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KOEBE DOMAINS FOR THE CLASS OF TYPICALLY REAL ODD FUNCTIONS

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In this paper we discuss the generalized Koebe domains for the class $T^{(2)}$ and the set $D \subset \Delta = \{z \in \mathbb{C} : |z| < 1\}$, i.e. the sets of the form $\bigcap_{f \in TM} f(D)$. The main idea we work with is the method of the envelope. We determine the Koebe domains for $H = \{z \in \Delta : |z^2 + 1| > 2|z|\}$ and for special sets $\Omega_\alpha,\ \alpha \leq \frac{4}{3}$. It appears that the set $\Omega_{\frac{4}{3}}$ is the largest subset of Δ for which one can compute the Koebe domain with the use of this method. It means that the set $K_{T^{(2)}}(\Omega_{\frac{4}{3}}) \cup K_T(\Delta)$ is the largest subset of the still unknown set $K_{T^{(2)}}(\Delta)$ which we are able to derive.

Introduction

Let $\mathcal{A}$ denote the set of all functions which are analytic in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. The notion of the Koebe domain was generalized in [5] as follows. For a given class $A \subset \mathcal{A}$ and for a given domain $D \subset \Delta$, a set

$$\bigcap_{f \in A} f(D)$$

is called the Koebe domain for the class A and the set D. We denote this set by $K_A(D)$.

It is easy to observe that if a compact class A has the property

$$f \in A \quad \text{iff} \quad e^{-i\varphi} f(ze^{i\varphi}) \in A \quad \text{for each} \quad \varphi \in R, \tag{1}$$

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and $D = \Delta_r = \{z \in \mathbb{C} : |z| < r\}$, $r \in (0,1]$, then $K_A(\Delta_r)$ is a disk with the center in the origin. Moreover, if all functions belonging to A are univalent in Δ_r then the radius of this disk is equal to $\min\{|f(z)| : f \in A, |z| = r\}$.

The property (1) does not hold in classes consisting of functions with real coefficients, for example in the class of typically real functions

$$T = \{f \in \mathcal{A} : \mathrm{Im}\ddagger\mathrm{Im}\{(\ddagger) \geq \prime,\ \ddagger \in \cdot\}\ . \tag{2}$$

We use the notation:
$S = \{f \in \mathcal{A} : \{ \text{ is univalent in } \Delta\}$,
$A_R = \{f \in A : f \text{ has real coefficients}\}$,
$\langle\langle a, b\rangle\rangle$ - a line segment connecting $a, b \in \mathbb{C}$,
$\overline{(A)}$ - a closure of A,
$\mathrm{int}(A)$ - an interior of A.

Koebe domains have a few simple properties.

Theorem A [5]. *For a fixed compact class $A \subset \mathcal{A}$ the following properties of $K_A(D)$ are true:*

1. *if A satisfies (1) and $A \subset S$ then $K_A(\Delta_r) = \Delta_{m(r)}$, where $m(r) = \min\{|f(z)| : f \in A, z \in \partial\Delta_r\}$;*
2. *if $A \subset \mathcal{A}_{\mathcal{R}}$ and D is symmetric with respect to the real axis then $K_A(D)$ is symmetric with respect to the real axis;*
3. *if $A \subset \mathcal{A}_{\mathcal{R}}$, $f \in A \iff -f(-z) \in A$, and D is symmetric with respect to both axes then $K_A(D)$ is symmetric with respect to both axes;*
4. *if $D_1 \subset D_2$ then $K_A(D_1) \subset K_A(D_2)$;*
5. *if $A_1, A_2 \subset \mathcal{A}$ and $A_1 \subset A_2$ then $K_{A_2}(D) \subset K_{A_1}(D)$.*

In [3] classes of typically real functions with n-fold symmetry were discussed, i.e.

$$T^{(n)} = \{f \in T : f(\varepsilon z) = \varepsilon f(z), z \in \Delta\},$$

where $\varepsilon = e^{\frac{2\pi i}{n}}$, $n \in N$, $n \geq 2$. The main result of this paper is

Theorem B [3]. *For $k \in N$ we have*

$$T^{(2k-1)} = \{f : f(z) = \sqrt[2k-1]{g(z^{2k-1})}, g \in T\},$$

$$T^{(2k)} = \{f : f(z) = \sqrt[k]{g(z^k)}, g \in T^{(2)}\}.$$

According to this theorem, the Koebe domains for $T^{(n)}$ can be determined by the relation

$$K_{T^{(2k-1)}}(D) = \{z : z^{2k-1} \in K_T(D)\},$$

$$K_{T^{(2k)}}(D) = \{z : z^k \in K_{T^{(2)}}(D)\}.$$

Results concerning the class T were obtained in [5]. In order to determine the Koebe domains also for $T^{(n)}$, while n is odd, we need to know these domains for the class $T^{(2)}$.

Koebe domains for $T^{(2)}$

It is known (see for example [3]) that each function $T^{(2)}$ can be represented in the integral form by

$$f(z) = \int_0^1 \frac{z(1+z^2)}{(1+z^2)^2 - 4z^2 t} d\mu(t), \tag{3}$$

where $\mu \in P_{[0,1]}$, i.e. μ is a probability measure on $[0,1]$.

While researching $T^{(2)}$ it is useful to work with functions

$$g(w) = \int_0^1 \frac{w}{w^2 - 4t} d\mu(t), \tag{4}$$

for which $f(z) = g(z + \frac{1}{z})$.
Let $\mathcal{T}^{(\in)}$ denote a class of functions given by (4), i.e.

$$\mathcal{T}^{(\in)} = \{\} :\ \}(\sqsupseteq) = \int_{\prime}^{\infty} \frac{\sqsupseteq}{\sqsupseteq^{\in} - \triangle\sqcup} \lceil\mu(\sqcup)\,,\ \sqsupseteq \in \mathbb{C}\setminus[-\in,\in]\,,\ \mu \in \mathcal{P}_{[\prime,\infty]}\}\,.$$

For $z \neq 0$ let $z + \frac{1}{z} = w$, $|w| = \varrho$, $\arg w = \varphi$. The set $\{g(w) :\ g \in \mathcal{T}^{(\in)}\}$ is a convex hull of the curve $[0,1] \ni t \mapsto \frac{w}{w^2-4t}$ (see for example [1]). It is easy to observe that if w is a point of real or imaginary axis then the set $\{g(w) :\ g \in \mathcal{T}^{(\in)}\}$ coincides with a segment $\langle\langle \frac{1}{w}, \frac{w}{w^2-4} \rangle\rangle$ included in this axis. Otherwise, the boundary of the set $\{g(w) :\ g \in \mathcal{T}^{(\in)}\}$ consists of the arc of the circle

$$\left| w - \frac{1}{4\varrho}\left(\frac{1}{\cos\varphi} - i\frac{1}{\sin\varphi}\right)\right| = \left|\frac{1}{4\varrho}\left(\frac{1}{\cos\varphi} - i\frac{1}{\sin\varphi}\right)\right| \tag{5}$$

having end points $\frac{1}{\varrho}e^{-i\varphi}$ and $\frac{\varrho e^{i\varphi}}{\varrho^2 e^{2i\varphi}-4}$ and not containing 0, and the line segment $\langle\langle\frac{1}{\varrho}e^{-i\varphi}, \frac{\varrho e^{i\varphi}}{\varrho^2 e^{2i\varphi}-4}\rangle\rangle$. We conclude from this fact that $\min\{|g(w)| : g \in \mathcal{T}^{(\in)}, \arg\}(\sqsupseteq) = \beta\}$, $\beta \in [\prime, \frac{\pi}{\in}]$ is achieved by functions

$$g_\varepsilon(w) = (1-\varepsilon)\frac{1}{w} + \varepsilon\frac{w}{w^2-4} \quad , \quad w \in \mathbb{C}\setminus[-2,2]\,, \varepsilon \in [0,1]. \qquad (6)$$

In terms of function $f \in T^{(2)}$, the minimum of $\{|f(z)| : f \in T^{(2)}, \arg f(z) = \beta\}$, $\beta \in [0, \frac{\pi}{2}]$, is achieved by functions

$$f_\varepsilon(z) = (1-\varepsilon)\frac{z}{1+z^2} + \varepsilon\frac{z(1+z^2)}{(1-z^2)^2} \quad , \quad z \in \Delta\,, \varepsilon \in [0,1], \qquad (7)$$

which correspond with functions (6).

