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THE SUMMARY EQUATION FOR FUNCTIONS ANALYTICAL OUTSIDE FOUR SQUARES. APPLICATIONS

Abstract. We consider the lacunary Stieltjes moment problem

$$\int_0^\infty F(x)x^{4n+1}\exp(-x)\,dx=\beta_n,\ n=0,1,2.$$

We search for a solution in the class of entire functions of the exponential type that satisfy the condition $F(iz)=F(z)$. Their indicator diagram is a certain octagon. We construct nontrivial solutions to the corresponding homogeneous problem. The problem reduces studying a linear summary equation in the class of functions holomorphic outside four squares. At infinity, they have a zero of multiplicity at least three. The boundary values satisfy a Hölder condition on any compact that does not contain the vertices. At the vertices, we allow at most logarithmic singularities. We search for a solution in the form of a Cauchy-type integral with an unknown density over the boundary of those squares. We suggest a method for the regularization of the summary equation. An equivalence condition for this regularization is established. Additionally, we identify some special cases, in which the obtained Fredholm equation of the second kind is solvable. For this, we use the contraction mapping theorem in a Banach space.

Key words: *equivalent regularization, Carleman problem, moments of entire functions*

2010 Mathematical Subject Classification: *30D05*

1. Introduction and problem statement. Let D_1 be a unit square with vertices $t_1=\gamma-i/2$, $t_2=\gamma+1-i/2$, $t_3=\gamma+1+i/2$, $t_4=\gamma+i/2$, and sides ℓ_j, $j=\overline{1,4}$, taken in the order in which they occur on the boundary

$\Gamma_1 = \partial D_1$ described positively ($t \in \ell_1 \Rightarrow \operatorname{Im} t = -0.5$). Here $\gamma \in (0.5, 1)$. Consider four functions:

$$\sigma_m(z) = t_m + t_{m+1} - z, \quad m = \overline{1,4} \quad (t_5 = t_1). \tag{1}$$

They induce a Carleman involutive shift $\alpha(t) = \{\sigma_m(t),\ t \in \ell_m\}$, which maps each side into itself changing its orientation. Moreover, the middle point of the sides are fixed points of the shift.

Let us consider another three squares $D_j = i^{j-1} D_1$, $j = \overline{2,4}$. The functions $\sigma_m(z)$ are defined by (1) only for $z \in D_1$. Assume that $\sigma_m(z) = \sigma_m(i^{1-j} z)$, $j = \overline{2,4}$, for $z \in D_j$.

Thus, the shift $\alpha(t)$ has been defined on the whole set Γ. We will use the following notation: $D = \bigcup\limits_{k=1}^{4} D_k$, $\Gamma_k = \partial D_k$, $\Gamma = \bigcup\limits_{k=1}^{4} \Gamma_k$.

Consider the functional equation

$$(V f)(z) = \sum_{m=1}^{4} f\left[\sigma_m(z)\right] = g(z),\ z \in D, \tag{2}$$

under the following assumptions:

1) The solution $f(z)$ is holomorphic on D and has a zero of multiplicity at least three at infinity. Moreover,

$$f(iz) = -i f(z). \tag{3}$$

Its boundary value $f^-(t)$ satisfies the Hölder condition on any compact set that does not contain vertices of the squares. At the vertices, we allow at most logarithmic singularities. We denote this class of solutions by B.

2) The independent term $g(z)$ is a piecewise holomorphic function, i.e., $g(z) = g_j(z)$, $z \in D_j$, $j = \overline{1,4}$, and satisfies the condition given in (3). Each function $g_j(z)$ is holomorphic on D_j and its boundary value $g_j^+(t) \in H(\Gamma_j)$.

Note that if $z \in D$, then $\sigma_m(z) \notin \overline{D}$. Therefore, we will search for a solution in the class of functions that are holomorphic outside D. In the applications, it is convenient to assume that it vanishes at infinity. In other words, the solution and the independent term are in different classes of analytical functions. classes of analytical functions. Operator V anticommutes with the differentiation operator, so we cannot apply powerful classical methods (see [1]) to equation (2). On the other hand, the equation turns out to be well-suited for some applications (see part 4 below).

