|\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\frac{(\lambda+q^*)}{(1-p^*)}r(t_2^{\alpha} - t_1^{\alpha}) + \frac{1}{\Gamma(\alpha+1)}\frac{m^*}{(1-p^*)}\left(t_2^{\alpha-\gamma+1} - t_1^{\alpha-\gamma+1}\right)\right|. \qquad (24)
\end{aligned}
$$

Now, let $h(t) = t^{\sigma}$. By the Lagrange Mean-value theorem, there exists

$\xi \in [t_1, t_2]$, such that

$$\frac{h(t_2) - h(t_1)}{t_2 - t_1} = h'(\xi).$$

We get $|t_2^\alpha - t_1^\alpha| = \alpha\xi^{\alpha-1} |t_2 - t_1| \leq \alpha b^{\alpha-1} |t_2 - t_1|$, with $\xi \leq t_2 \leq b$ and

$$\left|t_2^{\alpha-\gamma+1} - t_1^{\alpha-\gamma+1}\right| = (\alpha - \gamma + 1) \xi^{\alpha-\gamma} |t_2 - t_1| \leq (\alpha - \gamma + 1) b^{\alpha-\gamma} |t_2 - t_1|,$$

with $\xi \leq t_2 \leq b$. Hence, (24) implies

$$\begin{aligned}
&\left|t_2^{1-\gamma}\Lambda y(t_2) - t_1^{1-\gamma}\Lambda y(t_1)\right| \leq \\
&\leq \left|\frac{E_{\alpha,\gamma}(\lambda t_2^\alpha) - E_{\alpha,\gamma}(\lambda t_1^\alpha)}{1 - E_{\alpha,1}(\lambda b^\alpha)} \left[\frac{\Gamma(\gamma)b^\alpha}{\Gamma(\alpha+1)} \frac{(\lambda + q^*)}{(1 - p^*)} r + \frac{b^{\alpha-\gamma+1}}{\Gamma(\alpha - \gamma + 2)} \frac{m^*}{(1 - p^*)}\right]\right| + \\
&\qquad + \frac{B(\alpha, \gamma)}{\Gamma(\alpha)} \frac{(\lambda + q^*) r}{(1 - p^*)} \alpha b^{\alpha-1} |t_2 - t_1| + \\
&\qquad\qquad + \frac{1}{\Gamma(\alpha+1)} \frac{m^*}{(1 - p^*)} (\alpha - \gamma + 1) b^{\alpha-\gamma} |t_2 - t_1|.
\end{aligned}$$

From (6) we see that, as $t_1 \longrightarrow t_2$, the right-hand side of the preceding inequality is independent of y and tends to zero; hence,

$$\left|t_2^{1-\gamma}\Lambda y(t_2) - t_1^{1-\gamma}\Lambda y(t_1)\right| \to 0, \ \ \forall \ |t_2 - t_1| \to 0, \ y \in B_r. \tag{25}$$

From the above claims, together with the Arzela-Ascoli theorem, we conclude that the operator Λ is completely continuous. In the remaining part of the proof, we only need to prove that the set

$$\Delta = \{y \in C_{1-\gamma}[0, b] : y = \delta\Lambda y, \ \text{for some } \delta \in (0, 1)\}$$

is a bounded set. For each $t \in (0, b]$, let $y \in \Delta$, and $y = \delta\Lambda y$ for some $\delta \in (0, 1)$. Then we have

$$y(t) < \Lambda y(t).$$

Hence, by virtue of step (2) and definitions of ω and ϱ, we obtain

$$\begin{aligned}
&\|y\|_{C_{1-\gamma}} < \|\Lambda y\|_{C_{1-\gamma}} \leq \\
&\qquad \leq \left(\frac{E_{\alpha,\gamma}(\lambda t^\alpha)}{1 - E_{\alpha,1}(\lambda b^\alpha)} \frac{1}{\Gamma(\alpha - \gamma + 2)} + \frac{1}{\Gamma(\alpha+1)}\right) \frac{m^* b^{\alpha-\gamma+1}}{(1 - p^*)} + \\
&+ \left(\frac{E_{\alpha,\gamma}(\lambda t^\alpha)}{1 - E_{\alpha,1}(\lambda b^\alpha)} \frac{\Gamma(\gamma)}{\Gamma(\alpha+1)} + \frac{B(\alpha, \gamma)}{\Gamma(\alpha)}\right) \frac{(\lambda + q^*) b^\alpha}{(1 - p^*)} \|y\|_{C_{1-\gamma}} = \omega + \varrho \|y\|_{C_{1-\gamma}}.
\end{aligned}$$

Since $\varrho < 1$, inequality

$$\|y\|_{C_{1-\gamma}} \leq \frac{\omega}{1-\varrho} \leq r$$

follows.

