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THE THEOREM ON EXISTENCE OF SINGULAR SOLUTIONS TO NONLINEAR EQUATIONS

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The aim of this paper is to present some applications of p-regularity theory to investigations of nonlinear multivalued mappings. The main result addresses to the problem of existence of solutions to nonlinear equations in the degenerate case when the linear part is singular at the considered initial point. We formulate conditions for existence of solutions of equation $F(x) = \mathbf{0}$ when first $p-1$ derivatives of F are singular.

§ 1. Introduction

In this paper we consider the problem of the solutions existence to the nonlinear equation:

$$F(x) = \mathbf{0}, \tag{1}$$

where $F : X \to Y$ and X, Y be Banach spaces. The problem (1) is called regular at the point x^* if $\mathrm{Im} F'(x^*) = Y$. Otherwise, the problem (1) is called irregular (nonregular, degenerate) at the point x^*.

The construction of p-regularity introduced in [9] (see also [4, 10]) gives new possibilities for description and investigation of solutions in the degenerate case.

Let p be a natural number and let $B : X \times \ldots \times X \to Y$ be a continuous p-multilinear mapping. A p-form associated to B is the map $B[\cdot]^p : X \to Y$ defined by $B[x]^p = B(\underbrace{x, \ldots, x}_{p})$ for $x \in X$. Alternatively, we may simply view $B[\cdot]^p$ as homogeneous polynomial map $B : X \to Y$ of degree p, i.e. $B(\alpha x) = \alpha^p \cdot B(x)$.

Throughout this paper we assume that the mapping $F : X \to Y$ is continuous and p-times Fréchet differentiable on X and its p-th order derivative at point $x \in X$ will be denoted as $F^{(p)}(x)$ (a symmetric multilinear map of p copies of X to Y) and the associated p-form is

$$F^{(p)}(x)[h]^p = F^{(p)}(x)[\underbrace{h, \ldots, h}_{p}].$$

Furthermore, we use the following notation

$$\mathrm{Ker}^p F^{(p)}(x) = \Big\{ h \in X : F^{(p)}(x)[h]^p = \mathbf{0} \Big\}$$

for the p-kernel of the mapping $F^{(p)}(x)$ (the zero locus of $F^{(p)}(x)$). Denote also $\mathcal{L}(X, Y)$ as a space of continuous linear operators from X to Y.

The set

$$M(x^*) = \{x \in U : F(x) = F(x^*)\}$$

is called the *solution set* for the mapping F in neighborhood U.

We call h *a tangent vector* to a set $M \subseteq X$ at $x^* \in M$ if there exist $\varepsilon > 0$ and a function $r : [0, \varepsilon] \to X$ with the property that for $t \in [0, \varepsilon]$, we have $x^* + th + r(t) \in M$ and

$$\lim_{t \to 0} \frac{\|r(t)\|}{t} = 0.$$

The collection of all tangent vectors at x^* is called *the tangent cone* to M at x^* and it is denoted by $T_1 M(x^*)$ (see e. g. [1]).

Let X and Y be sets. We denote by 2^Y the set of all subsets of the set Y. Any mapping $\Phi : X \to 2^Y$ is said to be *multimapping* (or a *multivalued mapping*) from X into Y. For a linear operator $\Lambda : X \to Y$, we denote by Λ^{-1} its *right inverse*, that is $\Lambda^{-1} : Y \to 2^X$, which any element $y \in Y$ maps on its complete inverse image at the mapping Λ :

$$\Lambda^{-1} y = \{x \in X : \Lambda x = y\}.$$

Of course,

$$\Lambda \Lambda^{-1} = I_Y.$$

Furthermore, we shall use the "norm"

$$\left\|\Lambda^{-1}\right\| = \sup_{\|y\|=1} \inf \{\|x\| : \Lambda x = y, \;\; x \in X\}. \tag{2}$$

Note, that when Λ is one-to-one, $\|\Lambda^{-1}\|$ can be considered as the usual norm of the element Λ^{-1} in the space $\mathcal{L}(Y, X)$.