Let us describe the problem. Associate a set $H_j = \bigcup_{k=1}^{4} \sigma_k(D_j)$ with each square. If $H_j \cap D_k \neq \varnothing$ for $k \neq j$, the problem becomes meaningless. Recall that the solution $f(z)$ is defined only outside D. Thus, we should set $\gamma > 0.5$, so that the squares are not "too close" to each other. On the other hand, it is, henceforth, essential that $H_j \cap H_{j+1} \neq \varnothing$, $j = \overline{1,4}$, $H_5 = H_1$. Therefore, $\gamma < 1$. Note that the disconnectedness of the sets $C \setminus H_j$, $j = \overline{1,4}$, is what makes the problem given in (2) nontrivial. Problem (2) in the case of a single arbitrary quadrangle D was first studied in [2].

This paper consists of four parts. In the first part, we give the problem statement. In the second part, we suggest a method for regularization of problem (2), and establish conditions for its equivalence. In the third part, we study some special cases of the given equation, when it is possible to prove that the obtained Fredholm equation of the second kind is unconditionally solvable. In the fourth part, we examine the Stieltjes moment problem for entire functions of the exponential type, which is associated with problem (2). For this, we apply the Borel transformation [3, § 1, 1.1].

2. Problem regularization. We look for a solution to problem (2) in the form of a Cauchy-type integral:

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} (\tau - z)^{-1} \varphi(\tau)\, d\tau, \ \ z \notin \bar{D},$$

with an unknown density $\varphi(\tau)$ that satisfies the conditions

$$\varphi(it) = -i\varphi(t) \tag{4}$$

and

$$\int_{\Gamma} \varphi(\tau)\, d\tau = 0 \Rightarrow \forall j \int_{\Gamma_j} \varphi(\tau)\, d\tau = 0. \tag{5}$$

We can assume, without loss of generality, that

$$\varphi(\tau) + \varphi[\alpha(\tau)] = 0. \tag{6}$$

Indeed, the density $\varphi(\tau)$ is defined on each boundary Γ_j up to a term $a_j^+(\tau)$, which is the boundary value of a function $a_j(z)$, holomorphic on D_j.

By suitably choosing this function, it is possible to fulfill condition (6). This condition is considered as a Carleman problem for the unknown function $a_j(z)$. The solvability of this problem is a consequence of the principle of locally conformal gluing [4] and condition (5). As a result,

$$(2) \Rightarrow (A\varphi)(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \varphi(\tau) E(z,\tau)\, d\tau = g(z),\ z \in D, \tag{7}$$

where

$$E(z,\tau) = \sum_{m=1}^{4} (\tau - \sigma_m(z))^{-1};\ z \in D.$$

Proceed to the limit in (7) as $z \to t \in \Gamma$. In view of (6), we obtain a formula similar to the Sokhotski–Plemelj formula; namely

$$\left(A^{+}\varphi\right)(t) \equiv 2^{-1}\varphi(t) + (A\varphi)(t) = g^{+}(t), \tag{8}$$

in which the singular integral $(A\varphi)(t)$ is the result of the formal substitution of the variable $z \in D$ by $t \in \Gamma$ in (7) and must be understood in the sense of the Cauchy principal value. Substitute in (8) the variable t by $\alpha(t)$ and subtract the obtained equality from the initial one. Taking into account (6), we have

$$(T\varphi)(t) \equiv \varphi(t) + \frac{1}{2\pi i} \int\limits_{\Gamma} K(t,\tau)\varphi(\tau)\, d\tau = g^{+}(t) - g^{+}[\alpha(t)], \tag{9}$$

where

$$K(t,\tau) = E(t,\tau) + E\left[\alpha(t), \alpha(\tau)\right]. \tag{10}$$

Lemma 1. *Integral equation (9) is a Fredholm equation of the second kind.*

Proof. It is enough to verify, by direct enumeration of all the possible arrangements of the points τ and t on the sides Γ, that the kernel function (10) is bounded. With no loss of generality, we can assume that, for instance, $t \in l_1 \Rightarrow \alpha(t) = t_1 + t_2 - t$. Then, the restrictions on γ imply that kernel (10) may be unbounded only when $\tau \in \Gamma_1$. Let us consider three different cases. For the sake of brevity, we assume that $u = \tau + t$.