We begin with finding the Koebe domain for $T^{(2)}$ and the lens $H = \{z \in \Delta : |z^2+1| > 2|z|\}$. The set H, as it is known, has special properties. Firstly, H is the domain of univalence of T [2]. Secondly, it is the only domain of univalence of $T^{(2)}$ that is symmetric with respect to both axes of the complex plane.

We know from [5] that $K_T(H) = \Delta_{\frac{1}{4}}$. By Theorem A point 4, $K_{T^{(2)}}(H) \supset K_T(H)$. The boundary of $K_T(H)$ consists of images of some points belonging to the boundary of H under the functions

$$f(z) = \varepsilon\frac{z}{(1-z)^2} + (1-\varepsilon)\frac{z}{(1+z)^2} \quad , \quad \varepsilon \in [0,1].$$

Since these functions are not in $T^{(2)}$ while $\varepsilon \neq \frac{1}{2}$, $\Delta_{\frac{1}{4}}$ is a proper subset of $K_{T^{(2)}}(H)$. Furthermore, $z = \frac{1}{4}i$ and $z = -\frac{1}{4}i$ are the only common points of $K_T(H)$ and $K_{T^{(2)}}(H)$.

Theorem 1. *The set $K_{T^{(2)}}(H)$ is a bounded domain, whose boundary is the curve $\Psi((-\pi,\pi])$, where*

$$\Psi(\varphi) = \frac{1}{2}\cos^3\varphi + i\frac{1}{2}(\frac{3}{2} - \sin^2\varphi)\sin\varphi\,,\ \varphi \in (-\pi,\pi]. \qquad (8)$$

Proof. All functions of $T^{(2)}$ are univalent in H. Hence

$$K_{T^{(2)}}(H) = \bigcap_{0\le\varepsilon\le 1} f_\varepsilon(H).$$

In order to determine the set $K_{T^{(2)}}(H)$ we will derive the envelope of the family of segments $\langle\langle f_0(z), f_1(z)\rangle\rangle$, while z ranges over the whole boundary of H. After that we will prove that this envelope is in fact the boundary of $K_{T^{(2)}}(H)$. Basing on Theorem A point 2 we can restrict determining the envelope to the first quadrant of the complex plane.

Let $z \in \partial H \cap \{z : \mathrm{Im} z > 0\}$, which is equivalent to $w = z + \frac{1}{z} = 2e^{i\varphi}$, $\varphi \in (-\pi, 0)$. Observe that for these z

$$g_1(w) = \frac{w}{w^2 - 4} = \frac{1}{4i \sin\varphi}$$

and

$$g_0(w) = \frac{1}{w} = \frac{1}{2}e^{-i\varphi}.$$

Straight lines going through points $g_0(w)$ and $g_1(w)$ for a fixed $w = 2e^{i\varphi}$, $\varphi \in (-\pi, 0)$ are of the form

$$w_\varphi(t) = \frac{1}{2}e^{-i\varphi} + t\left[-\frac{i}{4\sin\varphi} - \frac{1}{2}e^{-i\varphi}\right] \quad , \quad t \in \mathbb{R}\,,\ \varphi \in (-\pi, 0). \tag{9}$$

These lines are pairwise symmetric with respect to the imaginary axis (namely, for all $t \in \mathbb{R}$ and $\varphi \in (-\pi, 0)$ we have $w_{-\pi-\varphi}(t) = -\overline{w_\varphi(t)}$). Therefore, the envelope is symmetric with respect to the imaginary axis. This is a reason why we can restrict the set of variability of φ to the interval $(-\frac{\pi}{2}, 0)$. The line $\{w_{-\frac{\pi}{2}}(t) : t \in \mathbb{R}\}$ coincides with the imaginary axis.

The straight lines (9) can be written equivalently

$$x \cot 2\varphi - y - \frac{1}{4\sin\varphi} = 0 \quad , \quad \varphi \in \left(-\frac{\pi}{2}, 0\right).$$

From the system

$$\begin{cases} x \cot 2\varphi - y - \frac{1}{4\sin\varphi} = 0 \\ x\frac{-2}{\sin^2 2\varphi} + \frac{\cos\varphi}{4\sin^2\varphi} = 0 \end{cases}$$

we derive the equation of the envelope of lines (9) in the first quadrant

$$x = \frac{1}{2}\cos^3\varphi, \quad y = -\frac{1}{2}(\frac{3}{2} - \sin^2\varphi)\sin\varphi, \varphi \in (-\frac{\pi}{2}, 0). \tag{10}$$

Let $W(\varphi) = \frac{1}{2}\cos^3\varphi - i\frac{1}{2}(\frac{3}{2} - \sin^2\varphi)\sin\varphi$, $\varphi \in (-\frac{\pi}{2}, 0)$. Observe that

$$\arg W(\varphi) < \arg \frac{1}{4i\sin\varphi}$$

and

$$\arg W(\varphi) > \arg \frac{1}{2} e^{-i\varphi},$$

where the argument in the above is taken from $(-\pi, \pi]$.

The first inequality is obvious. The second one is equivalent to

$$\frac{\frac{3}{2} - \sin^2 \varphi}{\cos^2 \varphi} \tan \varphi < \tan \varphi,$$

which is true for $\varphi \in (-\frac{\pi}{2}, 0)$.

Hence, the curve (10) is the envelope of the family of the straight lines (9) for $\varphi \in (-\frac{\pi}{2}, 0)$, as well as the envelope of the family of the line segments $\langle\langle g_0(2e^{i\varphi}), g_1(2e^{i\varphi})\rangle\rangle$ while $\varphi \in (-\frac{\pi}{2}, 0)$.

Putting $\varphi = 0$ into (10) we get the point $\frac{1}{2} = f_0(1) = g_0(2)$, and putting $\varphi = -\frac{\pi}{2}$ we obtain $\frac{1}{4} i = f_1(i(\sqrt{2} - 1)) = g_1(-2i)$.

It follows from $\arg[g_1(2e^{i\varphi}) - g_0(2e^{i\varphi})] = -2\varphi + \frac{\pi}{2}$ that this argument is a decreasing function of $\varphi \in (-\frac{\pi}{2}, 0)$. Therefore, the bounded domain D for which the curve (10) and the intervals $[0, \frac{1}{2}]$ and $[0, \frac{1}{4} i]$ are its boundary is convex. Hence, each set $\{f(z) : f \in T^{(2)}\}$ for $z \in \partial H \cap \{z : \mathrm{Re} z \geq 0, \mathrm{Im} z \geq 0\}$, which is the same as $\{g(2e^{i\varphi}) : g \in \mathcal{T}^{(\in)}\}$ for $\varphi \in [-\frac{\pi}{2}, 0]$ is disjoint from D (has exactly one common point with the closure of the curve (10)). It means that $D \subset f(H \cap \{z : \mathrm{Re} z \geq 0, \mathrm{Im} z \geq 0\})$ for each $f \in T^{(2)}$.

