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ON A CLASS OF MAPPINGS HARMONIC IN THE HALF-PLANE

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Well known is the class of typically-real functions holomorphic in the unit disc $|z| < 1$, introduced by W. Rogosinski in 1932 ([6]). There were also investigated classes of typically-real functions harmonic in the unit disc ([1,4]). Whereas in the present paper we consider some class of typically-real functions harmonic in the right half-plane. The results are based on the considerations concerning typically-real functions holomorphic in this half-plane ([3,5]).

§ 1. Definition and basic properties of the class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$

Let $\Pi^+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and let first $H = H(\Pi^+)$ be the class of all functions f holomorphic in Π^+ and such that

$$\lim_{\Pi^+ \ni z \to \infty} (f(z) - z) = a_f \tag{1.1}$$

where a_f is some complex number.

Let $\mathcal{T}_{\mathcal{R}}$ be a subclass of functions of H which take real values on the positive real half-axis only, that is

$$f(z) = \overline{f(z)} \Leftrightarrow z = \overline{z}, \quad z \in \Pi^+. \tag{1.2}$$

Evidently, for $f \in \mathcal{T}_{\mathcal{R}}$, in (1.1) we have $a_f \in \mathbb{R}$.

For the class $\mathcal{T}_{\mathcal{R}}$, we have

Proposition 1. ([3]) *A function $f \in H$ belongs to the class $\mathcal{T}_{\mathcal{R}}$ if and only if*

$$\operatorname{Im} f(z) \begin{cases} > 0 & \text{when } \operatorname{Im} z > 0, \\ = 0 & \text{when } \operatorname{Im} z = 0, \quad z \in \Pi^+, \\ < 0 & \text{when } \operatorname{Im} z < 0. \end{cases} \tag{1.3}$$

Let now $\mathcal{H} = \mathcal{H}(\Pi^+)$ denote the class of complex functions F harmonic in Π^+ and satisfying normalization condition (1.1). As we know, each function F harmonic in Π^+ is of the form $F = h + \bar{g}$, where h, g are some functions holomorphic in Π^+, because Π^+ is a simply connected domain.

Let next $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$ denote the class of functions $F \in \mathcal{H}$, $F = h + \bar{g}$, such that $\operatorname{Im} F$ satisfies condition (1.3) and

$$\lim_{\Pi^+ \ni z \to \infty} \operatorname{Re} g(z) = \alpha_g, \quad \alpha_g \in \mathbb{R}. \tag{1.4}$$

The class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$ will be called a class of harmonic functions typically-real in Π^+.

It follows from normalization condition (1.1) and from (1.3) that $a_F \in \mathbb{R}$ for $F \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$.

Example 1.1. Let F_1 be a function of the form

$$F_1(z) = z - \frac{1}{z} + \frac{1}{\bar{z}}, \quad z \in \Pi^+. \tag{1.5}$$

Function (1.5) satisfies conditions (1.1), (1.3) and (1.4), thus $F_1 \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$.

Directly from the definition of the class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$ we have

Proposition 2. *If $F = h + \bar{g}$ belongs to the class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, then the functions $F_\eta(z) = F(z) + \eta$, $\eta \in \mathbb{R}$, $F_\delta(z) = F(z + \delta)$, $\delta > 0$, $F_\rho(z) = \frac{f(\rho z)}{\rho}$, $\rho > 0$, $z \in \Pi^+$, belong to $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$.*

Let us notice that

$$\mathcal{T}_{\mathcal{R}} \subset \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}. \tag{1.6}$$

Indeed, if $f \in \mathcal{T}_{\mathcal{R}}$, then $f = h + \bar{g}$, where $h \equiv f$, $g \equiv 0$ is a function harmonic in Π^+, with that we also have (1.3), (1.4). In virtue of the definition of $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, this proves inclusion (1.6).

We next have

Proposition 3. *Let $f \in \mathcal{T}_{\mathcal{R}}$ be a function such that $\operatorname{Re} f(z) > 0$, $z \in \Pi^+$, and $F \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$. Then the function $k = F \circ f$ belongs to the class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$.*

Proof. Let $f \in \mathcal{T}_{\mathcal{R}}$, $\operatorname{Re} f(z) > 0$, $z \in \Pi^+$ and $F \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$. Since $F = h + \bar{g}$ is harmonic in Π^+, the function $k = F \circ f = h \circ f + \overline{g \circ f}$ is harmonic in Π^+. Moreover, the function k satisfies condition (1.3) because we have (1.3) for f and F.

Let

$$a_f = \lim_{\Pi^+ \ni z \to \infty} (f(z) - z), \quad a_F = \lim_{\Pi^+ \ni z \to \infty} (F(z) - z),$$

$\alpha_g = \lim_{\Pi^+ \ni z \to \infty} \operatorname{Re} g(z)$. Hence $\lim_{\Pi^+ \ni z \to \infty} f(z) = \infty$, therefore

$$\lim_{\Pi^+ \ni z \to \infty} \operatorname{Re} g(f(z)) = \alpha_g \quad \text{and} \quad \lim_{\Pi^+ \ni z \to \infty} (k(z) - z) = a_F + a_f.$$

Consequently, the function $k = F \circ f$ satisfies all the conditions of the definition of the class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$.