Thus, the set Δ is bounded. Schaefer's fixed point theorem shows that Λ has a fixed point, which is a solution of the problem (2)–(3).

Finally,

$$K_y(t) := \lambda y(t) + f(t, y(t), K_y(t)), \text{ for each } t \in (0, b],$$

where

$$\begin{aligned} y(t) &= \frac{t^{\gamma-1} E_{\alpha,\gamma}(\lambda t^\alpha)}{1 - E_{\alpha,1}(\lambda b^\alpha)} \int_0^b (b-s)^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}(\lambda (b-s)^\alpha) K_y(s) ds + \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^\alpha) K_y(s) ds. \end{aligned}$$

This implies

$$^H D_{0^+}^{\alpha,\beta} y(t) = K_y(t).$$

Consequently,

$$^H D_{0^+}^{\alpha,\beta} y(t) - \lambda y(t) = f(t, y(t), ^H D_{0^+}^{\alpha,\beta} y(t)).$$

The proof is completed. □

3.2. A uniqueness Result Via Banach's Fixed Point Theorem. Here we give a uniqueness result via Banach's fixed point theorem:

Theorem 3. *Assume that (H_1)-(H_3). Then the Hilfer problem (2)–(3) has a unique solution in $C_{1-\gamma}([0, b], \mathbb{R})$.*

Proof. We already know that the operator Λ, defined by (17), is well-defined and continuous, see Theorem 2.

Next, we prove that Λ is a contraction map on $C_{1-\gamma}([0, 1], \mathbb{R})$ with respect to the norm $\|\cdot\|_{C_{1-\gamma}}$. For any $y, y^* \in C_{1-\gamma}([0, b], \mathbb{R})$ and any $t \in (0, b]$ we prove, using Lemmas 6, 7, that

$$\left| t^{1-\gamma} [\Lambda y(t) - \Lambda y^*(t)] \right| \leq$$

$$\leq \left| \frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1 - E_{\alpha,1}(\lambda b^{\alpha})} \int_0^b (b-s)^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}(\lambda (b-s)^{\alpha}) \left[K_y(s) - K_{y^*}(s)\right] ds + \right.$$

$$\left. + t^{1-\gamma} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^{\alpha}) \left[K_y(s) - K_{y^*}(s)\right] ds \right| \leq$$

$$\leq \frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1 - E_{\alpha,1}(\lambda b^{\alpha})} \frac{1}{\Gamma(\alpha-\gamma+1)} \int_0^b (b-s)^{\alpha-\gamma} \left|K_y(s) - K_{y^*}(s)\right| ds +$$

$$+ \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|K_y(s) - K_{y^*}(s)\right| ds. \quad (26)$$

So,

$$\begin{aligned} &\left|K_y(s) - K_{y^*}(s)\right| \leq \\ &\quad \leq \lambda \left|y(s) - y^*(s)\right| + \left|f(s, y(s), K_y(s)) - f(s, y^*(s), K_{y^*}(s))\right| \leq \\ &\qquad \leq (\lambda + M) \left|y(s) - y^*(s)\right| + L \left|K_y(s) - K_{y^*}(s)\right|. \end{aligned}$$

Since $0 < L < 1$, it follows that

$$\left|K_y(s) - K_{y^*}(s)\right| \leq \frac{\lambda + M}{1 - L} \left|y(s) - y^*(s)\right|. \quad (27)$$