Taking the interval $(-\pi, \pi]$ instead of $(-\frac{\pi}{2}, 0)$ in (10) we obtain a curve which is closed and symmetric with respect to both axes. Let us denote by E the set which has this curve as a boundary and which contains the origin.

From the above argument it follows that $E \subset f(H)$ for each $f \in T^{(2)}$. Since

$$E \subset \bigcap_{f \in T^{(2)}} f(H) \subset \bigcap_{\varepsilon \in [0,1]} f_\varepsilon(H) = E\,,$$

we have $E = K_{T^{(2)}}(H)$. $\square$

Substituting $\cos \varphi$ by $\sqrt[3]{2x}$ in (10) one can write the equation of the boundary of $K_{T^{(2)}}(H)$ in the form

$$y^2 = \frac{1}{4} \left(1 - \sqrt[3]{4x^2}\right) \left(\frac{1}{2} + \sqrt[3]{4x^2}\right)^2 .$$

Now, we consider some special sets Ω_α for which we determine Koebe domains. After that we will be able to indicate the largest Koebe domain for $T^{(2)}$ and some set D which will be possible to determine applying the method of the envelope.

We need the following notation:

$$l(z) = z + \tfrac{1}{z}\,,\ z \in \Delta \setminus \{0\}, \tag{11}$$

$$\Omega_\alpha = \{z \in \Delta : |(z + \tfrac{1}{z})^2 - \alpha| > 4 - \alpha\},\ \alpha \le \tfrac{4}{3}. \tag{12}$$

$$D_\alpha = \{w : |w^2 - \alpha| > 4 - \alpha, \mathrm{Re} w > 0, \mathrm{Im} w > 0\},\ \alpha \le \tfrac{4}{3}, \tag{13}$$

$$\Gamma_\alpha = \{w : |w^2 - \alpha| = 4 - \alpha, \mathrm{Re} w > 0, \mathrm{Im} w > 0\},\ \alpha \le \tfrac{4}{3}. \tag{14}$$

In particular, $\Omega_0 = H$ and Γ_0 is the arc of the circle $|w| = 2$ that is included in the first quadrant of the complex plane.

All domains Ω_α, $\alpha \le \frac{4}{3}$ are symmetric with respect to both axes and $l(\Omega_\alpha) \cap \{w : \mathrm{Re} w > 0, \mathrm{Im} w > 0\} = D_\alpha$. According to [3], $l^{-1}(D_{\frac{4}{3}})$ is the quarter of the domain of local univalence for $T^{(2)}$ included in the fourth quadrant of the complex plane. It was proved in [4] that

THEOREM C. *Each function $g \in \mathcal{T}^{(\in)}$ is univalent in $D_{\frac{4}{3}}$.*

Obviously, $\alpha < \beta \le \frac{4}{3} \Longrightarrow D_\alpha \subset D_\beta$. Hence, all functions $g \in \mathcal{T}^{(\in)}$ are univalent in every set D_α, $\alpha \le \frac{4}{3}$.

In order to determine $K_{T^{(2)}}(\Omega_\alpha)$ we need the envelope of the family of line segments $\langle\langle g_0(w), g_1(w)\rangle\rangle$ for w ranging over Γ_α.

For $w \in \Gamma_\alpha$ we have

$$w = \sqrt{\alpha + (4-\alpha)e^{i\varphi}}\,,\ \varphi \in (0, \pi), \tag{15}$$

where the branch of the square root is taken in such a way that $\sqrt{1} = 1$. Denote

$$I_\alpha = \begin{cases} [2, 2-\alpha] & \alpha < 0, \\ [2-\alpha, 2] & \alpha \in (0, \frac{4}{3}] \end{cases}$$

and

$$\begin{aligned}\Psi_\alpha(p) = \frac{2}{\alpha^2(4-\alpha)(3p^2 + 2(2-\alpha))} \Bigg[&[p^2 - \frac{1}{2}(3\alpha - 4)p + \frac{1}{4}\alpha^2] \times \\ &\sqrt{\alpha(2+p)(p-2+\alpha)^3} \\ -i\alpha[p^2 + \tfrac{1}{2}(3\alpha-4)p + \tfrac{1}{4}\alpha^2]&\sqrt{\tfrac{1}{\alpha}(2-p)(p+2-\alpha)^3}\Bigg],\end{aligned} \tag{16}$$

where $p \in I_\alpha, \alpha \in (-\infty, 0) \cup (0, \frac{4}{3}]$.

Theorem 2. *The envelope of the straight lines going through $g_0(w)$ and $g_1(w)$, while w is of the form (15) and $\alpha \in (-\infty, 0) \cup (0, \frac{4}{3}]$, coincides with the curve $\Psi_\alpha(I_\alpha)$.*

Proof. Let w be of the form (15) and $\alpha \in (-\infty, 0) \cup (0, \frac{4}{3}]$. Denote

$$\sqrt{\alpha + (4-\alpha)e^{i\varphi}} = \varrho e^{i\theta}, \tag{17}$$

where the branch of the square root is chosen as in (15).
From this we observe that $\theta \in (0, \frac{\pi}{2})$ and that the sign of $\varrho - 2$ depends on α. Namely, for $\alpha \in (0, \frac{4}{3}]$ we have $\varrho - 2 < 0$ and for $\alpha \in (-\infty, 0)$ we have $\varrho - 2 > 0$.
Applying (17) we obtain

$$g_0(w) = \frac{1}{\varrho}\cos\theta - i\frac{1}{\varrho}\sin\theta$$

and

$$g_1(w) = \frac{(\varrho - \frac{4}{\varrho})\cos\theta}{\varrho^2 + \frac{16}{\varrho^2} - 8\cos 2\theta} - i\frac{(\varrho + \frac{4}{\varrho})\sin\theta}{\varrho^2 + \frac{16}{\varrho^2} - 8\cos 2\theta}.$$

The real equation of straight lines going through $g_0(w)$ and $g_1(w)$ can be written in the form

$$[4 - \varrho^2(1 + 2\cos 2\theta)]x\tan\theta + [4 + \varrho^2(1 - 2\cos 2\theta)]y + 2\varrho\sin\theta = 0. \tag{18}$$

We conclude from (17) that

$$\begin{aligned} \varrho^2 &= \sqrt{(4-\alpha)^2 + 2\alpha(4-\alpha)\cos\varphi + \alpha^2}, \\ \cot 2\theta &= \frac{\alpha + (4-\alpha)\cos\varphi}{(4-\alpha)\sin\varphi}. \end{aligned} \tag{19}$$

For convenience let

$$p = \frac{1}{2}\sqrt{(4-\alpha)^2 + 2\alpha(4-\alpha)\cos\varphi + \alpha^2}. \tag{20}$$

Hence, if $\varphi \in [0, \pi]$ then $p \in I_\alpha$.