Before the next example let us denote by $\mathcal{R}$ a subclass of functions $f \in H$ mapping Π^+ conformally onto a domain convex in the direction of the positive real half-axis. The class $\mathcal{R}$ is called a class of functions convex in the direction of the real axis in Π^+ (see [7]). We have

THEOREM A ([7]). *A function $f \in H$ belongs to the class $\mathcal{R}$ if and only if*

$$\operatorname{Re} f'(z) > 0, \quad z \in \Pi^+. \tag{1.7}$$

Of course, by (1.7), all functions of the class $\mathcal{R}$ are univalent ([2], p. 88).

Let next $\mathcal{R}_{\mathcal{R}}$ denote a subclass of $\mathcal{R}$ of functions which take real values for $z = \bar{z} \in \Pi^+$. Obviously,

$$\mathcal{R}_{\mathcal{R}} \subset \mathcal{T}_{\mathcal{R}}. \tag{1.7$'$}$$

EXAMPLE 1.2. Let f_1 be a function defined by the formula

$$f_1(z; a, c) = z + a - \frac{1}{z + c}, \quad z \in \Pi^+, \quad c \geq 1, \quad ac \geq 1. \tag{1.8}$$

It can easily be proved that: a) $f_1 \in \mathcal{T}_{\mathcal{R}}$, b) $f_1(\Pi^+) \subset \Pi^+$, c) $f_1 \in \mathcal{R}_{\mathcal{R}}$.

From Proposition 1.3 and the above example it follows that if $F \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, then $f \circ f_1 \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$.

The following proposition holds.

Proposition 4. *Let $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$. Then the function $f = h - g$ belongs to the class $\mathcal{T}_{\mathcal{R}}$.*

PROOF. If $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, then, of course, the function $f = h - g$ is holomorphic in Π^+. Moreover, $\operatorname{Im} f = \operatorname{Im} F$, so, by (1.3), the function f

satisfies condition (1.2). Furthermore,

$$\begin{aligned} f(z) - z &= h(z) - g(z) - z = h(z) + \overline{g(z)} - z - 2\mathrm{Re}\, g(z) \\ &= F(z) - z - 2\mathrm{Re}\, g(z), \quad z \in \Pi^+. \end{aligned}$$

Since F satisfies condition (1.1) and g satisfies condition (1.4), from the above equalities we get $\lim\limits_{\Pi^+ \ni z \to \infty} (f(z) - z) = a_F - 2\alpha_g =: a_f \in \mathbb{R}$. Hence the function f belongs to $\mathcal{T}_{\mathcal{R}}$.

We also have

Proposition 5. *Given a pair of functions h, g holomorphic in Π^+ and such that $f = h - g \in \mathcal{T}_{\mathcal{R}}$, where g satisfies condition (1.4), the function $F = h + \bar{g}$ belongs to $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$.*

Proof. Let the functions h, g satisfy the assumptions. Then the function $F = h + \bar{g}$ is harmonic in Π^+; besides, we have (1.4). Moreover, $\mathrm{Im}\, F = \mathrm{Im}\, f$, so, in virtue of Proposition 1.1, the function F satisfies condition (1.3), and

$$F(z) - z = f(z) - z + 2\mathrm{Re}\, g(z), \quad z \in \Pi^+.$$

According to the fact that we have (1.1) for f and (1.4) for g, the function F satisfies normalization condition (1.1). Consequently, $F \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$.

In particular, we obtain

Proposition 6. *If $f \in \mathcal{T}_{\mathcal{R}}$, then the function $F(z) = 2\mathrm{Re}\, z - \overline{f(z)}$, $z \in \Pi^+$, belongs to the class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$.*

Proof. Let $f \in \mathcal{T}_{\mathcal{R}}$. Let us set $f = h - g$ where $h(z) = z$, $g(z) = z - f(z)$, $z \in \Pi^+$. Of course, the functions h, g are holomorphic in Π^+ and g satisfies condition (1.4), which follows from the fact that f satisfies (1.1). Therefore, by Proposition 1.5, the function $F = h + \bar{g}$ belongs to the class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, with that $F(z) = h(z) + \overline{g(z)} = 2\mathrm{Re}\, z - \overline{f(z)}$, $z \in \Pi^+$, which ends the proof.

The proposition below is also true.

Proposition 7. *If $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, whereas w is an arbitrary function holomorphic in Π^+ and satisfying condition (1.4), then*

$$F^w = h + w + \overline{g + w} \tag{1.9}$$

is a function of the class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$.

Proof. Let $F = h + \bar{g} \in \mathcal{T}^{\mathcal{H}}_{\mathcal{R}}$ and let w be a function holomorphic in Π^+ and satisfying (1.4). Of course, the function F^w of form (1.9) is harmonic in Π^+ and $\operatorname{Im} F^w = \operatorname{Im} F$, thus we have (1.3) for the function F^w. Furthermore,

$$F^w(z) - z = F(z) - z + 2\mathrm{Re}\, w(z), \quad z \in \Pi^+.$$

Since F satisfies condition (1.1) and w - (1.4), it follows that F^w also satisfies (1.1). Hence $F^w \in \mathcal{T}^{\mathcal{H}}_{\mathcal{R}}$.

. Let $F = h + \bar{g} \in \mathcal{T}^{\mathcal{H}}_{\mathcal{R}}$. We shall prove that the function

$$F^1(z) = F(z) + 2\mathrm{Re}\,(h(z) - z), \quad z \in \Pi^+,$$

also belongs to the class $\mathcal{T}^{\mathcal{H}}_{\mathcal{R}}$.