In our further considerations, under notion Λ^{-1} we shall mean just right inverse operator (multivalued) with the norm defined by (2). Moreover, unless otherwise stated we will assume that X and Y are Banach spaces.

§ 2. Elements of p-regularity theory

Let U be a neighborhood of a point $x^* \in X$. If $F : U \to Y$ is p-times differentiable mapping in U and

$$F^{(i)}(x^*) = \mathbf{0}, \text{ for all } i = 1, \dots, p-1,$$

then we say that F is *completely degenerate* at x^* up to order p.

A mapping F is called *regular* at some point x^* if

$$\mathrm{Im} F'(x^*) = Y. \tag{3}$$

The mapping F is called *nonregular* (*irregular, degenerate*) if regularity condition is not satisfied.

In this paper we consider the case when the regularity condition (3) does not hold, but the mapping F is p-regular. First of all, let us remind the definition of p-regularity and construction of p-factor-operator.

Definition 1. *Let $F : U \to Y$, $U \subseteq X$, and $F \in C^p(U)$. Then for any $x \in U$ and for any element $h \in X$, $h \neq \mathbf{0}$ we can associate a linear operator $F^{(p)}(x)[h]^{p-1} \in \mathcal{L}(X, Y)$, defined by the formula:*

$$F^{(p)}(x)[h]^{p-1}\xi = F^{(p)}(x)[h]^{p-1}[\xi], \;\; for \;\; any \;\; \xi \in X.$$

The linear operator

$$\Psi_p(x, h) = F^{(p)}(x)[h]^{p-1} \in \mathcal{L}(X, Y)$$

is called the p-factor operator.

A generalization of the regularity condition is given by the following condition of p-regularity:

DEFINITION 2. *Let $F : U \to Y$, $U \subseteq X$ where $F \in C^p(U)$ and is completely degenerate at x^*. Then F is called p-regular at the point x^* if either*

$$\mathrm{Im}\Psi_p(x^*, h) = Y, \quad for\ \ any\ \ h \in \mathrm{Ker}^p F^{(p)}(x^*) \setminus \{\mathbf{0}\},$$

or $\mathrm{Ker}^p F^{(p)}(x^*) = \{\mathbf{0}\}$.

DEFINITION 3. *We say that the mapping F is p-regular at x^* along (on) the element h, if* $\mathrm{Im}\Psi_p(x_0, h) = Y$.

The description of a solution set in degenerate case is given by the following theorem:

THEOREM (GENERALIZED LYUSTERNIK THEOREM) [9] *Let U be a neighborhood of a point $x^* \in X$. Assume that $F : X \to Y$, where $F \in C^p(U)$ is p-regular at x^*. Then*

$$T_1 M(x^*) = \mathrm{Ker}^p F^{(p)}(x^*).$$

Let Z be a metric space with distance ρ. If $A_1 \subset Z$ and $A_2 \subset Z$, then the number

$$\sigma(A_1, A_2) = \sup_{z \in A_1} \rho(z, A_2) = \sup_{z \in A_1} \inf_{\omega \in A_2} \rho(z, \omega)$$

is called the *deviation* of the set A_1 from the set A_2. The maximum of the deviations $\sigma(A_1, A_2)$ and $\sigma(A_2, A_1)$,

$$\mathbf{h}(A_1, A_2) = \max\left\{\sigma(A_1, A_2), \sigma(A_2, A_1)\right\},$$

is called the *Hausdorff distance* between the sets A_1 and A_2. It follows at once from the definition that, if $\mathbf{h}(A_1, A_2) < \alpha$, then, for every $z_1 \in A_1$, there exists a $z_2 \in A_2$ such that $\rho(z_1, z_2) < \alpha$.

Let Φ be a multimapping from the space Z into itself. We shall say that it is a *contraction* on a set $A \subset Z$ if there exists a number θ with $0 < \theta < 1$ such that the inequality

$$\mathbf{h}(\Phi(z_1), \Phi(z_2)) \leq \theta \rho(z_1, z_2)$$

holds for any z_1 and z_2 from A.

We shall give three auxiliary lemmas. The first of these lemmas is a "multivalued"generalization of the contraction mapping principle, and is independent interest.