Bringing (27) into (26), we obtain

$$\left|t^{1-\gamma}\left[\Lambda y(t) - \Lambda y^*(t)\right]\right| \leq$$

$$\leq \frac{\lambda + M}{1 - L} \frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1 - E_{\alpha,1}(\lambda b^{\alpha})} \frac{1}{\Gamma(\alpha-\gamma+1)} \int_0^b (b-s)^{\alpha-\gamma} \left|y(s) - y^*(s)\right| ds +$$

$$+ \frac{\lambda + M}{1 - L} \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|y(s) - y^*(s)\right| ds \leq$$

$$\leq \frac{\lambda + M}{1 - L} \frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1 - E_{\alpha,1}(\lambda b^{\alpha})} \frac{\Gamma(\gamma)}{\Gamma(\alpha+1)} b^{\alpha} \left\|y - y^*\right\|_{C_{1-\gamma}[0,b]} +$$

$$+ \frac{b^{1-\gamma+\alpha} B(\alpha, \gamma)}{\Gamma(\alpha)} \frac{\lambda + M}{1 - L} \left\|y - y^*\right\|_{C_{1-\gamma}[0,b]} \leq$$

$$\leq \Theta \left\|y - y^*\right\|_{C_{1-\gamma}[0,b]}.$$

Since $\Theta < 1$, it follows that Λ is a contraction map. As a consequence of the Banach contraction principle, we conclude that the Hilfer problem (2) – (3) has a unique solution in $C_{1-\gamma}([0,1],\mathbb{R})$. $\square$

3.3. Ulam-Hyers and Ulam-Hyers-Rassias Stabilities Via the Generalized Gronwall Inequality. In this part, we discuss different types of stability results for the Hilfer fractional implicit differential equation (2). Let $\epsilon > 0$ and assume that a solution $x \in C_{1-\gamma}([0,b],\mathbb{R})$ exists and satisfies the following inequality:

$$\left| {}^{H}D_{0^+}^{\alpha,\beta}x(t) - \lambda x(t) - f(t, x(t), {}^{H}D_{0^+}^{\alpha,\beta}x(t)) \right| \leq \epsilon, \quad t \in (0,b]. \tag{28}$$

Definition 5. *The problem (2) – (3) is Ulam-Hyers stable, if there exists a real number $\eta_f > 0$, such that for each $\epsilon > 0$ there exists a solution $x \in C_{1-\gamma}([0,b],\mathbb{R})$ of inequality (28) corresponding to a solution $y \in C_{1-\gamma}([0,b],\mathbb{R})$ of the problem (2) – (3) with*

$$|x(t) - y(t)| \leq \eta_f \epsilon, \quad t \in (0,b].$$

Definition 6. *The problem (2) – (3) is generalized Ulam-Hyers stable, if there exists $\psi_f \in C([0,\infty),[0,\infty))$, $\psi_f(0) = 0$, such that for each $\eta_f > 0$ there exists a solution $x \in C_{1-\gamma}([0,b],\mathbb{R})$ of inequality (28), corresponding to a solution $y \in C_{1-\gamma}([0,b],\mathbb{R})$ of the problem (2) – (3) with*

$$|x(t) - y(t)| \leq \psi_f(\epsilon), \quad t \in (0,b].$$

Definition 7. *The problem (2) – (3) is Ulam-Hyers-Rassias stable with respect to $\varphi_\alpha \in C_{1-\gamma}([0,b],\mathbb{R})$, if there exists a real number $\eta_{\varphi_\alpha} > 0$, such that for each $\epsilon > 0$ and for each solution $x \in C_{1-\gamma}([0,b],\mathbb{R})$ of the inequality*

$$\left| {}^{H}D_{0^+}^{\alpha,\beta}x(t) - \lambda x(t) - f(t, x(t), {}^{H}D_{0^+}^{\alpha,\beta}x(t)) \right| \leq \epsilon\varphi_\alpha(t), \quad t \in (0,b], \tag{29}$$

there exists a solution $y \in C_{1-\gamma}([0,b],\mathbb{R})$ of the problem (2) – (3) with

$$|x(t) - y(t)| \leq \eta_{\varphi_\alpha} \epsilon \varphi_\alpha(t), \quad t \in (0,b].$$

Definition 8. *The problem (2)–(3) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi_\alpha \in C_{1-\gamma}([0,b],\mathbb{R})$ if there exists a real number*