Let us first notice that, for a function $F = h + \bar{g} \in \mathcal{T}^{\mathcal{H}}_{\mathcal{R}}$, the function $h_1(z) = h(z) - z$, $z \in \Pi^+$, is holomorphic in Π^+ and satisfies condition (1.4). Indeed, $\lim\limits_{\Pi^+ \ni z \to \infty} \mathrm{Re}\, h_1(z) = \lim\limits_{\Pi^+ \ni z \to \infty} \mathrm{Re}\,(h(z) - z) = \lim\limits_{\Pi^+ \ni z \to \infty} [\mathrm{Re}\,(F(z) - z) - \mathrm{Re}\, g(z)] = a_F - \alpha_g \in \mathbb{R}$. From this, by Proposition 1.7, the function

$$F^1(z) = h(z) + h_1(z) + \overline{g(z) + h_1(z)} = F(z) + 2\mathrm{Re}\,(h(z) - z), \quad z \in \Pi^+,$$

as well as F, belongs to the class $\mathcal{T}^{\mathcal{H}}_{\mathcal{R}}$.

Similarly, each function

$$F^\beta(z) = F(z) + 2\beta\mathrm{Re}\,(h(z) - z), \quad z \in \Pi^+, \ \ \beta \in \mathbb{R},$$

belongs to $\mathcal{T}^{\mathcal{H}}_{\mathcal{R}}$.

Example 1.3. Let F_2 be a function of the form

$$F_2(z; a, b, \nu) = z + a + \frac{b}{\bar{z}^\nu},\ z \in \Pi^+,\ a \in \mathbb{R},\ b \geq 0,\ \nu \in (0, 2],\ 1^\nu = 1. \tag{1.10}$$

We can prove that function (1.10) belongs to the class $\mathcal{T}^{\mathcal{H}}_{\mathcal{R}}$. Moreover, by Proposition 1.7, the function

$$F_2^w(z; a, b, \nu) = z + a + w(z) + \overline{\frac{b}{z^\nu} + w(z)},\ a \in \mathbb{R},\ b \geq 0,\ \nu \in (0, 2],\ 1^\nu = 1,$$

belongs to $\mathcal{T}^{\mathcal{H}}_{\mathcal{R}}$ if the function w satisfies appropriate assumptions.

In the case $b > 0$, $\nu > 2$ the function F_2 of form (1.10) does not satisfy condition (1.3) and it is not a function of $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$.

We shall prove

Proposition 8. *The class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$ is convex.*

Proof. If $F_k = h_k + \bar{g}_k \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, $k = 0, 1$, then the function $F_\lambda = \lambda F_1 + (1-\lambda)F_0 = \lambda h_1 + (1-\lambda)h_0 + \overline{\lambda g_1 + (1-\lambda)g_0}$, $\lambda \in (0,1)$,is harmonic in Π^+. We also have $\lim\limits_{\Pi^+ \ni z \to \infty} (F_\lambda(z) - z) = \lambda a_1 + (1-\lambda)a_0$, where $a_k = \lim\limits_{\Pi^+ \ni z \to \infty} (F_k(z) - z)$, $k = 0, 1$. Furthermore, $\lim\limits_{\Pi^+ \ni z \to \infty} \operatorname{Re}[\lambda g_1(z) + (1-\lambda)g_0(z)] = \lambda\alpha_1 + (1-\lambda)\alpha_0$ where $\alpha_k = \lim\limits_{\Pi^+ \ni z \to \infty} \operatorname{Re} g_k(z)$, $k = 0, 1$, and $\operatorname{Im} F_\lambda = \lambda F_1 + (1-\lambda)\operatorname{Im} F_0$. Hence, in virtue of the definition of the class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, we infer that $F_\lambda \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$ for any $\lambda \in (0,1)$, which gives the assertion.

§ 2. Properties of the class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$ following from its relationship with the class $\mathcal{T}_{\mathcal{R}}$

In this part of the paper we use the known properties of the class $\mathcal{T}_{\mathcal{R}}$ to obtain the respective consequences for functions of the class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$.

Proposition 9. *If $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, then*

$$h^{(n)}(z) + \overline{g^{(n)}(z)} = \overline{h^{(n)}(z)} + g^{(n)}(z) \text{ for } z = \bar{z} > 0,\ n = 0, 1, 2, \ldots \quad (2.1)$$

Proof. Let $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$. By Proposition 1.4, the function $f = h - g$ belongs to $\mathcal{T}_{\mathcal{R}}$ and, in virtue of Theorem 1 from [3], we have $h^{(n)}(z) - g^{(n)}(z) = \overline{h^{(n)}(z) - g^{(n)}(z)}$ for $z = \bar{z} > 0$, $n = 0, 1, 2, \ldots$, which is equivalent to (2.1).

Moreover, we have

Proposition 10. *Let $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$. Then*

$$F(\bar{z}) = \overline{F(z)}, \quad z \in \Pi^+, \quad (2.2)$$

if and only if

$$h^{(n)}(z) - \overline{g^{(n)}(z)} = \overline{h^{(n)}(z)} - g^{(n)}(z) \text{ for } z = \bar{z} > 0,\ n = 0, 1, 2, \ldots \quad (2.3)$$