a) $\tau \in l_1 \Rightarrow \alpha(\tau) = t_1 + t_2 - \tau$. Then $K(t,\tau) = (u - t_2 - t_3)^{-1} + (u - {}$ ${}-t_3 - t_4)^{-1} + (u - t_1 - t_4)^{-1} - (u - t_2 - 2t_1 + t_3)^{-1} - (u - 2t_1 - 2t_2 +$

$+t_3+t_4)^{-1}-(u-t_1-2t_2+t_4)^{-1}$. The sum (10) does not contain the term $(u-t_1-t_2)^{-1}$, which might be unbounded for such τ and t.

b) $\tau\in l_3 \Rightarrow \alpha(\tau)=t_4+t_3-\tau \Rightarrow K(t,\tau)=0$.

c) $\tau\in l_2 \Rightarrow \alpha(\tau)=t_2+t_3-\tau$. Then $K(t,\tau)=(u-t_3-t_4)^{-1}+(u-t_1-t_4)^{-1}-(u+t_4-t_1-2t_2)^{-1}-(u+t_4-2t_2-t_3)^{-1}$. The sum (10) does not contain the "bad" terms $(u-t_2-t_3)^{-1}$ and $(u-t_1-t_2)^{-1}$.

The fourth case, namely $t\in l_4$, is similar to case c). This finishes the proof. □

Corollary. *Integral equation (9) has a finite number of solvability conditions.*

Assume that these conditions hold. Make the reverse transition from integral equation (9) to the initial problem (2). In the same manner, as it is done in [5], we can show that there exists a solution $\varphi(t)$ of equation (9) that satisfies conditions (4) and (6). Then (9) $\Rightarrow$ $(A^+\varphi)(t)-(A^+\varphi)(\alpha(t))=g^+(t)-g^+[\alpha(t)]$, i.e., $(A\varphi)(z)=g(z)+C_z$, $z\in D$, because the solution of the Carleman problem $(A^+\phi)(t)=(A^+\phi)(\alpha(t))$ can only be a constant. The piecewise constant C_z is constant on each square and, moreover, $C_{iz}=-iC_z$.

Theorem 1. *Problem (2) has only a finite number of solvability conditions; namely, the solvability conditions of integral equation (9) and the additional condition*

$$(A\varphi)(z_0)=g_1(z_0), \tag{11}$$

where $z_0\in D_1$: it ensures the equivalence of the regularization.

Assume that all conditions hold and problem (2) is solvable. The set $H_0=\bigcap_{j=1}^{4}H_j$ is a square with vertices at the points $(\gamma-1)(\pm1\pm i)$. The signs here do not match. Consider a point $z\in H_0$. Using the condition of problem (2) on D_1, we have

$$f(z)+f(z+1-i)+f(z+1+i)+f(z+2)=g_1(t_1+t_4-z).$$

In exactly the same manner, using the condition of problem (2) on D_3, we obtain that $f(z)+f(z-2)+f(z-1+i)+f(z-1-i)=g_3(-t_1-t_4-z)$. Computing the sum of the last two equalities, we have

$$2f(z)+f(z+1+i)+f(z-1-i)+f(z-1+i)+f(z-1-i)+$$
$$+f(z+2)+f(z-2)=g_1(t_1+t_4-z)+g_3(-t_1-t_4-z),\ z\in H_0.$$

Next, plugging in the conditions of problem (2) on D_2 and D_4, we obtain:

$$2f(z)+f(z+1-i)+f(z-1-i)+f(z+1+i)+f(z+1-i)+$$
$$+f(z-2i)+f(z+z_0)=g_2(it_1+it_4-z)+g_4(-it_1-it_4-z),\ z\in H_0.$$

Therefore,

$$f(z+2)+f(z-2)-f(z+2i)-f(z-2i)=g_0(z),\ z\in H_0, \tag{12}$$

where

$$g_0(z)=g_1(t_1+t_4-z)+g_3(-t_1-t_4-z)-g_2(it_1+it_4-z)-g_4(-it_1-it_4-z). \tag{13}$$

