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NONUNIFORM SUPER WAVELETS IN $L^2(\mathbb{K})$

Abstract. In this paper, we introduce the structure of nonuniform super wavelets over local fields. We shall also provide the characterization of nonuniform parseval frame, nonuniform semi-orthogonal pareseval multiwavelets, and nonuniform super wavelets over local fields.

Key words: *nonuniform super wavelet, Fourier transform, Local field, Parseval frame*

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1. Introduction. In the framework of mathematical analysis and linear algebra, redundant representations are obtained by analysing vectors with respect to an overcomplete system. Then the obtained vectors are interpreted using the frame theory as introduced by Duffin and Schaeffer [11] and recently studied at depth, see [8] and the compressive list of references therein. Most commonly used coherent/structured frames are wavelet, Gabor, and wave-packet frames, which are a mixture type of wavelet and Gabor frames [8], [14]. Frames provide a useful model to obtain signal decompositions in cases where redundancy, robustness, over-sampling, and irregular sampling play a role.

The concept of multiresolution is intuitively related to the study of signals or images at different levels of resolution — almost like a pyramid. The resolution of a signal is a qualitative description associated with its frequency content. For a low-pass signal, the lower its frequency content (the narrower the bandwidth), the coarser is its resolution. In signal processing, a low-pass and subsampled version of a signal is usually a good coarse approximation for many real world signals. Multiresolution is especially evident in image processing and computer vision, where coarse versions of an image are often used as a first approximation in computational algorithms. For images and, indeed, for all signals, the simultaneous existence of a multiscale may also be referred to as multiresolution. From the point of view of practical application, MRA is really an effective mathematical framework for hierarchical decomposition of an image (or signal) into components of different scales (or frequencies). Signals are in general non-stationary and a complete representation of these signals requires frequency analysis that is local in time, resulting in the time-frequency analysis of signals. In real-life application all signals are not obtained from uniform shifts; so, there is a natural question regarding analysis and decompositions of these types of signals by a stable mathematical tool. Gabardo and Nashed [15] filled this gap by the concept of nonuniform multiresolution analysis and nonuniform wavelets based on the theory of spectral pairs, for which the associated translation set $\Lambda = \{0, r/N\} + 2\,\mathbb{Z}$ is no longer a discrete subgroup of $\mathbb{R}$ but a spectrum associated with a certain one-dimensional spectral pair and the associated dilation is an even positive integer related to the given spectral pair. Super-wavelets and decomposable frame wavelets for the Euclidean spaces has been rigorously studied by various authors [10], [13], [16], [22]. Multiresolution analysis has tremendous application in data learning. Data preprocessing is an important step in learning forecasting models. It plays a significant role in determining the most relevant features and models. Multiresolution analysis is a data preprocessing step used to decompose time-series data on different scales to model the data according to several variations of representation. Multiple representations of data are generated depending on the scaling parameters. Using multiple data representations enables more information to be captured and can thus produce better forecasting results. The proposed methodology uses multiresolution analysis for data decomposition. However, finding the best configuration of parameters, that gives the highest possible performance, relies on conducting several experiments.

On the other hand, there is a substantial body of work that has been concerned with the construction of wavelets on local fields. For example, R. L. Benedetto and J. J. Benedetto [7] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. Recently, Shah and

Abdullah [19] have generalized the concept of multiresolution analysis on Euclidean spaces $\mathbb{R}^n$ to nonuniform multiresolution analysis on local fields of positive characteristic, in which the translation set acting on the scaling function associated with the multiresolution analysis to generate the subspace V_0 is no longer a group, but is the union of $\mathcal{Z}$ and a translate of $\mathcal{Z}$, where $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$ is a complete list of (distinct) coset representation of the unit disc $\mathfrak{D}$ in the locally compact Abelian group $\mathbb{K}^+$. More results in the direction of wavelets, frames and their applications can also be found in [1–5], [17], [18], [20] and the references therein.

Drawing inspiration from the above work, we introduce the structure of nonuniform super wavelets on local fields. The characterization of nonuniform parseval frame, nonuniform semi-orthogonal pareseval multiwavelets and nonuniform super wavelets over local fields are established.