Proof. Let $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$. For any $z \in \Pi^+$, there exists $z_0 = \bar{z}_0 > 0$ such that $z \in U_0 = \{z \in \mathbb{C} : |z - z_0| < z_0\}$. In virtue of the holomorphy of the functions h, g in Π^+, we get

$$F(z) = h(z_0)+\overline{g(z_0)}+\sum_{n=1}^{\infty}\frac{h^{(n)}(z_0)}{n!}(z-z_0)^n+\sum_{n=1}^{\infty}\frac{\overline{g^{(n)}(z_0)}}{n!}(\bar{z}-z_0)^n,\ z\in U_0,$$

so

$$\begin{aligned} F(\bar{z}) &= h(z_0)+\overline{g(z_0)}+\sum_{n=1}^{\infty}\frac{h^{(n)}(z_0)}{n!}(\bar{z}-z_0)^n \\ &+\textstyle\sum_{n=1}^{\infty}\frac{\overline{g^{(n)}(z_0)}}{n!}(z-z_0)^n, \quad z\in U_0, \qquad (2.4)\\ \overline{F(z)} &= \overline{h(z_0)}+g(z_0)+\textstyle\sum_{n=1}^{\infty}\frac{\overline{h^{(n)}(z_0)}}{n!}(\bar{z}-z_0)^n \\ &+\textstyle\sum_{n=1}^{\infty}\frac{g^{(n)}(z_0)}{n!}(z-z_0)^n, \quad z\in U_0. \qquad (2.5)\end{aligned}$$

If $z = z_0 + re^{i\varphi}$, $\varphi\in(-\pi,\pi]$, $r\in[0,z_0)$, then, by (2.4), (2.5), we obtain

$$\begin{aligned} F(\bar{z}) = h(z_0)+\overline{g(z_0)}+\sum_{n=1}^{\infty}\frac{r^n}{n!}\Big[\Big(h^{(n)}(z_0)+\overline{g^{(n)}(z_0)}\Big)\cos n\varphi + \\ i\Big(\overline{g^{(n)}(z_0)}-h^{(n)}(z_0)\Big)\sin n\varphi\Big], \qquad (2.6)\\ \overline{F(z)} = \overline{h(z_0)}+g(z_0)+\sum_{n=1}^{\infty}\frac{r^n}{n!}\Big[\Big(\overline{h^{(n)}(z_0)}+g^{(n)}(z_0)\Big)\cos n\varphi + \\ i\Big(g^{(n)}(z_0)-\overline{h^{(n)}(z_0)}\Big)\sin n\varphi\Big]. \qquad (2.7)\end{aligned}$$

By Proposition 2.1, conditions (2.1) hold.

If (2.2) takes place, then, in virtue of (2.6), (2.7) and of the arbitrariness of z and z_0, we obtain (2.3). Whereas if conditions (2.3) take place, then, in view of (2.1) and (2.6), (2.7), we have (2.2), which completes the proof.

. If relations (2.1) and (2.3) are satisfied simultaneously, then

$$h^{(n)}(z)=\overline{h^{(n)}(z)},\quad g^{(n)}(z)=\overline{g^{(n)}(z)}\ \text{for}\ z=\bar{z}>0,\ n=0,1,2,\dots, \qquad (2.8)$$

so the functions h, g and also their succesive derivatives take real values on the half-axis $z=\bar{z}>0$. Of course, if we have (2.8), then equalities (2.1), (2.3) hold. What is more, conditions (2.8) are analogues of the fact that if a function is holomorphic and typically-real in the unit disc, then it expands in a power series with real coefficients.

Proposition 11. *Let $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$. Then*

$$\operatorname{Im} h'(z) = \operatorname{Im} g'(z) \quad \text{for} \quad z = \bar{z} > 0, \tag{2.9}$$

$$\operatorname{Re} h'(z) > \operatorname{Re} g'(z) \quad \text{for} \quad z = \bar{z} > 0. \tag{2.10}$$

Proof. Let $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$. Then, in virtue of Proposition 1.4, the function $f = h - g$ belongs to the class $\mathcal{T}_{\mathcal{R}}$ and, according to the respective property of functions of the class $\mathcal{T}_{\mathcal{R}}$ ([3, Prop. 4]), we have $h'(z) - g'(z) > 0$ for $z = \bar{z} > 0$. From the above inequality we directly obtain relations (2.9) and (2.10).

We also have

Proposition 12. *If $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, then*

$$\frac{|h^{(n)}(z) - g^{(n)}(z)|}{n!(h'(z) - g'(z))} \leq \frac{n}{z^{n-1}} \quad \text{for } z = \bar{z} > 0,\ n = 1, 2, \ldots, \tag{2.11}$$

and

$$\lim_{\bar{z} = z \to \infty} \frac{h^{(n)}(z) - g^{(n)}(z)}{h'(z) - g'(z)} = 0, \quad n = 2, 3, \ldots \tag{2.12}$$

Proof. If $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, then the function $f = h - g$ belongs to $\mathcal{T}_{\mathcal{R}}$. Hence, by the respective property of the class $\mathcal{T}_{\mathcal{R}}$ ([3, Prop. 5]), we get (2.11). Obviously, (2.12) follows directly from (2.11).

Proposition 13. *If $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$ and*

$$\lim_{\bar{z} = z \to 0^+} (h(z) - g(z)) = 0, \tag{2.13}$$

then

$$\operatorname{Re} \frac{h(z) - g(z)}{z} > 0, \quad z \in \Pi^+. \tag{2.14}$$

Proof. If $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$ satisfies condition (2.13), then, for the function $f = h - g \in \mathcal{T}_{\mathcal{R}}$, in view of Theorem 6 from [3], we have $\operatorname{Re} \frac{f(z)}{z} > 0$, $z \in \Pi^+$, which is equivalent to (2.14).