$\eta_{\varphi_\alpha} > 0$, *such that for each* $\epsilon > 0$ *and for each solution* $x \in C_{1-\gamma}([0,b], \mathbb{R})$ *of the inequality*

$$\left| {}^H D_{0^+}^{\alpha,\beta} x(t) - \lambda x(t) - f(t, x(t), {}^H D_{0^+}^{\alpha,\beta} x(t)) \right| \leq \varphi_\alpha(t), \quad t \in (0,b],$$

there exists a solution $y \in C_{1-\gamma}([0,b], \mathbb{R})$ *of the problem (2)–(3) with*

$$|x(t) - y(t)| \leq \eta_{\varphi_\alpha} \varphi_\alpha(t), \quad t \in (0,b].$$

Remark 2. *A function* $x \in C_{1-\gamma}([0,b], \mathbb{R})$ *is a solution of inequality (28) if and only if there exists a function* $z_x \in C_{1-\gamma}([0,b], \mathbb{R})$, *such that*

(i) $|z_x(t)| \leq \epsilon, \ \ t \in (0,b]$;

(ii) ${}^H D_{0^+}^{\alpha,\beta} x(t) - \lambda x(t) = f(t, x(t), {}^H D_{0^+}^{\alpha,\beta} x(t)) + z_x(t), \ \ t \in (0,b]$.

Lemma 10. *Let* $x \in C_{1-\gamma}([0,b], \mathbb{R})$ *satisfy inequality (28). Then* x *satisfies the following integral inequality:*

$$\left| x(t) - A_x - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^\alpha) K_x(s) ds \right| \leq$$
$$\leq \left(\frac{E_{\alpha,\gamma}(\lambda t^\alpha)}{1 - E_{\alpha,1}(\lambda b^\alpha)} \frac{b^\alpha}{\Gamma(\alpha - \gamma + 2)} + \frac{b^\alpha}{\Gamma(\alpha+1)} \right) \epsilon,$$

where

$$A_x = \frac{t^{\gamma-1} E_{\alpha,\gamma}(\lambda t^\alpha)}{1 - E_{\alpha,1}(\lambda b^\alpha)} \int_0^b (b-s)^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}(\lambda (b-s)^\alpha) K_x(s) ds,$$

and $K_x(t) := \lambda x(t) + f(t, x(t), K_x(t))$.

Proof. Indeed, by Remark 2 and Theorem 2, we have

$$x(t) = \frac{t^{\gamma-1} E_{\alpha,\gamma}(\lambda t^\alpha)}{1 - E_{\alpha,1}(\lambda b^\alpha)} \Bigg(\int_0^b (b-s)^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}(\lambda (b-s)^\alpha) K_x(s) ds +$$
$$+ \int_0^b (b-s)^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}(\lambda (b-s)^\alpha) z_x(s) ds \Bigg) +$$

$$+\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^{\alpha}) K_x(s)ds+$$
$$+\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^{\alpha}) z_x(s)ds.$$

Thus,

$$\left| x(t) - A_x - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^{\alpha}) K_x(s)ds \right| =$$
$$= \left| \frac{t^{\gamma-1} E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})} \int_0^b (b-s)^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}(\lambda (b-s)^{\alpha}) z_x(s)ds + \right.$$
$$\left. + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^{\alpha}) z_x(s)ds \right| \le$$
$$\le \frac{1}{\Gamma(\alpha-\gamma+1)} \frac{t^{\gamma-1} E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})} \int_0^b (b-s)^{\alpha-\gamma} |z_x(s)| ds +$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z_x(s)| ds \le$$
$$\le \Big(\frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})} \frac{b^{\alpha}}{\Gamma(\alpha-\gamma+2)} + \frac{b^{\alpha}}{\Gamma(\alpha+1)} \Big) \epsilon.$$

The proof is complete. □

Theorem 4. *Assume that (H_1) and (H_3) are satisfied. Then the problem (2) – (3) is Ulam-Hyers stable and generalized Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$ and $x \in C_{1-\gamma}([0,b],\mathbb{R})$ satisfy inequality (28) and $y \in C_{1-\gamma}([0,b],\mathbb{R})$ be a unique solution of the implicit fractional differential equation

$$^{H}D_{0^+}^{\alpha,\beta} y(t) - \lambda y(t) = f(t, y(t), ^{H}D_{0^+}^{\alpha,\beta} y(t)), \ t \in (0,b], \tag{30}$$

with

$$I_{0^+}^{1-\gamma} y(0) = I_{0^+}^{1-\gamma} x(0), \quad I_{0^+}^{1-\gamma} x(b) = I_{0^+}^{1-\gamma} y(b). \tag{31}$$