The remaining paper is structures as follows. In section 2, preliminaries on local fields are discussed and various operators along with their properties are discussed. In section 3, characterization of nonuniform Parseval frame and nonuniform semi-orthogonal Parseval frame multiwavelets are established. In Section 4, we introduce the notion of nonuniform super wavelets on local fields and obtain their complete characterization in $L^2(\mathbb{K})$.

2. Preliminaries on Local Fields.

2.1. Local Fields.

A local field $\mathbb{K}$ is a locally compact, non-discrete, and totally disconnected field. If it is of characteristic zero, then it is a field of p-adic numbers $\mathbb{Q}_p$ or its finite extension. If $\mathbb{K}$ is of positive characteristic, then $\mathbb{K}$ is a field of formal Laurent series over a finite field $GF(p^c)$. If $c = 1$, it is a p-series field, while for $c \neq 1$, it is an algebraic extension of degree c of a p-series field. Let $\mathbb{K}$ be a fixed local field with the ring of integers $\mathfrak{D} = \{x \in \mathbb{K} : |x| \leqslant 1\}$. Since $\mathbb{K}^+$ is a locally compact Abelian group, we choose a Haar measure dx for $\mathbb{K}^+$. The field $\mathbb{K}$ is locally compact, nontrivial, totally disconnected and complete topological field endowed with non–Archimedean norm $|\cdot| : \mathbb{K} \to \mathbb{R}^+$ satisfying

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|x\,y| = |x||y|$ for all $x, y \in \mathbb{K}$;

(c) $|x + y| \leqslant \max\{|x|, |y|\}$ for all $x, y \in \mathbb{K}$.

Property (c) is called the ultrametric inequality. Let $\mathfrak{B} = \{x \in \mathbb{K} : |x| < 1\}$

be the prime ideal of the ring of integers $\mathfrak{D}$ in $\mathbb{K}$. Then, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. Since $\mathbb{K}$ is totally disconnected and $\mathfrak{B}$ is both prime and principal ideal, so there exists a prime element $\mathfrak{p}$ of $\mathbb{K}$, such that $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in \mathbb{K} \colon |x| = 1\}$. Clearly, $\mathfrak{D}^*$ is a group of units in $\mathbb{K}^*$ and if $x \neq 0$, then we can write $x = \mathfrak{p}^n y, y \in \mathfrak{D}^*$. Moreover, if $\mathcal{U} = \{a_m \colon m = 0, 1, \ldots, q-1\}$ denotes the fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in \mathbb{K}$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \, \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Recall that $\mathfrak{B}$ is compact and open, so each fractional ideal $\mathfrak{B}^k = \mathfrak{p}^k \mathfrak{D} = \left\{x \in \mathbb{K} \colon |x| < q^{-k}\right\}$ is also compact and open and is a subgroup of K^+. We use the notation in Taibleson's book [21]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let χ be a fixed character on $\mathbb{K}^+$ that is trivial on $\mathfrak{D}$ but non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$, so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(y, x)$, $x \in \mathbb{K}$. Suppose that χ_u is any character on $\mathbb{K}^+$; then the restriction $\chi_u|\mathfrak{D}$ is a character on $\mathfrak{D}$. Moreover, as characters on $\mathfrak{D}, \chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. Hence, if $\{u(n) \colon n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in $\mathbb{K}^+$, then, as it has been proved in [21], the set $\left\{\chi_{u(n)} \colon n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$, where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that the span $\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \leqslant n < q, \; n = a_0 + a_1 p + \ldots + a_{c-1} p^{c-1}, \; 0 \leqslant a_k < p, \text{ and } k = 0, 1, \ldots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \ldots + a_{c-1}\zeta_{c-1})\, \mathfrak{p}^{-1}.$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \ldots + b_s q^s$, $n \in \mathbb{N}_0$, $0 \leqslant b_k < q$, $k = 0, 1, 2, \ldots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \ldots + u(b_s)\mathfrak{p}^{-s}.$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. However, if r, $k \in \mathbb{N}_0$ and $0 \leqslant s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) \colon k \in \mathbb{N}_0\} = \{u(k) \colon k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter, we use the notation $\chi_n = \chi_{u(n)}$, $n \geqslant 0$.