Lemma 1. (contraction multimapping principle) [3] *Let Z be a complete metric space with distance ρ. Assume that we are given a multimapping*

$$\Phi : U_\varepsilon(z_0) \to 2^Z,$$

on a ball $U_\varepsilon(z_0) = \{z : \rho(z, z_0) < \varepsilon\}$ $(\varepsilon > 0)$ where the sets $\Phi(z)$ are non-empty and closed for any $z \in U_\varepsilon(z_0)$. Further, assume that there exists a number $\theta, 0 < \theta < 1$, such that

1) $\mathbf{h}(\Phi(z_1), \Phi(z_2)) \le \theta\rho(z_1, z_2)$ *for any* $z_1, z_2 \in U_\varepsilon(z_0)$,
2) $\rho(z_0, \Phi(z_0)) < (1 - \theta)\varepsilon$.

Then, for every number ε_1 which satisfies the inequality

$$\rho(z_0, \Phi(z_0)) < \varepsilon_1 < (1 - \theta)\varepsilon,$$

there exists an element $z \in B_{\varepsilon_1/(1-\theta)}(z_0) = \{\omega : \rho(\omega, z_0) \le \varepsilon_1/(1 - \theta)\}$ such that

$$z \in \Phi(z) \tag{4}$$

Lemma 2. **[3]** *Let M_1 and M_2 be linear manifolds in X which are translations of a single subspace L. Then*

$$\mathbf{h}(M_1, M_2) = \sigma(M_1, M_2) = \sigma(M_2, M_1) =$$
$$= \inf\{\|x_1 - x_2\| : x_1 \in M_1, x_2 \in M_2\}.$$

Lemma 3. **[3]** *Let $\Lambda \in \mathcal{L}(X, Y)$. We set*

$$C(\Lambda) = \sup_{\|y\|=1} \inf\{\|x\| : x \in X,\ \Lambda x = y\}.$$

If $\mathrm{Im}\Lambda = Y$, *then* $C(\Lambda) < \infty$.

§ 3. Regular case

We quote one modification of the theorem on existence of solutions to the equations with non-degenerate mappings (see in [2, 5, 6]) with proof which the main idea is based on the similar construction to this in singular case.

Consider a mapping $F : X \to Y$ and the problem of existence of such point x^* that $F(x^*) = \mathbf{0}$, i.e. x^* is a solution of (1). This equation is solved by using non-degenerated condition of operator $F'(x)$. Throughout this section we assume that $[F'(x_0)]^{-1}$ is the multivalued mapping, and that $F'(x_0)$ is nonsingular, i.e. $F'(x_0)X = Y$.

THEOREM 1. *Let* $\|F(x_0)\| = \eta$ *and* $\left\|[F'(x_0)]^{-1}\right\| = \delta$, *and assume that*

$$\sup_{x \in U_\varepsilon(x_0)} \|F''(x)\| = C,$$

where $F \in C^2(U_\varepsilon(x_0))$. *Moreover, assume the following inequalities:*

1) $\delta \cdot C \cdot \varepsilon \leq \frac{1}{6}$,
2) $\delta \cdot \eta \leq \frac{\varepsilon}{2}$,
3) $\varepsilon \leq 1$.

Then the equation $F(x) = \mathbf{0}$ *has a solution* $x^* \in U_\varepsilon(x_0)$.

PROOF. Define a multivalued mapping

$$\Psi : U_\varepsilon(x_0) \to 2^X, \quad U_\varepsilon(x_0) \subset X$$

$$\Psi(x) = x - [F'(x_0)]^{-1} F(x), \quad x \in U_\varepsilon(x_0).$$

We shall show now that there exists a number q, $0 < q < 1$, such that

$$\mathbf{h}(\Psi(s_1), \Psi(s_2)) \leq q \cdot \|s_1 - s_2\|, \tag{5}$$

for any $s_1, s_2 \in U_\varepsilon(x_0)$.