In view of Theorem 2, we have

$$y(t) = A_y + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda\,(t-s)^{\alpha})\, K_y(s)ds,$$

where

$$A_y = \frac{t^{\gamma-1}E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})} \int_0^b (b-s)^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}(\lambda\,(b-s)^{\alpha})\, K_y(s)ds$$

and $K_y(t) = \lambda y(t) + f(t, y(t), K_y(t))$. By Lemma 6 and Eq. (31), we easily show that $A_y = A_x$. Indeed,

$$\begin{aligned}
&|A_y - A_x| \leq \\
&\leq \frac{t^{\gamma-1}E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})} \int_0^b (b-s)^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}(\lambda\,(b-s)^{\alpha})\, |K_y(s) - K_x(s)|\, ds \leq \\
&\quad \leq \frac{t^{\gamma-1}}{\Gamma(\alpha-\gamma+1)} \frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})} \int_0^b (b-s)^{\alpha-1}\, |K_y(s) - K_x(s)|\, ds \leq \\
&\quad \leq \frac{\lambda+M}{1-L} \frac{t^{\gamma-1}}{\Gamma(\alpha-\gamma+1)} \frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})} \int_0^b (b-s)^{\alpha-\gamma}\, |y(s) - x(s)|\, ds \leq \\
&\qquad \leq \frac{\lambda+M}{1-L} t^{\gamma-1} \frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})} I_{0+}^{\alpha-\gamma+1}\, |y(b) - x(b)| \leq \\
&\qquad\quad \leq \frac{\lambda+M}{1-L} t^{\gamma-1} \frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})} I_{0+}^{\alpha} I_{0+}^{1-\gamma}\, |y(b) - x(b)| = 0.
\end{aligned}$$

Thus, $A_y = A_x$.
Then we have

$$y(t) = A_x + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda\,(t-s)^{\alpha})\, K_y(s)ds.$$

We have, from Lemma 10:

$$\left| x(t) - A_x - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda\,(t-s)^{\alpha})\, K_x(s)ds \right| \leq$$

$$\leq \left(\frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\frac{1}{\Gamma(\alpha-\gamma+2)}+\frac{1}{\Gamma(\alpha+1)}\right)b^{\alpha}\epsilon. \tag{32}$$

Hence, for any $t \in (0, b]$

$$\begin{aligned}|x(t)-y(t)| \leq & \left|x(t)-A_x-\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda\,(t-s)^{\alpha})\,K_x(s)ds\right|+ \\ & +\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda\,(t-s)^{\alpha})\,|K_x(s)-K_y(s)|\,ds \leq \\ & \leq \left(\frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})}\frac{b^{\alpha}}{\Gamma(\alpha-\gamma+2)}+\frac{b^{\alpha}}{\Gamma(\alpha+1)}\right)\epsilon+ \\ & +\frac{\lambda+M}{1-L}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,|x(s)-y(s)|\,ds \leq \\ & \leq \left(\frac{1}{\Gamma(\gamma)\,(1-E_{\alpha,1}(\lambda b^{\alpha}))}\frac{b^{\alpha}}{\Gamma(\alpha-\gamma+2)}+\frac{b^{\alpha}}{\Gamma(\alpha+1)}\right)\epsilon+ \\ & +\frac{\lambda+M}{1-L}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,|x(s)-y(s)|\,ds.\end{aligned}$$

By utilizing Lemma 8 and Remark 1, we get

$$\begin{aligned}|x(t)-y(t)| &\leq U\epsilon+\int_0^t \left(\sum_{n=1}^{\infty}\frac{\left(\frac{\lambda+M}{1-L}\right)^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}U\epsilon\right)ds = \\ &= U\epsilon\left(1+\sum_{n=1}^{\infty}\frac{\left(\frac{\lambda+M}{1-L}\right)^n}{\Gamma(n\alpha+1)}(t\right)^{n\alpha} = \\ &= U\epsilon E_{\alpha}\left(\frac{\lambda+M}{1-L}t^{\alpha}\right) := \eta_f\epsilon, \end{aligned} \tag{33}$$

where $U := \left(\frac{1}{\Gamma(\gamma)(1-E_{\alpha,1}(\lambda b^{\alpha}))}\frac{b^{\alpha}}{\Gamma(\alpha-\gamma+2)}+\frac{b^{\alpha}}{\Gamma(\alpha+1)}\right)$ and $\eta_f := UE_{\alpha}(\frac{\lambda+M}{1-L}t^{\alpha})$.