Let the local field K be of characteristic $p>0$ and $\zeta_0,\zeta_1,\zeta_2,\ldots,\zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu=0 \text{ and } j=1, \\ 1, & \mu=1,\ldots,c-1 \text{ or } j\neq 1. \end{cases}$$

2.2. Fourier Transforms on Local Fields.

The Fourier transform of $f\in L^1(K)$ is denoted by $\hat{f}(\xi)$ and defined by

$$\mathcal{F}\{f(x)\} = \hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\,dx.$$

Note that

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)\,dx.$$

The properties of Fourier transforms on local field K are much similar to those of on the classical field $\mathbb{R}$. In fact, the Fourier transform on local fields of positive characteristic have the following properties:

- The map $f\to\hat{f}$ is a bounded linear transformation of $L^1(K)$ into $L^\infty(K)$, and $\|\hat{f}\|_\infty \leqslant \|f\|_1$.
- If $f\in L^1(K)$, then $\hat{f}$ is uniformly continuous.
- If $f\in L^1(K)\cap L^2(\mathbb{K})$, then $\|\hat{f}\|_2 = \|f\|_2$.

The Fourier transform of a function $f\in L^2(\mathbb{K})$ is defined by

$$\hat{f}(\xi) = \lim_{k\to\infty}\hat{f}_k(\xi) = \lim_{k\to\infty}\int_{|x|\leqslant q^k} f(x)\overline{\chi_\xi(x)}\,dx,$$

where $f_k = f\,\Phi_{-k}$ and Φ_k is the characteristic function of $\mathfrak{B}^k$. Furthermore, if $f\in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\hat{f}\big(u(n)\big) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}\,dx.$$

The series $\sum_{n\in\mathbb{N}_0}\hat{f}\big(u(n)\big)\chi_{u(n)}(x)$ is called the Fourier series of f. From the standard L^2-theory for compact Abelian groups, we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\|f\|_2^2 = \int_{\mathfrak{D}} \big|f(x)\big|^2 dx = \sum_{n\in\mathbb{N}_0}\Big|\hat{f}\big(u(n)\big)\Big|^2.$$

2.3. Nonuniform MRA on Local Fields.

For an integer $N \geqslant 1$ and an odd integer r with $1 \leqslant r \leqslant qN-1$, such that r and N are relatively prime, we define

$$\Lambda = \left\{0, \frac{u(r)}{N}\right\} + \mathcal{Z},$$

where $\mathcal{Z} = \{u(n) \colon n \in \mathbb{N}_0\}$. It is easy to verify that Λ is not a group on local field K, but is the union of $\mathcal{Z}$ and a translate of $\mathcal{Z}$. Following is the definition of nonuniform multiresolution analysis (NUMRA) on local fields of positive characteristic given by Shah and Abdullah [19].

Definition 1. *For an integer $N \geqslant 1$ and an odd integer r with $1 \leqslant r \leqslant qN-1$, such that r and N are relatively prime, an associated NUMRA on local field K of positive characteristic is a sequence of closed subspaces $\{V_j \colon j \in \mathbb{Z}\}$ of $L^2(\mathbb{K})$, such that the following properties hold:*

- *(a)* $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$;
- *(b)* $\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{K})$;
- *(c)* $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
- *(d)* $f(\cdot) \in V_j$ *if and only if* $f(\mathfrak{p}^{-1}N\cdot) \in V_{j+1}$ *for all* $j \in \mathbb{Z}$;
- *(e) There exists a function* ϕ *in* V_0, *such that* $\{\phi(\cdot - \lambda) \colon \lambda \in \Lambda\}$ *is a complete orthonormal basis for* V_0.

If we replace the term "orthonormal basis" by "nonuniform Parseval frame" in the last axiom, then the concept above is known as *nonuniform Parseval frame MRA*.

It is worth noticing that, when $N = 1$, one recovers from the definition above the definition of an MRA on local fields of positive characteristic $p > 0$. When, $N > 1$, the dilation is induced by $\mathfrak{p}^{-1}N$ and $|\mathfrak{p}^{-1}| = q$ ensures that $qN\Lambda \subset \mathcal{Z} \subset \Lambda$.