By Lemma 2 and Lemma 3, we have

$$\mathbf{h}(\Psi(s_1), \Psi(s_2)) = \inf \{\|z_1 - z_2\| : z_i \in \Psi(s_i), i = 1, 2\} =$$

$$= \inf \{\|z_1 - z_2\| : F'(x_0)z_i = F'(x_0)s_i - F(s_i), i = 1, 2\} =$$

$$= \inf \{\|z\| : F'(x_0)z = F'(x_0)[s_1 - s_2] - F(s_1) + F(s_2)\} =$$

$$= \delta \cdot \|F(s_1) - F(s_2) - F'(x_0) \cdot [s_1 - s_2]\|.$$

From the Taylor expansion of F, we have

$$F(s_1) - F(s_2) = F'(s_2)[s_1 - s_2] + \omega(s_1, s_2),$$

where

$$\|\omega(s_1,s_2)\| \le \sup_{x\in U_\varepsilon(x_0)} \left\|F''(x)[s_1-s_2]^2\right\| \le C\cdot\|s_1-s_2\|^2,$$

and moreover

$$F'(s_2)=F'(x_0)+\xi(s_2,x_0),$$

where

$$\|\xi(s_2,x_0)\| \le \sup_{x\in U_\varepsilon(x_0)} \|F''(x)[s_2-x_0]\| \le C\cdot\|s_2-x_0\|.$$

Taking into account the above and the assumption 1 we obtain

$$\begin{gathered}\mathbf{h}(\Psi(s_1),\Psi(s_2)) = \delta\cdot\|\xi(s_2,x_0)[s_1-s_2]+\omega(s_1,s_2)\| \le \\ \le \delta\cdot(\|\xi(s_2,x_0)\|\cdot\|s_1-s_2\|+\|\omega(s_1,s_2)\|) \le \\ \le \delta\cdot\left(C\cdot\|s_2-x_0\|\cdot\|s_1-s_2\|+C\cdot\|s_1-s_2\|^2\right) < \\ < 3\cdot C\cdot\varepsilon\cdot\delta\cdot\|s_1-s_2\| \le \frac{1}{2}\cdot\|s_1-s_2\|.\end{gathered}$$

Now, we claim that

$$\rho(\Psi(x_0),x_0) < (1-q)\cdot\varepsilon, \quad \text{where } \ q=\frac{1}{2}. \tag{6}$$

Indeed,

$$\begin{gathered}\rho(\Psi(x_0),x_0)=\inf\{\|z\| : F'(x_0)z=-F(x_0)\} \le \delta\cdot\|F(x_0)\| \le \\ \le \delta\cdot\eta \le \frac{\varepsilon}{2} < (1-q)\cdot\varepsilon.\end{gathered}$$

It means that all the conditions of Lemma 1 are fulfilled which gives (4), that is there exists such point $x^* \in \Psi(x^*)$. It means that

$$x^* \in x^* + [F'(x_0)]^{-1}(-F(x^*)),$$

so

$$\mathbf{0} \in [F'(x_0)]^{-1}(-F(x^*))$$

and $F'(x_0)\mathbf{0}=-F(x^*)$, further $\mathbf{0}=-F(x^*)$ and finally $F(x^*)=\mathbf{0}$. □

We conclude the discussion of the regular case by the following example, which is very simple, but serves to illustrate the basic idea of Theorem 1.

Example. Consider the problem (1) with $(x_0, y_0) = (0, 0)$ and

$$F : \mathbb{R}^2 \to \mathbb{R},$$

where $F(x, y) = x^3 + x - y + \frac{1}{100}$. Moreover, $\|F(0, 0)\| = \eta = \frac{1}{100}$,

$$\left\|[F'(0, 0)]^{-1}\right\| = \delta = \frac{\sqrt{2}}{2}, \qquad \sup_{(x,y)\in U_\varepsilon(0,0)} \|F''(x, y)\| \le C = 6 \cdot \varepsilon,$$

where $\varepsilon = 0, 1$. The assumptions of Theorem 1 are fulfilled, so there exists such point (x^*, y^*), that $F(x^*, y^*) = 0$, and for instance $(x^*, y^*) = (0, \frac{1}{100})$.