§ 3. Remarks on the compactness of the classes $\mathcal{T}_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$

It is known ([3, Th. 5]), that the class $\mathcal{T}_{\mathcal{R}}$ is not compact. Using the proof of this fact and inclusion (1.6), we obtain

Proposition 14. *The class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$ is not compact.*

The proof of the non-compactness of the class $\mathcal{T}_{\mathcal{R}}$ was based on the construction of a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions of the class $\mathcal{T}_{\mathcal{R}}$, divergent uniformly on compact subsets of Π^+ to infinity.

There arises a question: does there exist a sequence of functions of the class $\mathcal{T}_{\mathcal{R}}$, convergent uniformly on compact subsets of Π^+ to a finite function $f \notin \mathcal{T}_{\mathcal{R}}$? An analogous question concerns the class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$. The partial answer is given by

1. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions of the class $\mathcal{T}_{\mathcal{R}}$, convergent uniformly on compact subsets of Π^+ and on some set $D_R = \{z \in \Pi^+ : |z| \geq R\}$, $R > 0$, to a finite function f. Then $f \in \mathcal{T}_{\mathcal{R}}$.*

Proof. Let us consider a sequence $\{f_n\}_{n\in\mathbb{N}}$ satisfying the assumptions of Theorem 3.1. In virtue of the theorem of Weierstrass, the limit function f is holomorphic in Π^+.

It appears that the function f satisfies normalization condition (1.1). Indeed, let $a_n = \lim\limits_{\Pi^+\ni z\to\infty}(f_n(z) - z)$, $a_n \in \mathbb{R}$, $n = 1, 2, \ldots$ The point ∞ is an accumulation point of both Π^+ and D_R, so, in view of the uniform convergence of the sequence $\{f_n\}_{n\in\mathbb{N}}$ on D_R, by the known theorem on sequences of functions, the sequence $\{a_n\}_{n\in\mathbb{N}}$ is convergent and $\lim\limits_{D_R\ni z\to\infty}(f(z) - z) = \lim\limits_{n\to\infty} a_n =: a$. In virtue of the definitions of the sets Π^+ and D_R, we have

$$\lim_{\Pi^+\ni z\to\infty}(f(z) - z) = \lim_{D_R\ni z\to\infty}(f(z) - z) = a,$$

which gives condition (1.1) for the function f, so $f \in H$.

Let us next notice that since the functions f_n, $n = 1, 2, \ldots$, satisfy condition (1.3), therefore

$$\operatorname{Im} f(z) \begin{cases} \geq 0 & \text{if} \quad \operatorname{Im} z > 0, \\ = 0 & \text{if} \quad \operatorname{Im} z = 0, \quad z \in \Pi^+, \\ \leq 0 & \text{if} \quad \operatorname{Im} z < 0. \end{cases} \tag{3.1}$$

But if there existed a point $z_0 \in \Pi^+$, $\operatorname{Im} z_0 > 0$, such that $\operatorname{Im} f(z_0) = 0$, we would have $\operatorname{Im} f(z) = 0$ for $z \in \Pi^+$, $\operatorname{Im} z > 0$, by the minimum principle for harmonic functions. In view of this fact and of the holomorphy of the function f, there would be $\operatorname{Im} f(z) = 0$, $z \in \Pi^+$, so $f(z) = c$, $z \in \Pi^+$ where $c \in \mathbb{R}$ by (3.1). Hence we would have a contradiction to the fact that f satisfies (1.1). Therefore $\operatorname{Im} f(z) > 0$ for $z \in \Pi^+$, $\operatorname{Im} z > 0$. In an analogous way one can prove that $\operatorname{Im} f(z) < 0$ for $z \in \Pi^+$, $\operatorname{Im} z < 0$. Summing up, we infer that the function f satisfies condition (1.3), so, in view of Proposition 1.1, it belongs to the class $\mathcal{T}_{\mathcal{R}}$.

We shall next prove

. *Let $F_n = h_n + \bar{g}_n \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, $n = 1, 2, \ldots$, and let the sequences $\{F_n\}_{n\in\mathbb{N}}$, $\{\operatorname{Re} g_n\}_{n\in\mathbb{N}}$ be uniformly convergent on compact subsets of Π^+ and on some set $D_R = \{z \in \Pi^+ : |z| \geq R\}$, $R > 0$, to the finite functions F, φ, respectively. Then $F \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$.*

Proof. Let functions $F_n = h_n + \bar{g}_n \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, $n = 1, 2, \ldots$, satisfy the above assumptions.

Let us first notice that, analogously as in the proof of Theorem 3.1, we can state that the complex function F satisfies normalization condition (1.1).

Since $F_n = h_n + \bar{g}_n \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, $n = 1, 2, \ldots$, therefore $\lim\limits_{\Pi^+ \ni z \to \infty} \operatorname{Re} g_n(z) = \alpha_n$, $n = 1, 2, \ldots$ In virtue of the uniform convergence of the sequence $\{\operatorname{Re} g_n\}_{n\in\mathbb{N}}$ in D_R, proceeding as before, we obtain that the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ is convergent and

$$\lim_{\Pi^+ \ni z \to \infty} \varphi(z) = \lim_{D_R \ni z \to \infty} \varphi(z) = \lim_{n\to\infty} \alpha_n =: \alpha \in \mathbb{R}. \tag{3.2}$$

Of course,

$$\varphi(z) \in \mathbb{R}, \quad z \in \Pi^+. \tag{3.3}$$

We shall prove that the function F satisfies condition (1.3). As it is known from Proposition 1.4, $f_n = h_n - g_n \in \mathcal{T}_{\mathcal{R}}$, $n = 1, 2, \ldots$ Moreover, $f_n = F_n - 2\operatorname{Re} g_n$, $n = 1, 2, \ldots$ From the assumptions about the sequences $\{F_n\}_{n\in\mathbb{N}}$, $\{\operatorname{Re} g_n\}_{n\in\mathbb{N}}$ it follows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ satisfies the assumptions of Theorem 3.1, so the limit function $f = F - 2\varphi$ of this sequence belongs to the class $\mathcal{T}_{\mathcal{R}}$. Hence and from (3.3) we have

$$\operatorname{Im} F(z) = \operatorname{Im} f(z) \begin{cases} > 0 & \text{if} \quad \operatorname{Im} z > 0, \\ = 0 & \text{if} \quad \operatorname{Im} z = 0, \\ < 0 & \text{if} \quad \operatorname{Im} z < 0. \end{cases} \quad z \in \Pi^+,$$

Thus F really satisfies condition (1.3).