For every $j \in \mathbb{Z}$, define W_j to be the orthogonal complement of V_j in V_{j+1}. Then we have

$$V_{j+1} = V_j \oplus W_j \quad \text{and} \quad W_\ell \perp W_{\ell'} \quad \text{if } \ell \neq \ell'.$$

It follows that for $j > J$,

$$V_j = V_J \oplus \bigoplus_{\ell=0}^{j-J-1} W_{j-\ell}\,,$$

where all these subspaces are orthogonal. By virtue of condition (b) in the Definition 1, this implies

$$L^2(\mathbb{K}) = \bigoplus_{j\in\mathbb{Z}} W_j$$

a decomposition of $L^2(\mathbb{K})$ into mutually orthogonal subspaces.

As in the standard scheme, one expects the existence of $qN-1$ number of functions so that their translation by elements of Λ and dilations by the integral powers of $\mathfrak{p}^{-1}N$ form an orthonormal basis for $L^2(\mathbb{K})$.

Let a and b be any two fixed elements in K. Then, for any prime $\mathfrak{p}$ and $m, n \in \mathbb{N}_0$, let $D_\mathfrak{p}$, $T_{u(n)a}$ and $E_{u(m)b}$ be the unitary operators acting on $f \in L^2(\mathbb{K})$ defined by:

$$\begin{aligned} T_{u(n)a}f(x) &= f\big(x - u(n)a\big) && \text{(Translation)},\\ E_{u(m)b}f(x) &= \chi\big(u(m)bx\big)f(x) && \text{(Modulation)},\\ D_\mathfrak{p}f(x) &= \sqrt{qN}f\left(\mathfrak{p}^{-1}Nx\right) && \text{(Dilation)}. \end{aligned}$$

Then for any $f \in L^2(\mathbb{K})$, the following results can easily be verified:

$$\begin{aligned} \mathcal{F}\big\{T_{u(n)a}f(x)\big\} &= E_{-u(n)a}\mathcal{F}\big\{f(x)\big\},\\ \mathcal{F}\big\{E_{u(m)b}f(x)\big\} &= T_{u(m)b}\mathcal{F}\big\{f(x)\big\},\\ \mathcal{F}\big\{D_{\mathfrak{p}^j}f(x)\big\} &= D_{\mathfrak{p}^{-j}}\mathcal{F}\big\{f(x)\big\},\\ D_{\mathfrak{p}^j}T_{u(n)a} &= T_{(qN)^{-j}u(n)a}D_{\mathfrak{p}^j}. \end{aligned}$$

3. Nonuniform Parseval frame multiwavelets sets in $L^2(\mathbb{K})$.

For a given $\Psi = \{\psi_\ell \colon 1 \leqslant \ell \leqslant qN-1\} \subset L^2(\mathbb{K})$, define the nonuniform wavelet system

$$\mathcal{W}(\Psi) = \Big\{\psi_{\ell,j,\lambda} =\colon (qN)^{j/2}\psi_\ell\big((\mathfrak{p}^{-1}N)^j x - \lambda\big);\ j\in\mathbb{Z}, \lambda\in\Lambda\Big\}. \qquad (1)$$

Taking the Fourier transform, the system 1 can be rewritten as

$$\hat{\psi}_{\ell,j,\lambda}(\xi) = (qN)^{-j/2}\hat{\psi}_\ell\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)\overline{\chi_\lambda\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)}. \qquad (2)$$

We call the nonuniform wavelet system $\mathcal{W}(\Psi)$ a nonuniform Parseval frame multiwavelet frame for $L^2(\mathbb{K})$ if the system given by (1) forms a nonuniform Parseval frame for $L^2(\mathbb{K})$, i. e., for every $f \in L^2(\mathbb{K})$,

$$\|f\|^2 = \sum_{\ell=1}^{qN-1} \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \left|\langle f, D_{\mathfrak{p}^j} T_\lambda \psi_\ell\rangle\right|^2 .$$

If the system $\mathcal{W}(\Psi)$ given by (1) is an orthonormal basis for $L^2(\mathbb{K})$, then Ψ is called an *nonuniform orthonormal multiwavelet* of order $qN-1$ in $L^2(\mathbb{K})$. If the system $\mathcal{W}(\Psi)$ given by (1) is a Parseval frame for $L^2(\mathbb{K})$, then Ψ is known as *nonuniform Parseval frame wavelet*. Moreover, a nonuniform Parseval frame multiwavelet Ψ is said to be *semi-orthogonal* if $D_{\mathfrak{p}^j}\Gamma \perp D_{\mathfrak{p}^{j'}}\Gamma$, for $j \neq j'$, where $\Gamma = \overline{\operatorname{span}}\{T_\lambda \psi \colon \lambda\in\Lambda, \psi\in\Psi\}$.