§ 4. p-regular (singular) case

Let us mention one of the consequences of the Mean Value Theorem, which is important for our further investigations. For the proof we refer the reader to [7, 8].

LEMMA 4. *Let $F : U \to Y$, where $U \subseteq X$, such that $[a, b] \subset U$, then*

$$\|F(b) - F(a) - \Lambda(b - a)\| \le \sup_{\xi\in[a,b]} \|F'(\xi) - \Lambda\| \cdot \|a - b\|,$$

for any $\Lambda \in \mathcal{L}(X, Y)$.

Next lemma follows from homogeneity properties of p-form (see [4]).

LEMMA 5. **[4]** *Let $U(x_0)$ – neighborhood of x_0 in X, $F : U(x_0) \to Y$, $F \in C^{p+1}(U(x_0))$, and $F^{(i)}(x_0) = \mathbf{0}$, for $i = 1, \dots, p-1$. Then for every $\epsilon > 0$ there exist $\delta > 0$ and $R > 0$ $\left(R = \max\left\{1, \frac{2}{\epsilon} \cdot \frac{1}{(p-1)!} \left\|F^{(p)}(x_0)\right\|\right\}\right)$ such that for any $x \in X$, $\|x\| \le \delta$ and for any $x_1, x_2 \in X$, $\|x_j\| \le \|x\|/R$, $j = 1, 2$, the following condition*

$$\left\|F(x_0 + x + x_1) - F(x_0 + x + x_2) - \frac{1}{(p-1)!} F^{(p)}(x_0)[x]^{p-1}(x_1 - x_2)\right\| \le$$

$\le \epsilon \cdot \|x\|^{p-1} \cdot \|x_1 - x_2\|$ *holds.*

DEFINITION 4. (BANACH CONDITION) *Let $F : X \to Y$, and let $F \in C^p(X)$. Then for any $y \in Y$, $\|y\| = 1$ there exists $x \in X$, such that*

$$F^{(p)}(x_0)[x]^p = y, \qquad \|x\| \le c,$$

where $c > 0$ is a constant independent of y.

Let us introduce following additional notations

$$\delta = \|F(x_0)\| \neq 0,$$

$$\hat{h} = \frac{h}{\|h\|}, \quad \text{where} \;\; h \in \left\{ \left[\frac{1}{p!} F^{(p)}(x_0) \right]^{-1} (-F(x_0)) \right\}, \quad h \neq \mathbf{0},$$

as a generalization of (2) we have

$$\eta = \left\| \left[F^{(p)}(x_0) \right]^{-1} \right\| = \sup_{\|y\|=1} \inf \left\{ \|x\| : F^{(p)}(x_0)[x]^p = y, \;\; x \in X \right\},$$

$$C = \sup_{x \in U_\varepsilon(x_0)} \left\| F^{(p+1)}(x) \right\|,$$

$$C_1 = \left\| \left[F^{(p)}(x_0)[\hat{h}]^{p-1} \right]^{-1} \right\|$$

and

$$C_2 = \left\| F^{(p)}(x_0) \right\|.$$

We can now formulate our main result which is a generalization of Theorem 1 in the degenerate case.

THEOREM 2. *Let $F : X \to Y$ and assume that for $F \in C^{p+1}(U_\varepsilon(x_0))$ the Banach condition holds and F is p-regular mapping at the point x_0 along $\hat{h}$ and $F^{(i)}(x_0) = \mathbf{0}$, for $i = 1, \dots, p-1$. Moreover assume the following inequalities*

1) $p! \cdot \eta \cdot \delta^{\frac{1}{p}} \leq \frac{\varepsilon}{2}$,
2) $4^p \cdot (p-1)! \cdot C \cdot C_1 \cdot \varepsilon \leq \frac{1}{2}$,
3) $\varepsilon < 1$.

Then the equation $F(x) = \mathbf{0}$ has a solution $x^* \in U_\varepsilon(x_0)$.

ДОКАЗАТЕЛЬСТВО.