From (19) we derive

$$\begin{aligned} \varrho^2 &= 2p, \\ \varrho^2\cos 2\theta &= \tfrac{2}{\alpha}(p^2 - 4 + 2\alpha), \\ \varrho^2\sin 2\theta &= \tfrac{2}{|\alpha|}\sqrt{(4-p^2)(p^2 - (2-\alpha)^2)}, \end{aligned} \tag{21}$$

and then

$$\begin{aligned} \varrho\cos\theta &= \tfrac{1}{|\alpha|}\sqrt{\alpha(p+2)(p-2+\alpha))}, \\ \varrho\sin\theta &= \tfrac{1}{|\alpha|}\sqrt{\alpha(2-p)(p+2-\alpha))}. \end{aligned} \tag{22}$$

Applying (19) and (22) in (18) we obtain the equation equivalent to (18):

$$\sqrt{\frac{\alpha(p+2)}{p-2+\alpha}}(4-\alpha-2p)x+\sqrt{\frac{\alpha(2-p)}{p+2-\alpha}}(4-\alpha+2p)y+\alpha=0. \tag{23}$$

The envelope of the family of these lines is obtained as the solution of the system

$$\begin{cases} \sqrt{\frac{\alpha(p+2)}{p-2+\alpha}}(4-\alpha-2p)x+\sqrt{\frac{\alpha(2-p)}{p+2-\alpha}}(4-\alpha+2p)y+\alpha=0 \\ \sqrt{\frac{p-2+\alpha}{\alpha(p+2)}}\frac{4p^2+2(3\alpha-4)p+\alpha^2}{(p-2+\alpha)^2}x+\sqrt{\frac{p+2-\alpha}{\alpha(2-p)}}\frac{-4p^2+2(3\alpha-4)p-\alpha^2}{(p+2-\alpha)^2}=0. \end{cases}$$

In this way we get the curve given by (16). □

REMARK. **1.** *In the limit case, taking into account* $\lim_{\alpha\to 0}\Psi_\alpha(I_\alpha)$, *we obtain the curve* $\Psi((-\pi,\pi])$ *defined in Theorem 1. One can calculate this limit putting* $s=\frac{p-2+\alpha}{\alpha}$ *into the equation of* Ψ_α *(in this case* $s\in[0,1]$*).*
2. *The curve* $\Psi_\alpha(I_\alpha)$ *has one singularity for*

$$p_0 = \frac{1}{12}\sqrt{3(-112+80\alpha-9\alpha^2+\sqrt{(3\alpha-4)(27\alpha^3-508\alpha^2+3280\alpha-4672)}} \tag{24}$$

while $\alpha<-12$ *(if* $\alpha=-12$ *then* $p_0=2$*). It can be concluded from*

$$(x\prime(p))^2+(y\prime(p))^2=$$

$$\frac{p[24p^4+(9\alpha^2-80\alpha+112)p^2-2(2-\alpha)(\alpha^2-16\alpha+16)]}{8\alpha(4-\alpha)(4-p^2)(3p^2+4-2\alpha)^4}$$

and the fact that p_0 *is the only zero of this expression in* $(2,2-\alpha)$*.*
3. *Using (22) one can obtain a new complex parametric equation of* Γ_α

$$w(p)=\frac{1}{|\alpha|}\left(\sqrt{\alpha(p+2)(p-2+\alpha))}+i\sqrt{\alpha(2-p)(p+2-\alpha))}\right)\ ,\ p\in I_\alpha, \tag{25}$$

which is useful in the following consideration.

First we need

Lemma 1. *Let* $h(p) = \arg[g_1(w(p)) - g_0(w(p))]$, *where* g_0, g_1 *are given by (6), the argument is taken from the interval* $(-2\pi, 0]$, *and let* $w(p)$ *be given by (25). Then the range of* h *is* $[-\frac{3\pi}{2}, -\frac{\pi}{2})$. *Moreover, if* $\alpha < 0$ *then* h *is decreasing in* $(2, 2-\alpha)$, *and if* $\alpha \in (0, \frac{4}{3}]$ *then* h *is increasing in* $(2-\alpha, 2)$.

Proof. Let w be defined by (15) and let $k(\varphi) = \arg[g_1(w) - g_0(w)]$. The function k is decreasing for $\varphi \in (0, \pi)$ because

$$k(\varphi) = -[\frac{1}{2}\arg(\alpha + (4-\alpha)e^{i\varphi}) + \arg(e^{i\varphi} - 1)]\,.$$

.

Furthermore, by (20) for $\alpha \in (0, \frac{4}{3}]$, p is a decreasing function of φ. Hence, there exists its inverse function $\varphi = \varphi(p)$ and it is decreasing for $p \in [2-\alpha, 2]$. Combining these facts, we conclude that $h(p) = k(\varphi(p))$ is an increasing function for $p \in (2-\alpha, 2)$.

In the second case, for $\alpha < 0$, it follows from (20) that p is an increasing function of φ, and consequently, $h(p) = k(\varphi(p))$ is a decreasing function for $p \in (2-\alpha, 2)$. □

Theorem 3. *The envelope of the line segments* $\langle\langle g_0(w), g_1(w)\rangle\rangle$, *where* g_0, g_1 *are given by (6) and* w *is given by (15), is a convex curve of the form*

1. $\Psi_\alpha\left((2-\alpha, 2)\right)$ for $\alpha \in (0, \frac{4}{3}]$,
2. $\Psi_\alpha\left((2, d_\alpha)\right)$ for $\alpha \in [-2, 0)$,
3. $\Psi_\alpha\left((c_\alpha, d_\alpha)\right)$ for $\alpha \in (-\infty, -2)$,

where Ψ_α *is given by (16) and*

$$c_\alpha = \sqrt{-\frac{1}{2}(2-\alpha)\alpha}, \quad d_\alpha = \sqrt{\frac{1}{2}(\alpha^2 - 8\alpha + 8)}.$$

In this theorem and further on, the convexity of a curve means that the tangent line to this curve lies below the curve.

Proof. According to Theorem 2, $\Psi_\alpha\left(I_\alpha\right)$, $\alpha \in (-\infty, 0) \cup (0, \frac{4}{3}]$ is the envelope of straight lines going through $g_0(w)$ and $g_1(w)$, $w \in \Gamma_\alpha$.

This curve (whole or only a part of it) is also the envelope of line segments $\langle\langle g_0(w), g_1(w)\rangle\rangle$, but only for these p which satisfy the inequality

$$\arg g_1(w(p)) \le \arg \Psi_\alpha(p) \le \arg g_0(w(p))\ ,\ p \in I_\alpha. \tag{26}$$

For $w \in \Gamma_\alpha$ (w is of the form (15)) we have

$$\begin{aligned} \arg g_1(w) &= \arg[(\varrho^2-4)\cos\theta - i(\varrho^2+4)\sin\theta], \\ \arg g_0(w) &= \arg[\cos\theta - i\sin\theta] = -\theta. \end{aligned} \tag{27}$$

Let $\alpha \in (0, \frac{4}{3}]$ and let Ψ_α be given by (16).
It follows from (27) that $\arg g_1(w) \in (-\pi, -\frac{\pi}{2})$ and $\arg g_0(w) \in (-\frac{\pi}{2}, 0)$. Moreover, $\arg \Psi_\alpha(p) \in (-\frac{\pi}{2}, 0)$ for $p \in (2-\alpha, 2)$. Since the left hand side of (26) is fulfilled, it is sufficient to discuss only the right hand side inequality. We rewrite it as follows

$$\frac{p^2+\frac{1}{2}(3\alpha-4)p+\frac{1}{4}\alpha^2}{p^2-\frac{1}{2}(3\alpha-4)p+\frac{1}{4}\alpha^2}\sqrt{\frac{(2-p)(p+2-\alpha)^3}{(2+p)(p-2+\alpha)^3}} \geq \sqrt{\frac{(2-p)(p+2-\alpha)}{(2+p)(p-2+\alpha)}}$$

and equivalently

$$[p^2+\frac{1}{2}(3\alpha-4)p+\frac{1}{4}\alpha^2](p+2-\alpha) \geq [p^2-\frac{1}{2}(3\alpha-4)p+\frac{1}{4}\alpha^2](p-2+\alpha),$$

and further on

$$p^2+\frac{1}{2}\alpha(2-\alpha) \geq 0,$$

which holds for $\alpha \in (0, \frac{4}{3}]$ and $p \in [2-\alpha, 2]$.
Therefore, the inequality(26) is true for $\alpha \in (0, \frac{4}{3}]$ and $w \in \Gamma_\alpha$. We conclude from this that the curve $\Psi_\alpha(I_\alpha)$ is really the envelope of line segments $\langle\langle g_0(w), g_1(w)\rangle\rangle$ for $\alpha \in (0, \frac{4}{3}]$.