The following is a necessary and sufficient condition for the system $\mathcal{W}(\Psi)$ given by (1) to be a nonuniform Parseval frame for $L^2(\mathbb{K})$ [4]:

Theorem 1. *Suppose $\Psi = \{\psi_\ell \colon 1 \leqslant \ell \leqslant qN-1\} \subset L^2(\mathbb{K})$. Then the nonuniform affine system $\mathcal{W}(\Psi)$ is a nonuniform Parseval frame for $L^2(\mathbb{K})$ if and only if the following holds:*

$$\sum_{\ell=1}^{qN-1} \sum_{j\in\mathbb{Z}} \left|\widehat{\psi}_\ell\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)\right|^2 = 1, \tag{3}$$

$$\sum_{\ell=1}^{qN-1} \sum_{j\in\mathbb{N}_0} \widehat{\psi}_\ell\left((\mathfrak{p}^{-1}N)^{-j}\xi\right) \overline{\widehat{\psi}_\ell\left((\mathfrak{p}^{-1}N)^{-j}(\xi+\lambda\right)} = 0, \text{ for } \lambda\in\Lambda\backslash qN\Lambda. \tag{4}$$

In particular, Ψ is a nonuniform multiwavelet in $L^2(\mathbb{K})$ if and only if $\|\psi_\ell\| = 1$, for $1 \leqslant \ell \leqslant qN-1$, and the above conditions (3) and (4) hold.

The following is a Characterization of nonuniform Parseval frame:

Theorem 2. *Let $\varphi \in L^2(\mathbb{K})$. Then a necessary and sufficient condition for the system $\{\varphi(\cdot-\lambda)\colon \lambda\in\Lambda\}$ to be a nonuniform Parseval frame for $\overline{\operatorname{span}}\{\varphi(\cdot-\lambda)\colon \lambda\in\Lambda\}$ is as follows:*

$$0 \leqslant \sum_{\lambda\in\Lambda} |\widehat{\varphi}(\xi+\lambda)|^2 \leqslant 1, \quad a.e.\ \xi.$$

Proof. For every $f \in \overline{\operatorname{span}}\{\varphi(\cdot-\lambda)\colon \lambda\in\Lambda\} =: V_\varphi$, we have $\widehat{f}(\xi) = \gamma(\xi)\widehat{\varphi}(\xi)$, for some integral periodic function $\gamma \in L^2(\mathfrak{D}, w)$, where $w(\xi) = \sum_{\lambda\in\Lambda} |\widehat{\varphi}(\xi+\lambda)|^2$, and, hence,

$$\sum_{\lambda\in\lambda} |\langle f, T_\lambda\varphi\rangle|^2 = \sum_{\lambda\in\Lambda} \left| \int_{\mathbb{K}} \widehat{f}(\xi)\overline{\widehat{\varphi}(\xi)}\chi_\lambda(\xi)d\xi\right|^2 =$$

$$= \sum_{\lambda\in\Lambda} \Big| \sum_{\ell\in\mathbb{N}_0} \int_{\mathfrak{D}} \widehat{f}(\xi+u(\ell))\overline{\widehat{\varphi}(\xi+u(\ell))}\chi_\lambda(\xi+\ell)d\xi \Big|^2 =$$

$$= \sum_{\lambda\in\Lambda} \Big| \int_{\mathfrak{D}} \Big(\sum_{\ell\in\mathbb{N}_0} \gamma(\xi+u(\ell))|\widehat{\varphi}(\xi+u(\ell))|^2 \Big) \chi_\lambda(\xi)d\xi \Big|^2.$$