As in the proof of Theorem 1 consider a multivalued mapping

$$\Psi : U_\varepsilon(x_0) \to 2^X, \;\; U_\varepsilon(x_0) \subset X,$$

$$\Psi(x) = x - \left[\frac{1}{(p-1)!}F^{(p)}(x_0)[h]^{p-1}\right]^{-1}[F(x)], \quad x \in U_\varepsilon(x_0).$$

Let us first prove that there exists a number q, $0 < q < 1$, such that

$$\mathbf{h}(\Psi(s_1), \Psi(s_2)) \leq q \cdot \|s_1 - s_2\|, \tag{7}$$

for any $s_1, s_2 \in U_\varepsilon(x_0)$, such that $s_1 = x_0 + h + u_1$, $s_2 = x_0 + h + u_2$, where $\|u_i\| \leq \frac{\|h\|}{R}$, $i = 1, 2$ and $R = \max\left\{1, \frac{2}{\varepsilon \cdot C} \cdot \frac{1}{(p-1)!}\left\|F^{(p)}(x_0)\right\|\right\}$. By Lemmas 2 and 3, we have

$$\mathbf{h}(\Psi(s_1), \Psi(s_2)) = \inf\left\{\|z_1 - z_2\| : z_i \in \Psi(s_i), i = 1, 2\right\} =$$

$$= \inf\{\|z_1 - z_2\| : \frac{1}{(p-1)!}F^{(p)}(x_0)[h]^{p-1}[z_i] =$$

$$= \frac{1}{(p-1)!}F^{(p)}(x_0)[h]^{p-1}[s_i] - F(s_i), i = 1, 2\} \leq$$

$$\leq \inf\{\|z\| : \frac{1}{(p-1)!}F^{(p)}(x_0)[h]^{p-1}[z]$$

$$= \frac{1}{(p-1)!}F^{(p)}(x_0)[h]^{p-1}[s_1 - s_2] - F(s_1) + F(s_2)\} \leq$$

$$\leq \frac{(p-1)!}{\|h\|^{p-1}}C_1 \cdot \left\|F(s_1) - F(s_2) - \frac{1}{(p-1)!}F^{(p)}(x_0)[h]^{p-1}[s_1 - s_2]\right\|.$$

Further, taking into account Lemma 4, we have

$$\left\|F(s_1) - F(s_2) - \frac{1}{(p-1)!}F^{(p)}(x_0)[h]^{p-1}[s_1 - s_2]\right\| \leq$$

$$\leq \sup_{\theta \in [0,1]}\left\|F'(s_2 + \theta(s_1 - s_2)) - \frac{1}{(p-1)!}F^{(p)}(x_0)[h]^{p-1}\right\| \cdot \|s_1 - s_2\|. \tag{8}$$

From the Taylor expansion, we obtain

$$\begin{aligned} &F'(s_2 + \theta(s_1 - s_2)) = F'(x_0) + \ldots + \\ &+ \tfrac{1}{(p-1)!}F^{(p)}(x_0)[s_2 - x_0 + \theta(s_1 - s_2)]^{p-1} + \omega(h, u_1, u_2, \theta) = \\ &= \tfrac{1}{(p-1)!}F^{(p)}(x_0)[s_2 - x_0 + \theta(s_1 - s_2)]^{p-1} + \omega(h, u_1, u_2, \theta), \end{aligned} \tag{9}$$

where

$$\|\omega(h,u_1,u_2,\theta)\| \le \sup_{x\in U_\varepsilon(x_0)} \left\|F^{(p+1)}(x)[h+u_2+\theta(s_1-s_2)]^p\right\|.$$

On account of R and $\|u_i\|$, $i=1,2$, we have

$$\|h+u_2+\theta(s_1-s_2)\| \le 4\cdot\|h\|,$$

and from assumption, $\|h\| \le \frac{\varepsilon}{2}$. By the above,

$$\|\omega(h,u_1,u_2,\theta)\| \le C\cdot\|h\|^p \le 4^p\cdot C\cdot\frac{\varepsilon}{2}\cdot\|h\|^{p-1}. \tag{10}$$

