Труды Петрозаводского государственного университета
Серия "Математика" Выпуск 18, 2011

УДК 517

ON CHARACTERIZATIONS OF MAIN PARTS OF SOME MEROMORPHIC CLASSES OF AREA NEVANLINNA TYPE IN THE UNIT DISK

Romi Shamoyan, Haiying Li

We characterize main parts of Loran expansions of certain meromorphic spaces in the unit disk defined with the help of Nevanlinna characteristic.

§ 1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. Let $\mathrm{T} = \{z, |z| = 1\}$ be unit circle. Let $H(\mathbb{D})$ be the space of all functions holomorphic in $\mathbb{D}$, $dm_2(z)$ be a normalized Lebesgue measure in $\mathbb{D}$, $T(\tau, f)$ be Nevanlinna characteristic of $f \in H(\mathbb{D})$ (see [4]). We introduce several area Nevanlinna-type spaces that will be mentioned in this paper. Let further

$$N^p_{\alpha,\beta} = \left\{ f \in H(\mathbb{D}) : \|f\|^p_{N_{\alpha,\beta}} = \int_0^1 \left[\int_{|z|\le R} \ln^+ |f(z)|(1-|z|)^\alpha dm_2(z) \right]^p (1-R)^\beta dR < +\infty \right\},$$

where $\alpha > -1$, $\beta > -1$, $0 < p < \infty$.

$$(NA)_{p,\gamma,v} = \left\{ f \in H(\mathbb{D}) : \|f\|_{(NA)_{p,\gamma,v}} = \int_0^1 \left[\sup_{0<\tau<R} T(\tau, f)(1-\tau)^\gamma \right]^p (1-R)^v dR < +\infty \right\},$$

where $\gamma \geq 0, v > -1, 0 < p < \infty$.

$$N^{\infty,p}_{\alpha,\beta} = \{f \in H(\mathbb{D}) :$$

$$\|f\|_{N^{\infty,p}_{\alpha,\beta}} = \sup_{0 \leq R < 1} \int_0^R \left[\int_{\mathrm{T}} \ln^+ |f(|z|\xi)| d\xi \right]^p (1 - |z|)^\alpha d|z| (1 - R)^\beta < +\infty \Bigg\},$$

where $\alpha > -1, \beta \geq 0, 0 < p < \infty$.

It is not difficult to prove that all mentioned above analytic classes are topological vector spaces with complete invariant metric.

Let $M(\mathbb{D})$ be the space of all meromorphic functions in $\mathbb{D}$ and $X \subset M(\mathbb{D})$ be its subspace. Let also a sequence $\{z_k\}_{k=1}^\infty$ be from a certain class of sequences ($|z_k| < 1$). The problem that we will consider in this paper is the following one. We would like to find sharp conditions on $\{a_{k,n}\}$ and $\{z_k\}$ sequences of complex numbers so that under it there always exists a function $f \in X$ with main parts

$$H(z, z_k, a_k) = \frac{a_{k,n}}{(z - z_k)^n} + \ldots + \frac{a_{k,1}}{(z - z_k)^1}.$$

For meromorphic functions of bounded type such problems were considered and solved previously by A. Naftalevich in [3]. We also remark that the recently such a problem was considered in [1] for some meromorphic $M_\alpha \subset M(\mathbb{D})$ classes. Our intention is to develop these ideas from [1] solving such a problem for mentioned above meromorphic algebras of $N^p_{\alpha,\beta}$ and $N^{\infty,p}_{\alpha,\beta}, (NA)_{p,\gamma,v}$ type.

We define new classes of meromorphic functions in the unit disk. Let further

$$MN^p_{\alpha,\beta} = \{f \in M(\mathbb{D}) :$$

$$\int_0^1 \left(\int_{|z| \leq R} \log^+ |f(z)| (1 - |z|)^\alpha dm_2(z) \right)^p (1 - R)^\beta dR < +\infty \Bigg\},$$

where $\alpha > -1, \beta > -1, 0 < p < \infty$.

$$M(NA)_{p,\gamma,v} = \Bigg\{ f \in M(\mathbb{D}) :$$

$$\int_0^1 \left(\sup_{0 < \tau < R} T(\tau, f)(1 - \tau)^\gamma \right)^p (1 - R)^v dR < +\infty \Bigg\},$$

where $\gamma \geq 0$, $v > -1$, $0 < p < \infty$.

$$MN^{\infty,p}_{\alpha,\beta} = \Bigg\{ f \in M(\mathbb{D}) : \sup_{0 \leq R < 1} \int_0^R \left(\int_{\mathrm{T}} \ln^+ |f(|z|\xi)| d\xi \right)^p (1-|z|)^\alpha d|z| (1-R)^\beta < +\infty \Bigg\},$$

where $\alpha > -1, \beta \geq 0, 0 < p < \infty$.

Let further $B(z)$ be a classical Blaschke product (see [2]). For our exposition we will need two types of special sequences in the unit disk $\mathbb{D}$ sampling sequences and Carleson sequences.