Moreover, if we set $\psi(\epsilon) = \eta_f\ \epsilon$, with $\psi(0) = 0$ in (33), then problem (2) is generalized Ulam-Hyers stable. □

Now, we need to introduce the following assumption:

(H_4) There exists an increasing function $\varphi_\alpha \in C_{1-\gamma}([0,b],\mathbb{R})$ and there exists $\delta_{\varphi_\alpha} > 0$, such that for any $t \in (0,b]$

$$I_{0+}^{\alpha}\varphi_\alpha(t) \leq \delta_{\varphi_\alpha}\varphi_\alpha(t).$$

Remark 3. *A function $x \in C_{1-\gamma}([0,b],\mathbb{R})$ is a solution of inequality (29) if and only if there exists a function $z_x \in C_{1-\gamma}([0,b],\mathbb{R})$ (where z depends on the solution x), such that*

(i) $|z_x(t)| \leq \epsilon\varphi_\alpha(t)$ *for all* $t \in (0,b]$,

(ii) ${}^H D_{0+}^{\alpha,\beta}x(t) - \lambda x(t) = f(t, x(t), {}^H D_{0+}^{\alpha,\beta}x(t)) + z_x(t), \quad t \in (0,b].$

Theorem 5. *Assume that (H_1), (H_3), and (H_4) are satisfied. If $(\lambda + M)\delta_{\varphi_\alpha} \neq 1 - L$, then the problem (2) – (3) is Ulam–Hyers–Rassias stable with respect to φ_α, as well as generalized Ulam–Hyers–Rassias stable.*

Proof. Let $\epsilon > 0$ and $x \in C_{1-\gamma}([0,b],\mathbb{R})$ satisfy the inequality

$$\left|{}^H D_{0+}^{\alpha,\beta}x(t) - \lambda x(t) - f(t, x(t), {}^H D_{0+}^{\alpha,\beta}x(t))\right| \leq \epsilon\varphi_\alpha(t), \quad t \in (0,b]. \tag{34}$$

Using Remark 3 and (H_4) in a similar way to Theorem 4, we can find:

$$\left|x(t) - A_x - \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)K_x(s)ds\right| \leq$$
$$\leq \left(\frac{E_{\alpha,\gamma}(\lambda b^\alpha)}{1 - E_{\alpha,1}(\lambda b^\alpha)}\frac{1}{\Gamma(2-\gamma)} + 1\right)\epsilon\delta_{\varphi_\alpha}\varphi_\alpha(t), \tag{35}$$

where $K_x(s) = \lambda x(s) + f(t, x(s), K_x(s))$.

Let $y \in C_{1-\gamma}([0,b],\mathbb{R})$ be a unique solution of the problem (30) – (31). In view of Lemma 10, in a similar way to Theorem 4, we find:

$$y(t) = A_x + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)K_x(s)ds, \tag{36}$$

where

$$A_x = t^{\gamma-1}\frac{E_{\alpha,\gamma}(\lambda t^\alpha)}{1 - E_{\alpha,1}(\lambda b^\alpha)}\int_0^b (b-s)^{\alpha-\gamma}E_{\alpha,\alpha+1-\gamma}(\lambda(b-s)^\alpha)K_x(s)ds.$$