Moreover,

$$\begin{gathered} F^{(p)}(x_0)[h+u_2+\theta(s_1-s_2)]^{p-1} = \\ \sum_{i=0}^{p-1} C_{p-1}^i F^{(p)}(x_0)[h]^{p-1-i}[u_2+\theta(s_1-s_2)]^i = \\ F^{(p)}(x_0)[h]^{p-1} + \sum_{i=1}^{p-1} C_{p-1}^i F^{(p)}(x_0)[h]^{p-1-i}[u_2+\theta(s_1-s_2)]^i, \end{gathered} \tag{11}$$

where

$$\|u_2+\theta(s_1-s_2)\| \le 3\cdot\|h\|/R. \tag{12}$$

Taking into account the choice of R,

$$\begin{gathered} \left\|\sum_{i=1}^{p-1} C_{p-1}^i F^{(p)}(x_0)[h]^{p-1-i}[u_2+\theta(s_1-s_2)]^i\right\| \le \\ \le \left\|F^{(p)}(x_0)\right\|\cdot\sum_{i=1}^{p-1} C_{p-1}^i\|h\|^{p-1-i}(3\cdot\|h\|)^i/R^i \le \\ \le \left\|F^{(p)}(x_0)\right\|\cdot\|h\|^{p-1}\cdot 4^{p-1}/R \le 4^p\cdot(p-1)!\cdot\frac{\varepsilon}{2}\cdot C\cdot\|h\|^{p-1}. \end{gathered} \tag{13}$$

And finally, applying (9)–(13) in (8) we obtain

$$\begin{gathered} \left\|F(s_1)-F(s_2)-\frac{1}{(p-1)!}F^{(p)}(x_0)[h]^{p-1}\cdot[s_1-s_2]\right\| \le \\ \le 4^p\cdot\varepsilon\cdot C\cdot\|h\|^{p-1}\cdot\|s_1-s_2\|. \end{gathered}$$

Hence

$$\mathbf{h}(\Psi(s_1), \Psi(s_2)) \le \frac{(p-1)!}{\|h\|^{p-1}} \cdot 4^p \cdot C \cdot C_1 \cdot \varepsilon \cdot \|h\|^{p-1} \cdot \|s_1 - s_2\| =$$

$$= 4^p \cdot (p-1)! \cdot C \cdot C_1 \cdot \varepsilon \cdot \|s_1 - s_2\| \le q \cdot \|s_1 - s_2\|, \text{ where } q = \frac{1}{2}.$$

Let

$$x_1 \in x_0 - \left[\frac{1}{p!} F^{(p)}(x_0)\right]^{-1} (F(x_0)) \tag{14}$$

Thus this element x_1 is such that $\|x_1 - x_0\| \le \frac{\varepsilon}{2}$. Indeed,

$$\|x_1 - x_0\| \le \left\| \left[\frac{1}{p!} F^{(p)}(x_0)\right]^{-1} \right\| \cdot \|F(x_0)\|^{\frac{1}{p}} \le p! \cdot \delta^{\frac{1}{p}} \cdot \eta \le \frac{\varepsilon}{2},$$

so $x_1 \in U_\varepsilon(x_0)$. We have to verify that

$$\rho(\Psi(x_1), x_1) < (1-q) \cdot \varepsilon,$$

where $q = \frac{1}{2}$. So

$$\rho(\Psi(x_1), x_1) = \inf\{\|z\| : z \in \Psi(x_1) - x_1\} =$$

$$= \inf\left\{\|z\| : z \in x_1 - \left[\frac{1}{(p-1)!} F^{(p)}(x_0)[h]^{p-1}\right]^{-1} F(x_1) - x_1\right\} =$$

$$= \inf\left\{\|z\| : \left[\frac{1}{(p-1)!} F^{(p)}(x_0)[h]^{p-1}\right]^{-1} z = -F(x_1)\right\} \le$$

$$\frac{(p-1)!}{\|h\|^{p-1}} \cdot C_1 \cdot \|F(x_1)\|.$$

From the Taylor expansion of $F(x_1)$ and under an assumption of complete degeneration of F:

$$F(x_1) = F(x_0) + \frac{F^{(p)}(x_0)}{p!}[x_1 - x_0]^p + \omega_1(x_1, x_0),$$

where

$$\|\omega_1(x_1, x_0)\| \le \sup_{x \in U_\varepsilon(x_0)} \left\| F^{(p+1)}(x)[h]^{p+1} \right\| \le C \cdot \|h\|^{p+1},$$

and $x_1 - x_0 \in \left[\frac{1}{p!}F^{(p)}(x_0)\right]^{-1}(-F(x_0))$, hence $\frac{1}{p!}F^{(p)}(x_0)[x_1 - x_0]^p = -F(x_0)$, we finally obtain:

$$\rho(\Psi(x_1), x_1) \leq \frac{(p-1)!}{\|h\|^{p-1}} \cdot C_1 \cdot \left\| F(x_0) + \frac{F^{(p)}(x_0)}{p!}[x_1 - x_0]^p + \omega_1(x_1, x_0) \right\| =$$

$$= \frac{(p-1)!}{\|h\|^{p-1}} \cdot C_1 \cdot \|F(x_0) - F(x_0) + \omega_1(x_1, x_0)\| \leq$$

$$\leq \frac{(p-1)!}{\|h\|^{p-1}} \cdot C_1 \cdot C \cdot \|\omega_1(x_1, x_0)\| \leq$$

$$\leq \frac{(p-1)!}{\|h\|^{p-1}} \cdot C_1 \cdot C \cdot \|h\|^{p+1} \leq (p-1)! \cdot C_1 \cdot C \cdot \|x_1 - x_0\|^2 \leq$$

$$\leq (p-1)! \cdot C \cdot C_1 \cdot \frac{\varepsilon^2}{4} = \frac{1}{8} \cdot \varepsilon.$$

It follows that all conditions of Lemma 1 and inclusion (4), so there exists an element x^* such that $x^* \in \Psi(x^*)$. It means that

$$x^* \in x^* + \left[\frac{1}{(p-1)!}F^{(p)}(x_0)[h]^{p-1}\right]^{-1}(-F(x^*)),$$

hence

$$\mathbf{0} \in \left[\frac{1}{(p-1)!}F^{(p)}(x_0)[h]^{p-1}\right]^{-1}(-F(x^*))$$

and $\frac{1}{(p-1)!}F^{(p)}(x_0)[h]^{p-1}\mathbf{0} = -F(x^*)$, further $\mathbf{0} = -F(x^*)$ and finally $F(x^*) = \mathbf{0}$. $\square$

As in regular case we give simple examples illustrating the application of Theorem 2. More detailed and profound applications we omit with respect to the limit of this paper.

Example 1. Consider situation, where the p-regularity condition is fulfilled, but the Banach condition

$$F''(x_0)[X]^2 = Y \tag{15}$$

does not hold.

Namely, let $F : \mathbb{R} \to \mathbb{R}$, and $F(x) = x^2 + \varepsilon$, where $\varepsilon > 0$. Moreover, if $x_0 = 0$, and $p = 2$, we have that F is p-regular at x_0, but the equation

$F(x) = 0$ has no solution. This follows from the fact that the condition (15) fails.

Example 2. Let $F : \mathbb{R}^2 \to \mathbb{R}$, and $F(x, y) = x^2 - y^2 + x^3 + x^7 + \frac{1}{10^6}$. Consider the problem (1) with $(x_0, y_0) = (0, 0)$, and $p = 2$. It is easy to prove that the mapping F is 2-regular at the point (x_0, y_0). Moreover, we have $\delta = \frac{1}{10^6}$, $h = (0, \frac{1}{10^3})$, $\hat{h} = (0, 1)$, $\eta = 1$, $C = 6 + 210\varepsilon^4$, $C_1 = \frac{1}{2}$, $C_2 = 2\varepsilon^2$. All of the assumptions of the Theorem 2 are fulfilled. Hence there exists a solution (x^*, y^*) of the problem (1), and one of such points from a ε-neighborhood $U_\varepsilon(0)$, where $\varepsilon = \frac{1}{100}$, is $(0, \frac{1}{10^3})$.

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