It remains to prove that F is a function harmonic in Π^+, i.e. $F = h+\bar{g}$ where h, g are some functions holomorphic in Π^+, and that the function g satisfies condition (1.4).

Let us observe that the functions $\operatorname{Re} g_n$, $n = 1, 2, \dots$, as real parts of the functions holomorphic in Π^+, are real functions harmonic in Π^+. From this, in view of the uniform convergence of the sequence $\{\operatorname{Re} g_n\}_{n\in\mathbb{N}}$ on compact subsets of Π^+ to the function φ and of the respective properties of harmonic functions, φ is a real function harmonic in Π^+. Π^+ is a simply connected domain, so there exists a real function ψ harmonic in Π^+, conjugate to the function φ. Thus we have $F = f + 2\varphi = \operatorname{Re} f + \varphi + i(\operatorname{Im} f + \psi) + \varphi - i\psi$, so $F = h + \bar{g}$ where $h = \operatorname{Re} f + \varphi + i(\operatorname{Im} f + \psi)$, $g = \varphi + i\psi$. The functions φ, ψ are harmonic 0 and mutually conjugate in Π^+, therefore g is holomorphic in Π^+, with that $\varphi = \operatorname{Re} g$ and (3.2) holds. Furthermore, since f is a function holomorphic in Π^+, $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic functions mutually conjugate in Π^+. In consequence, the functions $\operatorname{Re} h = \operatorname{Re} f + \varphi$, $\operatorname{Im} h = \operatorname{Im} f + \psi$ are a pair similar to φ, ψ, so the function h is also holomorphic in Π^+. From these considerations it follows that the function F is a complex function harmonic in Π^+ and (1.4) holds.

In virtue of the facts presented above we infer that the function F belongs to $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$.

§ 4. The class $\mathcal{S}_{\mathcal{R}}^{\mathcal{H}}$

Let $\mathcal{S}_{\mathcal{R}}^{\mathcal{H}}$ denote a class of functions $F = h + \bar{g} \in \mathcal{H}$ univalent in Π^+ and satisfying condition (1.4) and equalities (2.1), (2.3), thus in view of Remark 2.1, conditions (2.8).

. Let us notice that, by the known lemma ([3, Lemma 1]), in order that, for a function $F = h + \bar{g} \in \mathcal{H}$, relations (2.8) hold, it is sufficient that there exists a point $z_0 = \bar{z}_0 > 0$ such that $h^{(n)}(z_0) = \overline{h^{(n)}(z_0)}$, $g^{(n)}(z_0) = \overline{g^{(n)}(z_0)}$, $n = 0, 1, 2, \dots$, because the functions h, g are holomorphic in Π^+. This fact can be used in the definition of the class $\mathcal{S}_{\mathcal{R}}^{\mathcal{H}}$.

We shall prove

1. *The inclusion*

$$\mathcal{S}_{\mathcal{R}}^{\mathcal{H}} \subset \mathcal{T}_{\mathcal{R}}^{\mathcal{H}} \tag{4.1}$$

takes place.

Proof. Let $F = h + \bar{g} \in \mathcal{S}_{\mathcal{R}}^{\mathcal{H}}$. In order to state that $F \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, it suffices to show that condition (1.3) holds.

For the function $F \in \mathcal{S}_{\mathcal{R}}^{\mathcal{H}}$, we have (2.1) and (2.3), so $F(z) = \overline{F(z)}$ for $z = \bar{z} > 0$ and $\overline{F(z)} = F(\bar{z})$ for an arbitrary $z \in \Pi^+$, which follows from the analogous considerations as in the proof of Proposition 2.2. From the univalence of the function F we deduce that $F(\bar{z}) = F(z)$ if and only if $z = \bar{z} > 0$. From the above and by normalization condition (1.1) for F we have $\lim_{\Pi^+ \ni z \to \infty} (\operatorname{Im} F(z) - \operatorname{Im} z) = 0$. From this, in particular, for any fixed $x_0 > 0$, $\lim_{y \to +\infty, x_0 > 0} (\operatorname{Im} F(x_0 + iy) - y) = 0$, so there exists $z_0 = x_0 + iy_0$, $x_0 > 0$, such that $\operatorname{Im} F(z_0) > 0$. Therefore, in virtue of the continuity of the function F in Π^+, if $\operatorname{Im} z > 0$, $z \in \Pi^+$, then $\operatorname{Im} F(z) > 0$, whereas if $\operatorname{Im} z < 0$, $z \in \Pi^+$, then $\operatorname{Im} F(z) < 0$. This means that the function F satisfies condition (1.3) and, in consequence, F belongs to the class $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, which completes the proof.