Since the system $\{\mathfrak{D}+\lambda\colon \lambda\in\Lambda\}$ is a measurable partition of $\mathbb{K}$, and for all $\lambda\in\Lambda$, $\ell\in\mathbb{N}_0$, $\chi_\lambda(u(\ell))=1$. Using the periodicity of the function γ, the above expression can be written as:

$$\sum_{\lambda\in\Lambda} |\langle f, T_\lambda\varphi\rangle|^2 = \sum_{\lambda\in\mathbb{N}_0} \Big| \int_{\mathfrak{D}} \gamma(\xi)w(\xi)\chi_\lambda(\xi)d\xi \Big|^2 = \int_{\mathfrak{D}} |\gamma(\xi)|^2|w(\xi)|^2 d\xi,$$

Therefore, it follows that

$$\int_{\mathfrak{D}} |\gamma(\xi)|^2|w(\xi)|d\xi = \int_{\mathfrak{D}} |\gamma(\xi)|^2|w(\xi)|^2 d\xi.$$

For every $f\in V_\varphi$, we have

$$\|f\|^2 = \int_{\mathfrak{D}} |\gamma(\xi)|^2|w(\xi)|d\xi.$$

This means that

$$\int_{\mathfrak{D}} |\gamma(\xi)|^2 w(\xi) \left(\chi_\Omega(\xi) - w(\xi)\right) d\xi = 0,$$

holds for all integral periodic functions $\gamma\in L^2(\mathfrak{D},w)$ if and only if $w(\xi)=\chi_\Omega(\xi)$, a. e. ξ, where

$$\Omega = \operatorname{supp} w \equiv \{\xi\in\mathbb{K}\colon w(\xi)\neq 0\}.$$

Now, it is enough to show that $f\in V_\varphi$ if and only if

$$\widehat{f}(\xi) = \gamma(\xi)\widehat{\varphi}(\xi),$$

for some integral periodic function $\gamma\in L^2(\mathfrak{D},w)$. This clearly follows by noting that $V_\varphi=\overline{\mathcal{S}_\varphi}$, $L^2(\mathfrak{D},w)=\overline{\mathcal{P}_\varphi}$ and the operator $U\colon \mathcal{S}_\varphi\to\mathcal{P}_\varphi$ defined by $U(f)(\xi)=\gamma(\xi)$ is an isometry that is onto, where

$$\mathcal{S}_\varphi = \operatorname{span}\{T_\lambda\varphi\colon \lambda\in\Lambda\},$$

and $\mathcal{P}_\varphi$ is the space of all integral periodic trigonometric polynomials γ with the $L^2(\mathfrak{D}, w)$ norm

$$\|\gamma\|^2 = \int_{\mathfrak{D}} |\gamma(\xi)|^2 w(\xi) d\xi.$$

Here, $f \in \mathcal{S}_\varphi$ if and only if for $\gamma \in \mathcal{P}_\varphi$, $\widehat{f}(\xi) = \gamma(\xi)\widehat{\varphi}(\xi)$, where

$$\gamma(\xi) = \sum_{\lambda \in \Lambda} a_\lambda \overline{\chi_\lambda(\xi)},$$

for a finite number of non-zero elements of $\{a_\lambda\}_{\lambda \in \Lambda}$. Using the periodicity of γ, and decompose the integral into cosets of $\mathfrak{D}$ in $\mathbb{K}$, we obtain

$$\|f\|_2^2 = \int_{\mathfrak{D}} \sum_{\lambda \in \Lambda} \left| \widehat{f}(\xi + \lambda) \right|^2 d\xi = \int_{\mathfrak{D}} |\gamma(\xi)|^2 \sum_{\lambda \in \Lambda} |\widehat{\varphi}(\xi + \lambda)|^2 d\xi = \|\gamma\|^2,$$

which clearly shows that the operator U is an isometry. This completes the proof of the theorem. □

Now the following theorem gives a characterization of nonuniform Semi-orthogonal Parseval frame multiwavelets in $L^2(\mathbb{K})$:

Theorem 3. *Let $\Psi = \{\psi_\ell\}_{\ell=1}^{qN-1} \subset L^2(\mathbb{K})$ be such that for each $1 \leqslant \ell \leqslant qN - 1$, $|\widehat{\psi_\ell}| = \chi_{\Gamma_\ell}$, and $\Gamma = \bigcup_{\ell=1}^{qN-1} \Gamma_\ell$ is a disjoint union of measurable subsets of $\mathbb{K}$. Then Ψ is a nonuniform semi-orthogonal Parseval frame multiwavelet in $L^2(\mathbb{K})$ if and only if the following conditions hold:*