3. Study of the Fredholm integral equation in some special cases.

In what follows, we assume that $\gamma \geq 0.9$. Let us prove that equation (9) is solvable. Consider the corresponding homogeneous equation:

$$T\varphi=0. \tag{14}$$

We assume that the operator T is defined in the Banach space $\widetilde{C}(\Gamma)$. This is the set of functions continuous on the closure of each side of the square, with a norm defined in the natural manner, namely

$$M=\max|\varphi(t)|\,,\ t\in\Gamma. \tag{15}$$

At this, the vertices might be, at most, discontinuities of the first kind. Since $A(t,\tau)=A(\tau,t)$, we have $T'=T$. We can construct a fundamental system of solutions (f.s.s.) for equation (14) in such a way that each function of the system satisfies either condition (6), or the opposite condition, i.e., $\varphi(t)=\varphi[\alpha(t)]$ (on this matter, see [4]). The solutions that agree with the last condition, are automatically orthogonal to the right-hand side of (9), since (6) $\Rightarrow$ (5).

Lemma 2. *The f.s.s. of equation* (14) *does not contain functions satisfying condition* (6).

Proof. Assume the opposite and, for definiteness, set $\gamma=0.9$. By virtue of (6), we can consider the kernel $K_1(t,\tau)=2^{-1}[K(t,\tau)-K(t,\alpha(\tau))]$ instead of kernel (10). Let us estimate the absolute value of the kernel from above. Owing to the symmetry of Γ, it is enough to consider only two cases:

I. The maximum (15) is achieved at some point $t \in \ell_4$. Then

$$A(t,\tau) = (u-2.8+i)^{-1} + (u-3.8)^{-1} + (u-2.8-i)^{-1} + (u-1.8)^{-1},$$
$$u = \tau + t; \quad \alpha(t) = 1.8 - t, \quad t = 0.9 + iy, \quad y \in [-0.5; 0.5].$$

We need to consider four subcases:

1.1. $\tau \in \ell_1 \Rightarrow K(t,\tau) = (u-2.8-i)^{-1} + (u-3.8)^{-1} - (u-0.8+i)^{-1} - (u-1.8+2i)^{-1} \Rightarrow |K_1| \leq 0.08$;

1.2. $\tau \in \ell_2 \Rightarrow K(t,\tau) = 0$;

1.3. $\tau \in \ell_3$. By virtue of the symmetry, we can affirm that the estimate $|K_1| \leq 0.08$ is valid.

1.4. $\tau \in \ell_4 \Rightarrow K(t,\tau) = (u-2.8+i)^{-1} + (u-3.8)^{-1} + (u-2.8-i)^{-1} - (u-0.8-i)^{-1} - (u+0.2)^{-1} - (u-0.8+i)^{-1} \Rightarrow |K_1| \leq 0.13$.

For $\tau \in \Gamma_1$, the kernel $E(t,\tau)$ contains terms whose integral must be understood in the sense of Cauchy principal value. But kernel (10) is bounded. Therefore, kernel (10) is given explicitly (subcases 1.1–1.4). There are no such terms, when $\tau \notin \Gamma_1$, i.e., there is no need to cancel such terms.

Assume that $\tau \in \Gamma_2$. Denote the lower, right, upper, and left sides of the square by ℓ_j, $j = \overline{1,4}$, respectively. If $\tau \in \ell_j$, then $|K_1| \leq C_j$, with $C_1 = 0.5$, $C_2 = 0.32$, $C_3 = 0.15$, and $C_4 = 0.18$.

Suppose that $\tau \in \Gamma_3$. In this case, we obtain $C_1 = C_3 = 0.12$, $C_2 = 0.19$, $C_4 = 0.08$.

The estimates in the case $\tau \in \Gamma_4$ are the same as in the case $\tau \in \Gamma_2$. Sum out all the numbers, to see that the sum is less than 2π, i.e., $\varphi \equiv 0$. Thus, we obtain a contradiction if we suppose that the maximum (15) is achieved at a point $t \in \ell_1$.