The sampling sequence in $\mathbb{D}$ is a sequence in unit disk $\mathbb{D}$ such that for $\tau \in (0,1], \mathcal{D} = \bigcup_\tau \mathcal{D}(a_k, \tau)$, the sets $\mathcal{D}(a_k, \frac{\tau}{4})$ are mutually disjoint; each point $z \in \mathbb{D}$ belongs to at most N of the sets $\mathcal{D}(a_k, 2\tau)$, where N is independent from $\{a_k\}$ for fixed $\tau \in (0,1]$. Such a sequence in $\mathbb{D}$ exists (see [5]). Let $\{z_k\}_{k=0}^\infty$ be a sequence of complex numbers in $\mathbb{D}$, $\{z_k\}_{k=0}^\infty$ is a Carleson sequence (see [2]) if

$$\inf_{k \geq 1} \prod_{j \neq k}^{\infty} \left| \frac{z_j - z_k}{1 - \overline{z_j} z_k} \right| = \delta > 0.$$

We denote by C in this note all constants which depend only on various parameters like p, q, α.

§ 2. Main results

The goal of this section is to characterize main parts of Loran expansions of certain meromorphic spaces in the unit disk defined with the help of Nevanlinna characteristic.

The following theorem provides a solution of mentioned problem of description of main parts of Loran expansions of some classes of meromorphic functions on the unit disk $\mathbb{D}$ based on Nevanlinna characteristic. First we will consider the general situation.

Theorem 1. *Let $\{z_k\}_{k=1}^\infty$ be a sequence of unit disk $\mathbb{D}$. Let $X \subset M(\mathbb{D})$ be a class of meromorphic functions in the unit disk. We assume that $HX = X \bigcap H(\mathbb{D})$ is closed under the operation of differentiation (if $f \in HX$, then $f' \in HX$). Let $f \in X, f(z)(B(z))^n \in HX, n = 1, 2, \ldots$. Let*

$$\sum_{k=0}^{+\infty} (1 - |z_k|^2)^\alpha \ln^+ |f(z_k)| < +\infty \tag{1}$$

for some $\alpha > 0$, *for every* $f \in HX$. *Let also the* S *map,* S *is acting from* HX *to the class of all* $\{w_k\}_{k=1}^{\infty}$ *sequences such that*

$$\sum_{k=0}^{+\infty}(1-|z_k|^2)^{\alpha}\ln^+|w_k| < +\infty \tag{2}$$

be onto map, this is for every sequence of complex numbers with condition (2), there is a function $f, f \in HX$ *so that* $f(z_k) = w_k$. *Then for every expression of type*

$$H(z, z_k, a_k) = \frac{a_{k,n}}{(z-z_k)^n} + \ldots + \frac{a_{k,1}}{z-z_k}, k = 1, 2, \ldots$$

there is a function $G(z) \in X$ *with* $H(z, z_k, a_k)$ *as the main parts of it is Loran expansion if and only if*

$$\sum_{k=1}^{+\infty}(1-|z_k|^2)^{\alpha}\ln^+\frac{|a_{k,i}||B_k(z_k)|^n}{(1-|z_k|^2)^n} < +\infty,\ i = 1, 2, \ldots, n, \tag{3}$$

where $z_k \in \mathbb{D} = \{|z| < 1\}, \{a_{k,n}\} \in \mathbb{C}$, $B_k(z)$ *is an ordinary Blaschke product but without* k *factor*

$$B_k(z) = \prod_{j=1, z_j \neq z_k}^{\infty} \frac{z_j - z}{1-\overline{z_j}z} \cdot \frac{\overline{z_j}}{|z_j|}, k \geq 1.$$

In our exposition below as X we will take various concrete spaces of meromorphic functions in the unit disk for which (1) and (2) (but for some fixed z_k) can be checked directly.

THEOREM 2. *Let* $\{z_k\}_{k=1}^{\infty}$ *is a Carleson sequence in* $\mathbb{D}$. *Then for every expression of type*

$$H(z, z_k, a_k) = \frac{a_{k,n}}{(z-z_k)^n} + \ldots + \frac{a_{k,1}}{z-z_k}, k = 1, 2, \ldots$$

there is a function $G \in MN^1_{\alpha,\beta}, \alpha > -1, \beta > -1$ *with* $H(z, z_k, a_k)$ *as the main parts of it is Loran expansion if and only if*

$$\sum_{k=1}^{+\infty}(1-|z_k|)^{\alpha+\beta+3}\ln^+\frac{|a_{k,i}||B_k(z_k)|^n}{(1-|z_k|^2)^n} < +\infty,\ i = 1, 2, \ldots, n,$$

where $\{a_{k,n}\} \in \mathbb{C}$.

THEOREM 3. *Let* $\{z_k\}_{k=1}^{\infty}$ *is a Carleson sequence in* $\mathbb{D}$. *Then for every expression of type*

$$H(z, z_k, a_k) = \frac{a_{k,n}}{(z - z_k)^n} + \ldots + \frac{a_{k,1}}{z - z_k}, k = 1, 2, \ldots$$

there is a function $f \in M(NA)_{1,\gamma,v}, \gamma > 0, v > -1$ *with* $H(z, z_k, a_k)$ *as the main parts of it is Loran expansion if and only if*

$$\sum_{k=1}^{+\infty} (1 - |z_k|)^{\gamma+v+2} \ln^+ \frac{|a_{k,i}||B_k(z_k)|^n}{(1 - |z_k|^2)^n} < +\infty,\ i = 1, 2, \ldots, n,$$

where $\{a_{k,n}\} \in \mathbb{C}$.