On the other hand, by utilizing (36) and (35), we can get

$$
\begin{aligned}
|x(t)-y(t)| \leq & \left| x(t) - A_x - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^{\alpha}) K_x(s) ds \right| + \\
& + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^{\alpha}) \left| K_x(s) - K_y(s) \right| ds \leq \\
& \leq \left(\frac{E_{\alpha,\gamma}(\lambda b^{\alpha})}{1-E_{\alpha,1}(\lambda b^{\alpha})} \frac{1}{\Gamma(2-\gamma)} + 1 \right) \epsilon \delta_{\varphi_\alpha} \varphi_\alpha(t) + \\
& + \frac{\lambda+M}{1-L} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| x(s) - y(s) \right| ds \leq \\
\leq & \left(\frac{1}{\Gamma(2-\gamma)\Gamma(\gamma)\left(1-E_{\alpha,1}(\lambda b^{\alpha})\right)} + 1 \right) \epsilon \delta_{\varphi_\alpha} \varphi_\alpha(t) + \\
& \qquad + \frac{\lambda+M}{1-L} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| x(s) - y(s) \right| ds
\end{aligned}
$$

and, applying Lemma 8 and Remark 1, we derive:

$$
\begin{aligned}
& |x(t)-y(t)| \leq \\
& \leq \nabla \epsilon \delta_{\varphi_\alpha} \varphi_\alpha(t) + \int_0^t \left(\sum_{n=1}^{\infty} \frac{\left(\frac{\lambda+M}{1-L}\right)^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \nabla \epsilon \delta_{\varphi_\alpha} \varphi_\alpha(s) \right) ds \leq \\
& \leq \nabla \epsilon \delta_{\varphi_\alpha} \varphi_\alpha(t) + \nabla \epsilon \delta_{\varphi_\alpha} \int_0^t \left(\sum_{n=1}^{\infty} \frac{\left(\frac{\lambda+M}{1-L}\right)^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \varphi_\alpha(s) \right) ds \leq \\
& \qquad \leq \nabla \epsilon \delta_{\varphi_\alpha} \varphi_\alpha(t) + \nabla \epsilon \delta_{\varphi_\alpha} \sum_{n=1}^{\infty} \left(\frac{\lambda+M}{1-L} \delta_{\varphi_\alpha} \right)^n \varphi_\alpha(t) \leq \\
& \qquad \leq \nabla \left(\sum_{n=0}^{\infty} \left(\frac{\lambda+M}{1-L} \delta_{\varphi_\alpha} \right)^n \right) \epsilon \delta_{\varphi_\alpha} \varphi_\alpha(t) = \\
& \qquad \qquad = \nabla \epsilon \left(\frac{1}{1 - \frac{\lambda+M}{1-L} \delta_{\varphi_\alpha}} \right) \delta_{\varphi_\alpha} \varphi_\alpha(t) = \eta_{f,\varphi_\alpha} \epsilon \varphi_\alpha(t),
\end{aligned}
$$

where $\nabla := \left(\frac{1}{\Gamma(2-\gamma)\Gamma(\gamma)(1-E_{\alpha,1}(\lambda b^{\alpha}))} + 1\right)$ and $\eta_{f,\varphi_\alpha} := \dfrac{\nabla\delta_{\varphi_\alpha}}{1 - \frac{\lambda+M}{1-L}\delta_{\varphi_\alpha}}$. So

$$|x(t) - y(t)| \leq \eta_{f,\varphi_\alpha}\epsilon\varphi_\alpha(t). \tag{37}$$

Thus, the problem (2) – (3) is Ulam-Hyers Rassias stable.

Moreover, a similar argument, with $\epsilon = 1$ in Remark 3, we get

$$|x(t) - y(t)| \leq \eta_{f,\varphi_\alpha}\varphi_\alpha(t).$$

This proves that the problem (2)–(3) is generalized Ulam-Hyers Rassias stable. □

4. An example. In this section, we give one example to illustrate our result. Consider the following Hilfer fractional differential equation with an integral condition:

$$\begin{cases} {}^{H}D_{0^+}^{\frac{1}{2},\frac{1}{3}}y(t) = -\frac{1}{2}y(t) + \frac{t}{10}\left(1 + |y(t)| + \left|D_{0^+}^{\frac{1}{2},\frac{1}{5}}y(t)\right|\right), \quad t \in (0,1] \\ I_{0^+}^{\frac{1}{3}}y(0) = I_{0^+}^{\frac{1}{3}}y(1). \end{cases} \tag{38}$$

Here $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, $\gamma = \alpha + \beta - \alpha\beta = \frac{2}{3}$, $\lambda = -\frac{1}{2}$ and

$$f(t, y(t), {}^{H}D_{0^+}^{\frac{1}{2},\frac{1}{3}}y(t)) = \frac{t}{10}(1 + |y(t)| + |D_{0^+}^{\frac{1}{2},\frac{1}{3}}y(t)|).$$