Example 4.1. Let us come back to the function F_2 of form (1.10). For $b = 0$, we have $F_2(z; a, 0, \nu) = z + a$, $z \in \Pi^+$, so it is a function of the class $\mathcal{S}_{\mathcal{R}}^{\mathcal{H}}$. We can prove that, for $b > 0$, $\nu = 1$, the function F_2 does not belong to the class $\mathcal{S}_{\mathcal{R}}^{\mathcal{H}}$. Consequently, we obtain that

$$\mathcal{T}_{\mathcal{R}}^{\mathcal{H}} \setminus \mathcal{S}_{\mathcal{R}}^{\mathcal{H}} \neq \emptyset.$$

From the detailed considerations it follows that, for $b > 0$, $\nu = 1$ the image of the half-plane Π^+ in the mapping F_2 is the set $F_2(\Pi^+) = \{w \in \mathbb{C} : \operatorname{Re} w \ > a, |w - a| \geq 2\sqrt{b}\}$. In consequence, the function F_2 maps the domain Π^+ onto the set $F_2(\Pi^+)$ which is not a domain.

We shall prove

Proposition 15. *The class $\mathcal{S}_{\mathcal{R}}^{\mathcal{H}}$ is not compact.*

Proof. Let us consider the sequence $\{F_n\}_{n \in \mathbb{N}}$ of functions $F_n(z) = z + n$, $z \in \Pi^+$, $n = 1, 2, \ldots$ Of course, $F_n \in \mathcal{S}_{\mathcal{R}}^{\mathcal{H}}$, $n = 1, 2, \ldots$ We shall prove that the sequence $\{F_n\}_{n \in \mathbb{N}}$ is uniformly divergent on compact subsets of Π^+ to $\widetilde{F(z)} \equiv \infty$, with that $\widetilde{F} \notin \mathcal{S}_{\mathcal{R}}^{\mathcal{H}}$ obviously.

Let us take an arbitrary compact set $\Delta \subset \Pi^+$. Then there exists $R > 0$ such that $\Delta \subset \{z \in \Pi^+ : |z| < R\}$. Next, take any $M > 0$ and let $N = [M + R]$. Then, for $n > N$, we have $|F_n(z)| \geq |n - |z|| > n - R > M$ for $z \in \Delta$. From this we obtain the announced assertion.

From the definitions of the classes $\mathcal{S}_{\mathcal{R}}^{\mathcal{H}}$ and $\mathcal{R}_{\mathcal{R}}$ and (1.6), (1.7'), (4.1) we get

$$\mathcal{R}_{\mathcal{R}} \subset \mathcal{S}_{\mathcal{R}}^{\mathcal{H}}.$$

Let next $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$. Denote

$$F^{\bullet}(z) = h'(z) - g'(z), \quad z \in \Pi^{+}.$$

Then the following proposition is true.

Proposition 16. *If* $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$ *and*

$$\operatorname{Re} F^{\bullet}(z) > 0, \quad z \in \Pi^{+}, \tag{4.2}$$

then $f = h - g$ *is a function of the class* $\mathcal{R}_{\mathcal{R}}$.

Proof. Let $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$ satisfy condition (4.2). Then the function $f = h - g$ belongs (by Proposition 1.4) to the class $\mathcal{T}_{\mathcal{R}}$ and satisfies condition (1.7). In view of Theorem A and the definition of the class $\mathcal{R}_{\mathcal{R}}$, we infer that, indeed, $f \in \mathcal{R}_{\mathcal{R}}$.

Let us assume again that $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$ satisfies condition (4.2). Let $s > 0$. Consider the image of a line $z = s + iy$, $y \in \mathbb{R}$, in the mapping F. This is "a curve" of the equation

$$w(y) = F(s + iy), \quad y \in \mathbb{R}. \tag{4.3}$$

Let next $\omega(y) = \operatorname{Im} w(y)$, $y \in \mathbb{R}$. Then we have

$$\omega(y) = \operatorname{Im}[h(s + iy) - g(s + iy)], \quad y \in \mathbb{R},$$

and

$$\omega'(y) = \frac{\partial}{\partial y} \operatorname{Im}[h(s+iy) - g(s+iy)] = \operatorname{Re}[h'(s+iy) - g'(s+iy)], \quad y \in \mathbb{R}.$$

Since F satisfies (4.2), $\omega'(y) > 0$ for $y \in \mathbb{R}$. Moreover, by normalization condition (1.1) for the function F, we get

$$\lim_{y \to \pm\infty} (\omega(y) - y) = 0.$$

From the above facts it follows that the function ω is a function increasing continuously from $-\infty$ to $+\infty$. Hence we obtain

Proposition 17. *Let $F = h + \bar{g} \in \mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$ satisfy condition (4.2). Then, for any $s > 0$, the line of the equation (4.3) has the property that an arbitrary straight line parallel to the real axis in the (w)-plane has exactly one common point with it. Furthermore, the image of a half-plane $D_s = \{z \in \mathbb{C} : \operatorname{Re} z > s\}$ in the mapping F is a connected set lying on the right of line (4.3) in the (w)-plane.*

The last statement follows from the continuity of the function F and from normalization condition (1.1).

Proposition 18. *Let $\{c_n\}_{n\in\mathbb{N}}$ be an arbitrary fixed sequence of real numbers satisfying the condition*

$$\sum_{n=1}^{\infty} n|c_n| \leq 1. \tag{4.4}$$

Then the function: i) f of the form

$$f(z) = z + 1 + \sum_{n=1}^{\infty} \frac{c_n}{(z+1)^n}, \quad z \in \Pi^+, \tag{4.5}$$

belongs to the class $\mathcal{T}_{\mathcal{R}}$ and is univalent in Π^+; ii) F of the form

$$F(z) = z + 1 - \sum_{n=1}^{\infty} \frac{c_n}{(\bar{z}+1)^n}, \quad z \in \Pi^+, \tag{4.6}$$

belongs to $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$. Moreover, F is locally univalent and orientation-preserving in Π^+.