THEOREM 4. *Let* $\{z_k\}_{k=1}^{\infty}$ *is a Carleson sequence in* $\mathbb{D}$. *Then for every expression of type*

$$H(z, z_k, a_k) = \frac{a_{k,n}}{(z - z_k)^n} + \ldots + \frac{a_{k,1}}{z - z_k}, k = 1, 2, \ldots$$

there is a function $G \in MN_{\alpha,\beta}^{\infty,1}, \alpha > -1, \beta > 0$ *with* $H(z, z_k, a_k)$ *as the main parts of it is Loran expansion if and only if*

$$\sum_{k=1}^{+\infty} (1 - |z_k|)^{\alpha+\beta+2} \ln^+ \frac{|a_{k,i}||B_k(z_k)|^n}{(1 - |z_k|^2)^n} < +\infty,\ i = 1, 2, \ldots, n,$$

where $\{a_{k,n}\} \in \mathbb{C}$.

The proofs of mentioned assertions will be based on following propositions. These assertions as separate statements are interesting from our point of view as separate propositions.

LEMMA 1. *Let* $\{z_k\}_{k=1}^{\infty}$ *be a sampling sequence. Then*

$$\sum_{k=1}^{\infty} (1 - |z_k|)^{\tau+2} \ln^+ |f(z_k)| \le C\|f\|_{N_{\alpha,\beta}^p},$$

where $p \ge 1, \tau = (1 + \alpha) + \beta, \tau > 0, \alpha > -1, \beta > -1$.

$$\sum_{k=1}^{\infty} (1 - |z_k|)^{\tau} \ln^+ |f(z_k)| \le C\|f\|_{(NA)_{p,\gamma,v}},$$

where $0 < p < \infty, v = \tau p - \gamma p - 1, v > -1, \gamma \geq 0, \tau > 0$.

LEMMA 2. *Let* $\beta > -1, \gamma > -1, 0 < q < \infty$. *Let* $f \in H(\mathbb{D}), \widetilde{f} = \log^+ |f(w)|$. *Then*

$$\int\limits_0^1 (1-\tau)^\beta \Bigg(\int\limits_{|z|\leq\tau} \widetilde{f}(z)(1-|z|)^\gamma dm_2(z) \Bigg)^q d\tau \leq$$

$$\leq C \int\limits_0^1 (1-\tau)^{\beta+q(\gamma+1)} \Bigg(\int\limits_T \widetilde{f}(\tau\xi) d\xi \Bigg)^q d\tau.$$

LEMMA 3. *1) Let* $\{z_k\}_{k=1}^\infty$ *be a sampling sequence. Then for every sequence* $\{w_k\}_{k=1}^\infty$,

$$\sum_{k=1}^\infty (1-|z_k|)^\tau \ln^+ |w_k| < \infty,$$

there is a function $f(z)$ *so that* $\|f\|_{(NA)_{1,\gamma,v}} < \infty$ *and so that* $f(z_k) = w_k, k = 1, 2, \ldots, \tau = \gamma + v + 2, \gamma > 0, v > -1$.

2) Let $\{z_k\}_{k=1}^\infty$ *be a sampling sequence. Then for every sequence* $\{w_k\}_{k=1}^\infty$ *so that*

$$\sum_{k=1}^\infty (1-|z_k|)^\tau \ln^+ |w_k| < \infty,$$

there is a function $f(z)$ *so that* $\|f\|_{N_{\alpha,\beta}^{\infty,1}} < \infty$ *and so that* $f(z_k) = w_k, k = 1, 2, \ldots, \tau = \alpha + \beta + 2, \alpha > -1, \beta > 0$.

Complete proofs of provided assertions will be given by authors elsewhere.

Acknowledge. We thank Professor F. Shamoyan for various discussions concerning results of this note.

Список литературы

[1] Bednag V., Shamoyan F. *Descriptions of main parts of Loran expansion of some classes of meromorphic functions in the unit disk* // Vestnik BGU. 2004. V. 4. P. 84–92. (in Russian).

[2] Garnett John B. *Bounded Analytic Functions*. New York: Springer, 2006.

[3] Naftalevich A. G. *On interpolation of functions of bounded type* // Uchenie Zapiski Vilnus University. 1956. V. 18. P. 19–27. (in Russian).

[4] Nevanlinna R. *Analytic functions*. Moscow: GITTL, 1991.

[5] Zhu K. *Spaces of holomorphic functions in the unit ball*. New York: Springer-Verlag, 2005.

Bryansk State University,
Department of Mathematics,
241050, Bryansk, Russia
E-mail: rsham@mail.ru, shamoyan@tu-bryansk.ru

Henan Normal University,
College of Mathematics and Information Science,
453007, Xinxiang, P. R. China
E-mail: tslhy2001@yahoo.com.cn