Let now $\alpha \in (-\infty, 0)$ and $w \in \Gamma_\alpha$.
From (27) we obtain $\arg g_1(w) \in (-\frac{\pi}{2}, 0)$ and $\arg g_0(w) \in (-\frac{\pi}{2}, 0)$. The left hand side of (26) is equivalent to

$$-\frac{p^2+\frac{1}{2}(3\alpha-4)p+\frac{1}{4}\alpha^2}{p^2-\frac{1}{2}(3\alpha-4)p+\frac{1}{4}\alpha^2}\sqrt{\frac{(2-p)(p+2-\alpha)^3}{(2+p)(p-2+\alpha)^3}} \leq$$

$$\frac{p+2}{p-2}\sqrt{\frac{(2-p)(p+2-\alpha)}{(2+p)(p-2+\alpha)}}$$

and then

$$-[p^2+\frac{1}{2}(3\alpha-4)p+\frac{1}{4}\alpha^2][-p^2+\alpha p-2(2-\alpha)]$$

$$\geq [p^2 - \frac{1}{2}(3\alpha - 4)p + \frac{1}{4}\alpha^2][p^2 + \alpha p - 2(2-\alpha)].$$

After simple calculations it takes form

$$(4-\alpha)p[p^2 - \frac{1}{2}(\alpha^2 - 8\alpha + 8)] \geq 0. \tag{28}$$

The inequality (28), and in consequence, the inequality $\arg g_1(w) \leq \arg \Psi_\alpha(p)$ holds only for $p \in [2, \sqrt{\frac{1}{2}(\alpha^2 - 8\alpha + 8)}]$ because of $2 < \sqrt{\frac{1}{2}(\alpha^2 - 8\alpha + 8)} < 2 - \alpha$.

The right hand side of (26) turns to

$$\frac{p^2 + \frac{1}{2}(3\alpha - 4)p + \frac{1}{4}\alpha^2}{p^2 - \frac{1}{2}(3\alpha - 4)p + \frac{1}{4}\alpha^2}\sqrt{\frac{(2-p)(p+2-\alpha)^3}{(2+p)(p-2+\alpha)^3}} \leq -\sqrt{\frac{(2-p)(p+2-\alpha)}{(2+p)(p-2+\alpha)}}. \tag{29}$$

Hence

$$[p^2 + \frac{1}{2}(3\alpha - 4)p + \frac{1}{4}\alpha^2](p+2-\alpha) \leq [p^2 - \frac{1}{2}(3\alpha - 4)p + \frac{1}{4}\alpha^2](p-2+\alpha),$$

and then

$$p^2 + \frac{1}{2}\alpha(2-\alpha) \geq 0. \tag{30}$$

It is easy to check that if $\alpha \in (-2, 0)$ and $p \in [2, 2-\alpha]$, then (30) holds. It means that (29) is fulfilled. If $\alpha \in (-\infty, -2)$ then (30) is satisfied only for $p \in [\sqrt{-\frac{1}{2}\alpha(2-\alpha)}, 2-\alpha]$.

Our next goal is to prove the convexity of the above derived envelope of the line segments.

In view of Remark 2 the envelope of the straight lines going through $g_0(w)$ and $g_1(w)$ has no singularities for $\alpha \in [-12, 0) \cup (0, \frac{4}{3}]$. If $\alpha < -12$ then this envelope has the only singularity corresponding to p_0 given by(24), but $p_0 < c_\alpha$. Indeed,

$$-112 + 80\alpha - 9\alpha^2 + \sqrt{(3\alpha - 4)(27\alpha^3 - 508\alpha^2 + 3280\alpha - 4672)} <$$

$$-24(2-\alpha)\alpha$$

and then

$$(\alpha - 2)(7\alpha + 4)(\alpha - 4) > 0\,,$$

which is true for $\alpha < -12$.
Therefore, the envelope of the line segments $\langle\langle g_0(w), g_1(w)\rangle\rangle$ has no singularities, and, by Lemma 1, is convex. □

Let $\alpha < 0$ and

$$\Phi_\alpha(p) = \frac{1}{2(4-\alpha)}\left[\sqrt{\alpha\frac{p-2+\alpha}{p+2}} - i\sqrt{\alpha\frac{p+2-\alpha}{2-p}}\right],\ p \in (2, 2-\alpha], \tag{31}$$

$$\Upsilon_\alpha(p) = \frac{-1}{2\alpha p}\left[\sqrt{\alpha(p+2)(p-2+\alpha)} - i\sqrt{\alpha(2-p)(p+2-\alpha)}\right], \tag{32}$$

$p \in [2, 2-\alpha]$. For Φ_α and Υ_α we have

$\Phi_\alpha\left((2, 2-\alpha)\right) = \{g_1(w) : w \in \Gamma_\alpha\}$ and $\Upsilon_\alpha\left((2, 2-\alpha)\right) = \{g_0(w) : w \in \Gamma_\alpha\}$.

Let E_α be a bounded domain whose boundary is of the form:

- $[0, \frac{1}{2}) \cup \Psi_\alpha\left([2-\alpha, 2]\right) \cup \left(-i(0, \frac{\sqrt{4-2\alpha}}{8-3\alpha})\right)$ for $\alpha \in (0, \frac{4}{3}]$,
- $[0, \frac{1}{2}) \cup \Psi_\alpha\left([2, d_\alpha]\right) \cup \Phi_\alpha\left((d_\alpha, 2-\alpha]\right) \cup \left(-i(0, \frac{\sqrt{4-2\alpha}}{8-2\alpha})\right)$ for $\alpha \in [-2, 0)$,
- $[0, \frac{1}{2}) \cup \Upsilon_\alpha\left([2, c_\alpha)\right) \cup \Psi_\alpha\left([c_\alpha, d_\alpha]\right) \cup \Phi_\alpha\left((d_\alpha, 2-\alpha]\right) \cup \left(-i(0, \frac{\sqrt{4-2\alpha}}{8-2\alpha})\right)$ for $\alpha \in (-\infty, -2)$.

THEOREM 4. *For each $w \in \Gamma_\alpha$ and $\alpha \in (-\infty, 0) \cup (0, \frac{4}{3}]$:*

$$E_\alpha \cap \{g(w) :\ g \in \mathcal{T}^{(\in)}\} = \emptyset, \tag{33}$$

$$cl(E_\alpha) \cap \{g(w) :\ g \in \mathcal{T}^{(\in)}\} \mathrm{isaone-pointset.} \tag{34}$$

We need four lemmas to prove Theorem 4.