Proof. Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers satisfying condition (4.4). Then the series

$$g(z) = -\sum_{n=1}^{\infty} \frac{c_n}{(z+1)^n}$$

is convergent in the set $|z+1| > 1$, thus in Π^+. Consequently, f is a function holomorphic in Π^+, whereas F-harmonic in Π^+. From (4.4) and (4.5) it follows that $\operatorname{Re} f'(z) \geq 1 - \sum\limits_{n=1}^{\infty} \frac{n|c_n|}{|z+1|^{n+1}} > 0$ if $|z+1| > 1$. Hence ([2], p. 88) f is univalent in Π^+. We also have $\overline{f(z)} = f(z)$ for

$z = \bar{z} > 0$. Therefore $\operatorname{Im} f(z)$ has a fixed sign if $\operatorname{Im} z > 0$ or if $\operatorname{Im} z < 0$. Evidently, $\lim\limits_{\Pi^+ \ni z \to \infty} (f(z) - z) = 1$, so normalization condition (1.1) holds, and $\operatorname{Im} f(z) > 0$ if $\operatorname{Im} z > 0$ and $\operatorname{Im} f(z) < 0$ if $\operatorname{Im} z < 0$. In this way, i) has been proved.

ii) follows from Proposition 1.5, (4.6) and the fact that $\lim\limits_{\Pi^+ \ni z \to \infty} \operatorname{Re} g(z) = 0$. Furthermore, $J_F(z) = |h'(z)|^2 - |g'(z)|^2 > 0$ for $z \in \Pi^+$, where $h(z) = z + 1$, $z \in \Pi^+$. This ends the proof.

. 1). The form of function (4.5) comes from the function $\varphi(\zeta) = \zeta + \sum\limits_{n=1}^{\infty} \frac{c_n}{\zeta^n}$ holomorphic and univalent in the set $|\zeta| > 1$, thus from the function from the area theorem ([2, p. 29–30]). 2) For functions from the class $\mathcal{T}_{\mathcal{R}}$, thus from $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, we have no possibility of expanding them in a Laurent series, but we only dispose of normalization condition (1.1) or conditions (1.1), (1.4). 3) Moreover, from (4.5) and a consequence of the area theorem we obtain $|f(z)| \leq 2|z+1|$, $z \in \Pi^+$. 4) A special case is the case when $c_n \geq 0$, $n = 1, 2, \ldots$ Several classes of functions with coefficients of a fixed sign have been investigated in many papers.

Proposition 19. *Let $\alpha \in (0, \frac{\pi}{2})$, $n \in \mathbb{N}$, $c_k \in \mathbb{R}$, $\lambda_k > 0$ for $k = 1, 2, \ldots, n$ and*

$$\sum_{k=1}^{n} \lambda_k |c_k| \leq 1. \tag{4.7}$$

Then the function: i) f_n of the form

$$f_n(z) = z + \sum_{k=1}^{n} c_k e^{-\lambda_k z}, \quad z \in \Pi_\alpha^+ = \{z \in \mathbb{C} : -\alpha < \operatorname{Arg} z < \alpha\}, \tag{4.8}$$

is typically-real in Π_α^+; ii) F_n of the form

$$F_n(z) = z - \sum_{k=1}^{n} c_k e^{-\lambda_k \bar{z}}, \quad z \in \Pi_\alpha^+, \tag{4.9}$$

is typically-real harmonic in the domain Π_α^+ (in the sense of definitions of $\mathcal{T}_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}^{\mathcal{H}}$, modified to Π_α^+, respectively); F_n is also locally univalent and orientation-preserving.

Proof. Let the assumptions of Proposition 4.5 be satisfied. Then, of course, f_n of form (4.8) takes real values for $z = \bar{z} > 0$. Since $|e^{-\lambda_k z}| =$

$e^{-\lambda_k \operatorname{Re} z}$, therefore if $\Pi_\alpha^+ \ni z \to \infty$, then $\operatorname{Re} z \to \infty$ and $\lim_{\Pi_\alpha^+ \ni z \to \infty} |e^{-\lambda_k z}| = 0$. Consequently, by (4.8), $\lim_{\Pi_\alpha^+ \ni z \to \infty} (f_n(z) - z) = 0$. From (4.7) and (4.8) we also have $\operatorname{Re} f_n'(z) \geq 1 - \sum_{k=1}^{n} \lambda_k |c_k| e^{-\lambda_k \operatorname{Re} z} > 0$ if $z \in \Pi_\alpha^+$, so ([2], p. 88) f_n is a function univalent in Π_α^+ and therefore it is typically-real in this domain. ii) follows from Proposition 1.5 applied to the case of the set Π_α^+. Of course, $J_{F_n}(z) > 0$ for $z \in \Pi^+$.

. 1) The functions $f_n - z$, $n \in \mathbb{N}$, are the nth partial sums of the respective Dirichlet series.

2) In view of the normalization condition, we cannot consider a full Dirichlet series although Π^+ is a half-plane, i.e. a set of a type of sets of convergence for such series. 3) Also by this condition, we cannot consider f_n in Π^+ but only in an arbitrary fixed "angle-domain" contained in Π^+.

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