Clearly, the function f is continuous on $(0, 1]$. For all $t \in (0, b]$ and $u, v, \overline{u}, \overline{v} \in \mathbb{R}$, we have

$$|f(t, u, v) - f(t, \overline{u}, \overline{v})| \leq \frac{1}{10}\left[|u - \overline{u}| + |v - \overline{v}|\right].$$

Hence, the first hypothesis (H_1) is satisfied with $M = L = \frac{1}{10}$. For $\lambda = -\frac{1}{2}$, $\alpha = \frac{1}{2}, \gamma = \frac{2}{3}, b = 1$ and $M = L = \frac{1}{10}$ by direct calculations we conclude that $\Theta < 1$. It follows from Theorem 3, that the problem (38) has a unique solution on $(0, 1]$.

Moreover, for $u, v \in \mathbb{R}$ and $t \in (0, 1]$ we find that

$$|f(t, u, v)| \leq \frac{t}{10}\left(1 + |u(t)| + |v(t)|\right).$$

Thus, the assumption (H_2) is satisfied with $m(t) = q(t) = p(t) = \frac{t}{10}$. Clearly, the functions m, q and p are continuous on $[0, 1]$ and $m^* = q^* = p^* = \sup\limits_{t\in[0,1]} \frac{t}{10} = \frac{1}{10} < 1$. Now, by simple calculations, we get

$$\varrho = \left(\frac{E_{\alpha,\gamma}(\lambda t^{\alpha})}{1 - E_{\alpha,1}(\lambda b^{\alpha})}\frac{\Gamma(\gamma)}{\Gamma(\alpha + 1)} + \frac{B(\alpha, \gamma)}{\Gamma(\alpha)}\right)\frac{(\lambda + q^*)\, b^{\alpha}}{(1 - p^*)} < 1.$$

Using Theorem 2, we can conclude that the problem (38) has at least one solution on $(0,1]$.

For $t \in (0,1]$, let $\varphi_\alpha \in C_{\frac{1}{3}}([0,1],\mathbb{R})$ be such that $\varphi_\alpha(t) = t$. We have

$$I_{0^+}^{\frac{1}{2}}\varphi(t) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}} s ds \leq \frac{2}{\sqrt{\pi}}\varphi(t) = \delta_\varphi \varphi(t),$$

where $\delta_{\varphi_\alpha} = \frac{2}{\sqrt{\pi}}$. On the other hand, as shown in Theorem (5), for $\epsilon = 1$, if $x \in C_{\frac{1}{3}}([0,1],\mathbb{R})$ satisfies

$$\left| {}^H D_{0^+}^{\alpha,\beta} x(t) - \lambda x(t) - f(t, x(t), {}^H D_{0^+}^{\alpha,\beta} x(t)) \right| \leq t, \quad t \in (0,1],$$

there exists a unique solution $y(t) \in C_{\frac{1}{3}}([0,1],\mathbb{R})$ such that

$$|x(t) - y(t)| \leq \eta_{f,\varphi_\alpha} t.$$

where $\eta_{f,\varphi_\alpha} := \dfrac{\nabla \delta_{\varphi_\alpha}}{1 - \frac{\lambda+M}{1-L}\delta_{\varphi_\alpha}} \approx \dfrac{2\sqrt{\pi}}{1 + \frac{8}{9\sqrt{\pi}}}$, $\nabla \approx \pi$ and

$$E_{\frac{1}{2}}\left(-\frac{1}{2}\right) = e^{\frac{1}{4}}\left(1 - erf\left(\frac{1}{2}\right)\right).$$

It follows from Theorem (5) that the problem (38) is generalized Ulam–Hyers–Rassias stable.

5. Concluding remarks. In this paper, we have successfully established the existence and uniqueness results of fractional implicit differential equations with a periodic condition, involving Hilfer derivative. Moreover, we have discussed the different types of stability of solutions to such equations in the weighted space $C_{1-\gamma}([0,b],\mathbb{R})$. In addition, an example is presented to illustrate the results. In the future, we plan to extend the results to other fractional derivatives and boundary-value problems, especially, we discuss the global attractivity for the boundary-value problem using the generalized fractional derivative. This topic will be the subject of a forthcoming paper.

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