LEMMA 2. *For $\alpha < -2$ and $p \in (2, c_\alpha)$ we have $\arg\left[\Phi_\alpha(p) - \Upsilon_\alpha(p)\right] - \arg\Upsilon'_\alpha(p) > 0$, where the arguments of $\Phi_\alpha(p) - \Upsilon_\alpha(p)$ and $\Upsilon'_\alpha(p)$ are taken from the interval $(-\frac{3\pi}{2}, -\frac{\pi}{2})$.*

PROOF. Let $\alpha < -2$. Firstly, we are going to prove that $w(c_\alpha)$ is the only point, given by $w(p)$, $p \in (2, 2-\alpha)$, for which the tangent line to the curve $\Upsilon_\alpha(I_\alpha)$ coincides with the straight line going through $g_0(w)$ and $g_1(w)$.
Let us discuss the equation

$$\mathrm{Re}\left[\Phi_\alpha(p) - \Upsilon_\alpha(p)\right] \cdot \mathrm{Im}\Upsilon'_\alpha(p) = \mathrm{Im}\left[\Phi_\alpha(p) - \Upsilon_\alpha(p)\right] \cdot \mathrm{Re}\Upsilon'_\alpha(p)\ .$$

This equation, by (31) and (32), is equivalent to

$$[-\alpha p+4(2-\alpha)][4p^2-2\alpha p-2(4-\alpha)(2-\alpha)]=$$

$$[\alpha p+4(2-\alpha)][4p^2+2\alpha p+2(4-\alpha)(2-\alpha)]$$

and hence $p^2=-\frac{1}{2}\alpha(2-\alpha)$.
Thus the expression $\arg[\Phi_\alpha(p)-\Upsilon_\alpha(p)]-\arg\Upsilon'_\alpha(p)$ does not change the sign for all $p\in(2,c_\alpha)$.
Moreover,

$$\arg[\Phi_\alpha(p)-\Upsilon_\alpha(p)]=-\pi-\arctan\left(\frac{2p-(4-\alpha)}{2p+(4-\alpha)}\sqrt{\frac{(p+2)(p+2-\alpha)}{(p-2)(2-\alpha-p)}}\right),$$

and for $p\in(2,4)$ we have $\mathrm{Re}\Upsilon'_\alpha(p)<0$ and $\mathrm{Im}\Upsilon'_\alpha(p)<0$. Thus

$$\arg\Upsilon'_\alpha(p)=-\pi-\arctan\left(\frac{\alpha p+4(2-\alpha)}{\alpha p-4(2-\alpha)}\sqrt{\frac{(p+2)(2-\alpha-p)}{(p-2)(p+2-\alpha)}}\right).$$

For $p\in(2,4)$ the inequality

$$\arg[\Phi_\alpha(p)-\Upsilon_\alpha(p)]-\arg\Upsilon'_\alpha(p)>0 \tag{35}$$

is equivalent to

$$\frac{\alpha p+4(2-\alpha)}{\alpha p-4(2-\alpha)}\sqrt{\frac{(p+2)(2-\alpha-p)}{(p-2)(p+2-\alpha)}}>\frac{2p-(4-\alpha)}{2p+(4-\alpha)}\sqrt{\frac{(p+2)(p+2-\alpha)}{(p-2)(2-\alpha-p)}},$$

and, in consequence, to $p^2<-\frac{1}{2}\alpha(2-\alpha)$. Therefore, (35) holds for $p\in(2,4)\cap(2,c_\alpha)$.

The function $\arg[\Phi_\alpha(p)-\Upsilon_\alpha(p)]-\arg\Upsilon'_\alpha(p)$ is continuous for $p\in(2,2-\alpha)$, positive for $p\in(2,4)\cap(2,c_\alpha)$ and its only zero is c_α. Thus, (35) holds for $p\in(2,c_\alpha)$. □

Proofs of next three lemmas will be omitted.

LEMMA 3. *For* $\alpha<0$ *the function* $\arg\Upsilon_\alpha(p)$ *is decreasing in* $(2,2-\alpha)$.

LEMMA 4. *The function* $\arg\Upsilon'_\alpha(p)$ *is*

1. *increasing in* $(2,2-\alpha)$ *for* $\alpha\in[-4,0)$,
2. *decreasing in* $(2,b_\alpha)$ *and increasing in* $(b_\alpha,2-\alpha)$ *for* $\alpha<-4$,

where $b_\alpha = \sqrt{6(2-\alpha)}$.

LEMMA 5. *For* $\alpha < 0$ *the function* $\arg \Phi'_\alpha(p)$ *is increasing in* $(2, 2-\alpha)$.

PROOF OF THEOREM 4.

We will prove this theorem only for $\alpha < -4$. For other α the proof is easier, because we need only some elements of the argument presented below. Let $w = w(p)$ be given by (25). Denote by B_p a closed and convex set whose boundary consists of a ray l_p with the end point in $g_0(w(p))$ and going through $2g_0(w(p))$, a ray k_p with the end point in $g_1(w(p))$ and going through $2g_1(w(p))$, and a segment $\langle\langle g_0(w(p)), g_1(w(p))\rangle\rangle$.

The set $\{g(w(p)) :\ g \in \mathcal{T}^{(\in)}\}$, $p \in (2, 2-\alpha)$ is a segment of the disk whose boundary is given by (5). Moreover, $0 \notin \{g(w(p)) :\ g \in \mathcal{T}^{(\in)}\}$. From it and from the fact that $w = 0$ belongs to the circle (5) we conclude that for $p \in (2, 2-\alpha)$

$$\{g(w(p)) :\ g \in \mathcal{T}^{(\in)}\} \subset \mathcal{B}_{\surd}.$$

Let $p \in (2, b_\alpha]$.
From Lemmas 1 - 4 it yields that $B_p \subset$

$$\{u \in \mathbb{C} : \arg\left[g_1(w(p)) - g_0(w(p))\right] \le \arg\left[u - g_0(w(p))\right] \le \arg g_0(w(p))\}$$
$$\subset \{u \in \mathbb{C} : \arg\left[g_1(w(b_\alpha)) - g_0(w(b_\alpha))\right] \le$$
$$\arg\left[u - g_0(w(p))\right] \le \arg g_0(w(p))\}$$
$$\subset \{u \in \mathbb{C} : \arg \tfrac{d}{dp}\ g_0(w(p))|_{p=b_\alpha} \le \arg\left[u - g_0(w(p))\right] \le \arg g_0(w(p))\}.$$

Let $p \in (b_\alpha, c_\alpha)$. From Lemmas 1 - 4 we have $B_p \subset$

$$\{u \in \mathbb{C} : \arg\left[g_1(w(p)) - g_0(w(p))\right] \le \arg\left[u - g_0(w(p))\right] \le \arg g_0(w(p))\}$$
$$\subset \{u \in \mathbb{C} : \arg \tfrac{d}{dp}\ g_0(w(p)) \le \arg\left[u - g_0(w(p))\right] \le \arg g_0(w(p))\}.$$

It means that for $p \in (2, c_\alpha)$ there is $B_p \cap E_\alpha = \emptyset$. Therefore, $\{g(w(p)) :\ g \in \mathcal{T}^{(\in)}\} \cap \mathcal{E}_\alpha = \emptyset$ and $\{g(w(p)) :\ g \in \mathcal{T}^{(\in)}\} \cap \rfloor\updownarrow(\mathcal{E}_\alpha) = \{\}\text{,}(\sqsupseteq(\surd))\}$.

Let $p \in (c_\alpha, d_\alpha)$. From Lemma 1, from the inequalities $\arg g_0(w(c_\alpha)) < \arg g_0(w(p))$ and $\arg g_1(w(p)) < \arg g_1(w(d_\alpha))$ and from the fact that the segment $\langle\langle g_0(w(p)), g_1(w(p))\rangle\rangle$ is tangent to ∂E_α (or equivalently to $\Psi([c_\alpha, d_\alpha])$) we obtain $B_p \cap E_\alpha = \emptyset$, and thus $\{g(w(p)) :\ g \in \mathcal{T}^{(\in)}\} \cap \mathcal{E}_\alpha = \emptyset$. Furthermore, the only common point of $\{g(w(p)) :\ g \in \mathcal{T}^{(\in)}\}$ and $cl(E_\alpha)$ is a point of tangency.