II. The maximum (15) is achieved at some point $t \in \ell_j$, $j = \overline{1,3}$. All the estimates above can be obtained for this case also and they are, at least, not worse, since the point in this case is "not closer" to the squares D_j, $j = \overline{2,4}$, compared to case I.

Finally, assume that $\gamma > 0.9$. Then the squares are farther from each other than in the case we have just considered, and the estimates can only get better. This finishes the proof of the lemma. $\square$

Remark 1. *From the estimates obtained in the proof of Lemma 2, it follows that the lemma is also valid for some numbers $\gamma \in (0.5, 0.9)$. In this paper, however, we do not investigate how low this number γ can possibly be.*

Theorem 2. *If $\gamma \geq 0.9$, then problem (2) has the solvability condition (11), which is unique.*

Remark 2. *For each function $g(z)$, we can find a constant C, such that problem (2) is unconditionally solvable for $\widetilde{g}_1(z) = g_1(z) + C, z \in D_1$.*

4. Applications to the moment problem Now we consider some applications of problem (2). Let $F(z)$ be an entire function of the exponential type and assume that $F(z)$ is an upper function, Borel-associated with the lower function $f(z) \in B$. Its indicator diagram is an octagon D_0 with vertices $\pm t_2$, $\pm t_3$, $\pm it_2$, $\pm it_3$ (the signs do not match). Moreover,

$$F(iz) = F(z). \tag{16}$$

Now, use equality (3) to rewrite (12) in the form

$$2\int\limits_0^\infty F(x)\exp(-2x)\left[\operatorname{sh}(xz) + \sin(xz)\right]\, d\tau = g_0(z),\ \ z \in H_0, \tag{17}$$

in which the second term is defined by (13). Equate the Maclaurin coefficients of the first and second terms in (17). As a result, we obtain a Stieltjes moment problem for an entire function of exponential type with an exponential weight, namely

$$L\left[F, n\right] \equiv 4\int\limits_0^\infty F(x)\exp(-2x)x^{4k+1}\, dx = g_0^{(4k+1)}(0),\ \ k = \overline{0, \infty}.$$

Since $g_0^{(4k+1)}(0) = 4g_1^{(4k+1)}(2\gamma)$, consider the series

$$g_1(z) = \beta_0 + \sum_{k=1}^{\infty} \frac{\beta_k (z - 2\gamma)^k}{k!}. \tag{18}$$

Assume that its convergence radius $R > \sqrt{\gamma^2 + 0.25}$. Choose the coefficient β_0 in such a way that problem (2) be solvable.

Theorem 3. *The moment problem $L\left[F, n\right] = \beta_{4n+1}$, $n = 0, 1, 2, \ldots$, under the condition*

$$e\sqrt{\gamma^2 + 0.25}\ \overline{\lim_{n\to\infty}}\ \frac{\sqrt[4n+1]{\beta_{4n+1}}}{4n+1} < 1$$

is solvable in the class of entire functions $F(z)$ that are of the exponential type, satisfy condition (16) *holds, and are Borel-associated with the lower function $f(z) \in B$.*

Proof. Construct nontrivial solutions to the homogeneous problem $L[F, n] = 0$. It is enough to suppose that $\beta_n \neq 0$ in (18) for some $n > 0$ and $\beta_{4n+1} = 0$ for any n. $\square$

Remark 3. *Nontrivial solutions were constructed in [6] for the homogeneous lacunary moment problem*

$$\int\limits_0^\infty F(x)\exp(-x)x^{4n+3}\,dx = 0,\ \ n = 0, 1, 2, \ldots,$$

in the class of entire functions $F(z)$ of the exponential type that satisfy condition (16). *However, their indicator diagram is a square.*

Remark 4. *The conjugated indicator diagram may not be the octagon D_0, but some "smaller" convex set $D_0' \subset D_0$. However, this case is not interesting. Then problem* (2) *is overdetermined. Condition* (2) *holds not only for $z \in D$, but also in some neighborhood of infinity. A necessary (but not sufficient!) condition for this is the possibility of analytical continuation of $g_1(z)$ from D_1 to some neighborhood of infinity, with $g_1(\infty) = 0$. See [7], [8] for more details on this case.*

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