Let $p \in (d_\alpha, 2-\alpha)$. From Lemma 1, Lemma 4 and Lemma 5 we have

$$\begin{aligned} B_p \subset \{u \in \mathbb{C} : \arg g_1(w(p)) \le \arg[u - g_1(w(p))] \le \\ \le \arg[g_0(w(p)) - g_1(w(p))]\} \\ \subset \{u \in \mathbb{C} : \arg g_1(w(p)) \le \arg[u - g_1(w(p))] \le \\ \le \arg[g_0(w(d_\alpha)) - g_1(w(d_\alpha))]\} \\ = \{u \in \mathbb{C} : \arg g_1(w(p)) \le \arg[u - g_1(w(p))] \le \arg \frac{d}{dp}\, g_1(w(p))|_{p=d_\alpha}\}\,. \end{aligned}$$

Hence $B_p \cap E_\alpha = \emptyset$ and $\{g(w(p)) :\ g \in \mathcal{T}^{(\in)}\} \cap \mathcal{E}_\alpha = \emptyset$. Moreover, $\{g(w(p)) :\ g \in \mathcal{T}^{(\in)}\} \cap \rfloor\updownarrow(\mathcal{E}_\alpha) = \{\}_\infty(\sqsupseteq(\surd))\}$.

Let $A_\alpha = l^{-1}(D_\alpha)$, i.e. $A_\alpha = \Omega_\alpha \cap \{z \in \Delta : \mathrm{Re}z > 0, \mathrm{Im}z < 0\}$.

Corolary. $K_{T^{(2)}}(A_\alpha) = E_\alpha$.

Proof. For each $f \in T^{(2)}$ and $z \in \Delta$ there are $\mathrm{Im}z = 0 \Rightarrow \mathrm{Im}f(z) = 0$ and $\mathrm{Re}z = 0 \Rightarrow \mathrm{Re}f(z) = 0$. Hence

$$E_\alpha \cap \{f(z) :\ f \in T^{(2)},\ z \in \partial A_\alpha,\ z \ne 1,\ \mathrm{Re}z\mathrm{Im}z = 0\} = \emptyset\,.$$

This and Theorem 4 leads to

$$E_\alpha \cap \{f(z) :\ f \in T^{(2)},\ z \in \partial A_\alpha,\ z \ne 1\} = \emptyset\,.$$

Moreover, if $z = 1$ is a regular point of $f \in T^{(2)}$ then $f(1) \ge \frac{1}{2}$ (because for $x \in (0,1)$ there is $f(x) \ge \frac{x}{1+x^2}$). It means that $E_\alpha \subset f(A_\alpha)$ for each $f \in T^{(2)}$. Therefore, $E_\alpha \subset K_{T^{(2)}}(A_\alpha)$. By the definition of the Koebe domain, $K_{T^{(2)}}(A_\alpha) \subset \bigcap_{\varepsilon\in[0,1]} f_\varepsilon(A_\alpha)$.

The univalence of f_ε in A_α (by Theorem C) and (33) leads to $\bigcap\limits_{\varepsilon\in[0,1]} f_\varepsilon(A_\alpha) = E_\alpha$. From the above argument $E_\alpha \subset K_{T^{(2)}}(A_\alpha) \subset E_\alpha$. □

Theorem 5. *The set $K_{T^{(2)}}(\Omega_\alpha)$, $\alpha \in (-\infty, 0) \cup (0, \frac{4}{3}]$ is a bounded domain, symmetric with respect to both axes of the complex plane. The boundary of this domain in the fourth quadrant coincides with:*

1. $\Psi_\alpha([2-\alpha, 2])$ for $\alpha \in (0, \frac{4}{3}]$,

2. $\Psi_\alpha([2, d_\alpha]) \cup \Phi_\alpha((d_\alpha, 2-\alpha])$ for $\alpha \in [-2, 0)$,

3. $\Upsilon_\alpha([2, c_\alpha)) \cup \Psi_\alpha([c_\alpha, d_\alpha]) \cup \Phi_\alpha((d_\alpha, 2-\alpha])$ for $\alpha \in (-\infty, -2)$,

where Ψ_α, Φ_α, Υ_α are given by (15), (29), (30) respectively, and $c_\alpha = \sqrt{-\frac{1}{2}(2-\alpha)\alpha}$, $d_\alpha = \sqrt{\frac{1}{2}(\alpha^2 - 8\alpha + 8)}$.

Proof. Let us denote $G_\alpha = \mathrm{int}(\overline{(E_\alpha \cup \overline{E_\alpha} \cup (-E_\alpha) \cup (-\overline{E_\alpha}))}$.
I. Let $\alpha \in (0, \frac{4}{3}]$. For each $f \in T^{(2)}$ the set $f(\Omega_\alpha)$ is symmetric with respect to both axes of the complex plane. Therefore, by Corollary 1,

$$E_\alpha \cup \overline{E_\alpha} \cup (-E_\alpha) \cup (-\overline{E_\alpha}) \subset f(\Omega_\alpha)\,,$$

so

$$G_\alpha \setminus \{z \in \mathbb{C} :\ \mathrm{Re}z\mathrm{Im}z = 0\} \subset f(\Omega_\alpha)\,.$$

If $0 \le x < 1$ then $f(x) \ge \frac{x}{1+x^2}$ and if $-1 < x < 0$ then $f(x) \le \frac{x}{1+x^2}$. Hence for each $f \in T^{(2)}$

$$f\left((-1,1)\right) \supset (-\frac{1}{2}, \frac{1}{2}) = G_\alpha \cap \{z \in \mathbb{C} :\ \mathrm{Im}z = 0\}\,, \tag{36}$$

and then

$$G_\alpha \setminus \{z \in \mathbb{C} :\ \mathrm{Re}z = 0\} \subset f(\Omega_\alpha)\,.$$

This leads to

$$G_\alpha \setminus \{z \in \mathbb{C} :\ \mathrm{Re}z = 0\} \subset K_{T^{(2)}}(\Omega_\alpha)\,.$$

Our next goal is to prove that the line segment $G_\alpha \cap \{z \in \mathbb{C} :\ \mathrm{Re}z = 0\}$ is also included in $K_{T^{(2)}}(\Omega_\alpha)$.

Let us suppose that there exists a point iy_0 such that $iy_0 \notin K_{T^{(2)}}(\Omega_\alpha)$. It means there exists a function $f_\star \in T^{(2)}$ such that $f_\star(\Omega_\alpha) \not\ni iy_0$.

Let α_0 be taken in such a way that $0 < \alpha_0 < \alpha$ and $iy_0 \in G_{\alpha_0}$ (existence of such α_0 follows from the definition of G_α and from the fact that iy_0 is an interior point of this segment). We have $\Omega_{\alpha_0} \subset \Omega_\alpha$. Moreover, these sets have only two common points $z = -1$ and $z = 1$.

Since $f_\star$ is a typically real function, we can see that $f_\star(-1) \neq iy_0$ and $f_\star(1) \neq iy_0$. Hence there exists a neighborhood U of the point iy_0 such that $U \cap f_\star(\Omega_{\alpha_0}) = \emptyset$.
This gives $U \cap G_{\alpha_0} = \emptyset$, a contradiction, because

$$G_{\alpha_0} \setminus \{z \in \mathbb{C} :\ \mathrm{Re}z = 0\} \subset f(\Omega_{\alpha_0})\,.$$

The above given argument leads to $G_\alpha \subset K_{T^{(2)}}(\Omega_\alpha)$.

From Corollary 1 and the symmetry of $f(\Omega_\alpha)$ with respect to both axes of the complex plane we deduce $\bigcap_{\varepsilon \in [0,1]} f_\varepsilon(\Omega_\alpha) = G_\alpha$ and then $K_{T^{(2)}}(\Omega_\alpha) \subset G_\alpha$. Hence $K_{T^{(2)}}(\Omega_\alpha) = G_\alpha$.

II. Let $\alpha < 0$. We will prove that

$$G_\alpha \cap \{f(z): \ f \in T^{(2)}, \ z \in \partial\Omega_\alpha, \ z \neq \pm 1\} = \emptyset\,. \tag{37}$$

According to Theorem 4,

$$E_\alpha \cap \{f(z): \ f \in T^{(2)}, \ z \in \partial A_\alpha, \ \mathrm{Re}z\mathrm{Im}z \neq 0\} = \emptyset\,.$$

All functions belonging to $T^{(2)}$ are univalent in the lens H [1,3], and then in Ω_α (now $\Omega_\alpha \subset H$). From this we obtain

$$f(A_\alpha) \subset \{z \in \mathbb{C}: \ \mathrm{Re}z > 0\mathrm{Im}z < 0\}\,,$$

and, as a consequence,

$$G_\alpha \cap \{f(z): \ f \in T^{(2)}, \ z \in \partial\Omega_\alpha, \ \mathrm{Re}z\mathrm{Im}z \neq 0\} = \emptyset\,. \tag{38}$$

Observe that

$$\partial\Omega_\alpha \cap \{z \in \mathbb{C}: \ \mathrm{Re}z = 0\} = \left\{\pm\frac{1}{2}\left(\sqrt{8-2\alpha} - \sqrt{4-2\alpha}\right) i\right\}$$

and

$$\partial G_\alpha \cap \{z \in \mathbb{C}: \ \mathrm{Re}z = 0\} = \left\{\pm\frac{\sqrt{4-2\alpha}}{8-2\alpha} i\right\}\,.$$

Since $\frac{1}{i} f\left(\frac{1}{2}\left(\sqrt{8-2\alpha} - \sqrt{4-2\alpha}\right) i\right) \geq \frac{\sqrt{4-2\alpha}}{8-2\alpha}$, we have

$$\{f\left(\frac{1}{2}\left(\sqrt{8-2\alpha} - \sqrt{4-2\alpha}\right) i\right): \ f \in T^{(2)}\} \cap G_\alpha = \emptyset\,. \tag{39}$$

The inclusion (36) also holds for $\alpha < 0$. Combining (38), (39) and (36) we can write that for each $f \in T^{(2)}$

$$G_\alpha \subset f(\Omega_\alpha)\,.$$

Hence

$$G_\alpha \subset K_{T^{(2)}}(\Omega_\alpha) \subset \bigcap_{\varepsilon \in [0,1]} f_\varepsilon(\Omega_\alpha)\,.$$

From the univalence of f_ε in Ω_α it follows that $\bigcap_{\varepsilon\in[0,1]} f_\varepsilon(\Omega_\alpha) = G_\alpha$. □

The specific significance of the set $\Omega_{\frac{4}{3}}$ is presented in Theorem 6. We know that the equation $g'_\varepsilon(w) = 0$, where g_ε is defined by (6) and $\varepsilon \in$

$(0, \frac{8}{9})$, has four different solutions. From the univalence of $z \mapsto z+\frac{1}{z}$, while $z \in \Delta$, we conclude that the equation $f'_\varepsilon(z) = 0$, where f_ε is defined by (7) and $\varepsilon \in (0, \frac{8}{9})$, has also four different solutions in Δ: z_ε , $\overline{z_\varepsilon}$, $-z_\varepsilon$, $-\overline{z_\varepsilon}$ (we choose z_ε to satisfy $\mathrm{Re} z_\varepsilon > 0$, $\mathrm{Im} z_\varepsilon > 0$). Moreover, $z_0 = 1$, $z_{8/9} = \frac{\sqrt{3}}{3} i$ are the only solutions of $f'_0(z) = 0$ and $f'_{8/9}(z) = 0$ respectively, in the set $\{z \in \Delta : \mathrm{Re} z_\varepsilon \geq 0 \, , \, \mathrm{Im} z_\varepsilon \geq 0\}$.

THEOREM 6. $\partial K_{T^{(2)}}(\Omega_{\frac{4}{3}}) \cap \{z \in \mathbb{C} : \ \mathrm{Re} z \geq 0 \, , \, \mathrm{Im} z \geq 0\} = \{f_\varepsilon(z_\varepsilon) : \varepsilon \in [0, \frac{8}{9}]\}$.

PROOF. By definition of f_ε and g_ε, $f'_\varepsilon(z) = 0$ if and only if $g'_\varepsilon(z + \frac{1}{z}) = 0$. Let $w = z + \frac{1}{z}$. For $\varepsilon \in [0, \frac{8}{9}]$ we have $g'_\varepsilon(w) = 0$ iff

$$w = \pm\left(\sqrt{(\sqrt{1-\varepsilon}+1)(3\sqrt{1-\varepsilon}-1)} \pm i\sqrt{(1-\sqrt{1-\varepsilon})(1+3\sqrt{1-\varepsilon})}\right).$$

Since $\mathrm{Re} z_\varepsilon \geq 0$, $\mathrm{Im} z_\varepsilon \geq 0$, z_ε satisfies the equation

$$z + \frac{1}{z} = \sqrt{(\sqrt{1-\varepsilon}+1)(3\sqrt{1-\varepsilon}-1)} - i\sqrt{(1-\sqrt{1-\varepsilon})(1+3\sqrt{1-\varepsilon})}\,.$$

From this $f_\varepsilon(z_\varepsilon) =$

$$g_\varepsilon\left(\sqrt{(\sqrt{1-\varepsilon}+1)(3\sqrt{1-\varepsilon}-1)} - i\sqrt{(1-\sqrt{1-\varepsilon})(1+3\sqrt{1-\varepsilon})}\right) =$$
$$\frac{\left(4-3\varepsilon+i\sqrt{\varepsilon(8-9\varepsilon)}\right)\left(\sqrt{2-3\varepsilon+2\sqrt{1-\varepsilon}}+i\sqrt{-2+3\varepsilon+2\sqrt{1-\varepsilon}}\right)}{16\sqrt{1-\varepsilon}}.$$

Substituting $p = 2\sqrt{1-\varepsilon}$ (then $\varepsilon \in [0, \frac{8}{9}]$ iff $p \in [\frac{2}{3}, 2]$) in the above we obtain

$$f_\varepsilon(z_\varepsilon) = \frac{3\sqrt{3}}{32}\left[\sqrt{(2+p)(p-\frac{2}{3})^3} + i\sqrt{(2-p)(p+\frac{2}{3})^3}\right],$$

which completes the proof. □

One can define Ω_α also for $\alpha \in (\frac{4}{3}, 2]$. It is easily seen that if $\alpha_1 < \alpha_2 \leq 2$ then $\Omega_{\alpha_1} \subset \Omega_{\alpha_2}$. Let G_α, $\alpha \in (\frac{4}{3}, 2]$ be defined analogously as for $\alpha \in (0, \frac{4}{3}]$. Certainly, for $\alpha \leq \frac{4}{3}$ we have

$$G_\alpha = K_{T^{(2)}}(\Omega_\alpha) \subset K_{T^{(2)}}(\Omega_{\frac{4}{3}}) = G_{\frac{4}{3}}\,.$$

From Theorem 6 we know for $\alpha \in (\frac{4}{3}, 2]$ that

$$G_\alpha \subset G_{\frac{4}{3}}, G_\alpha \neq G_{\frac{4}{3}},$$

which means

$$G_\alpha \subset K_{T^{(2)}}(\Omega_\alpha), G_\alpha \neq K_{T^{(2)}}(\Omega_\alpha).$$

The above presented argument shows that the set $K_{T^{(2)}}(\Omega_{\frac{4}{3}})$ is the largest subset of $K_{T^{(2)}}(\Delta)$ (the set $K_{T^{(2)}}(\Delta)$ is still unknown) which one can compute applying the method of the envelope.

Список литературы

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