(i) $\{(\mathfrak{p}^{-1}N)^{-j}\Gamma \colon j \in \mathbb{Z}\}$ is a measurable partition of $\mathbb{K}$, and

(ii) for each $1 \leqslant \ell \leqslant qN - 1$, the set $\{\Gamma_\ell + \lambda \colon \lambda \in \Lambda\}$ is a measurable partition of a subset of $\mathbb{K}$.

Proof. Let $\Psi = \{\psi_\ell\}_{\ell=1}^{qN-1} \subset L^2(\mathbb{K})$ be such that $|\widehat{\psi_\ell}| = \chi_{\Gamma_\ell}$, where $\Gamma = \bigcup_{\ell=1}^{qN-1} \Gamma_\ell$ is a measurable subset of $\mathbb{K}$. By condition (3) of Theorem 1, it follows that $\bigcup_{j \in \mathbb{Z}} (\mathfrak{p}^{-1}N)^j \Gamma = \mathbb{K}$, a. e.; that is, equivalent to the part (i), which also gives that for $j \geqslant 0$, $|(\mathfrak{p}^{-1}N)^{-j}\Gamma_\ell \cap \Gamma_{\ell'}| = 0$ for each $\ell, \ell' \in \{1, 2, \ldots, qN - 1\}$, and $\ell \neq \ell'$. By virtue of Theorem 2, we can say that the system

$$\{\psi_\ell(\cdot - \lambda) \colon \lambda \in \Lambda\}, \ \ell \in \{1, 2, \ldots, qN - 1\}$$

is a nonuniform Parseval frame for $\overline{\operatorname{span}}\,\{\psi_\ell(\cdot-\lambda)\colon \lambda\in\Lambda\}$ in $L^2(\mathbb{K})$ if and only if

$$\sum_{\lambda\in\Lambda}\left|\widehat{\psi}_\ell(\xi+\lambda)\right|^2=\sum_{\lambda\in\Lambda}\chi_{\Gamma_\ell}(\xi+\lambda)\leqslant 1,\ \text{a. e. } \xi,$$

that is equivalent to the part (ii). In this case

$$\{f\in L^2(\mathbb{K})\colon \operatorname{supp}\widehat{f}\subset\Gamma\}=\overline{\operatorname{span}}\{\psi(\cdot-\lambda)\colon \psi\in\Psi,\lambda\in\Lambda\}=:\Gamma_0.$$

By scaling Γ_0 for any $j\in\mathbb{Z}$, we have

$$\begin{aligned}D_{\mathfrak{p}^j}\Gamma_0=\overline{\operatorname{span}}\{D^j\psi(\cdot-\lambda)\colon \psi\in\Psi,\lambda\in\Lambda\}=&\\ =\{f\in L^2(\mathbb{K})\colon \operatorname{supp}\widehat{f}\subset(\mathfrak{p}^{-1}N)^{-j}\Gamma\}.&\end{aligned}$$

Therefore, Ψ is a nonuniform semi-orthogonal Parseval frame multiwavelet in $L^2(\mathbb{K})$ if and only if $\bigoplus_{j\in\mathbb{Z}}D_{\mathfrak{p}^j}\Gamma_0=L^2(\mathbb{K})$ and (ii) hold, which is true if and only if (i) and (ii) hold. $\square$

4. Characterization of Nonuniform Super-wavelet of length n on Local Fields. The following definition of nonuniform super-wavelets on local fields is an analogue of the Euclidean case:

Definition 2. *Suppose that $\Phi=(\phi_1,\phi_2,\dots,\phi_n)$, where ϕ_i is a nonuniform Parseval frame wavelet for $L^2(\mathbb{K})$ for each $i\in\{1,2,\dots,n\}$. We call the n-tuple Φ a nonuniform super-wavelet of length n if*

$$\mathcal{F}(\Phi):=\left\{\bigoplus_{i=1}^{n}D_{\mathfrak{p}^j}T_\lambda\phi_i\equiv D_{\mathfrak{p}^j}T_\lambda\phi_1\oplus\ldots\oplus D_{\mathfrak{p}^j}T_\lambda\phi_n\colon j\in\mathbb{Z},\lambda\in\Lambda\right\}$$

is an orthonormal basis for $L^2(\mathbb{K})\oplus\ldots\oplus L^2(\mathbb{K})$ (say, $\bigoplus_n L^2(\mathbb{K})$). Each ϕ_i here is called a component of the nonuniform super-wavelet. In the case when $\mathcal{F}(\Phi)$ is a nonuniform Parseval frame for $\bigoplus_n L^2(\mathbb{K})$, the n-tuple Φ is called a nonuniform Parseval frame super-wavelet.

The theorem given below is a characterization of a nonuniform super-wavelet of length n on local fields.

Theorem 4. *Let $\phi_1,\dots,\phi_n\in L^2(\mathbb{K})$. Then $(\phi_1,\dots,\phi_n)$ is a super-wavelet of length n if and only if the following equations hold:*

(i) $\sum_{j\in\mathbb{Z}} |\widehat{\phi}_i\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)|^2 = 1,$ *for a. e.* $\xi \in K,\ i = 1, \dots, n,$

(ii) $\sum_{j=0}^{\infty} \widehat{\phi}_i\left((\mathfrak{p}^{-1}N)^{j}\xi\right) \overline{\widehat{\phi}_i\left((\mathfrak{p}^{-1}N)^{j}(\xi+u(\sigma)\right)} = 0,$ *for a. e.* $\xi \in K,$ $s \in \Lambda\backslash qN\Lambda, 1 \leqslant i \leqslant n,$

(iii) $\sum_{\lambda\in\Lambda}\sum_{i=1}^{n} \widehat{\phi}_i\left((\mathfrak{p}^{-1}N)^{j}(\xi+\lambda)\right) \overline{\widehat{\phi}_i(\xi+\lambda)} = \delta_{j,0},$ *for a. e.* $\xi \in \mathbb{K}, j \in \mathbb{N}_0.$

Proof. Suppose $(\phi_1, \dots, \phi_n)$ is a nonuniform super-wavelet of length n. Then the system $\mathcal{F}(\Phi)$ given by (1) is an orthonormal basis for $\bigoplus_n L^2(\mathbb{K})$. Therefore, the function ϕ_i is a nonuniform Parseval frame wavelet for $L^2(\mathbb{K})$ for each $1 \leqslant i \leqslant n$, and, hence, the conditions (i) and (ii) follow from equations (3) and (4). Now, condition (iii) follows from following descriptions:

Since

$$\left\langle \bigoplus_{i=1}^{n} D_{\mathfrak{p}^j}T_\lambda\phi_i, \bigoplus_{i=1}^{n} D_{\mathfrak{p}^{j'}}T_\sigma\phi_i \right\rangle = \delta_{\lambda,\sigma}\,\delta_{j,j'},\ \text{for } \lambda,\sigma\in\Lambda; j,j'\in\mathbb{Z},$$

is equivalent to

$$\left\langle \bigoplus_{i=1}^{n} D_{\mathfrak{p}^j}T_\lambda\phi_i, \bigoplus_{i=1}^{n} \phi_i \right\rangle = \delta_{\lambda,0}\delta_{j,0},\ \text{for } \lambda\in\Lambda; j\geqslant 0.$$

Now, let $j \geqslant 0$ and $\lambda \in \Lambda$. Since for each $\lambda \in \Lambda, m \in \mathbb{N}_0,\ \chi_\lambda(u(m)) = 1$, and the system $\{\mathfrak{D} + \lambda\colon \lambda \in \Lambda\}$ is a measurable partition of $\mathbb{K}$, we have

$$\left\langle \bigoplus_{i=1}^{n} D_{\mathfrak{p}^j}T_\lambda\phi_i, \bigoplus_{i=1}^{n} \phi_i \right\rangle = \sum_{i=1}^{n} \langle D_{\mathfrak{p}^j}T_\lambda\phi_i, \phi_i\rangle = \sum_{i=1}^{n} \left\langle \widehat{D_{\mathfrak{p}^j}T_\lambda\phi_i}, \widehat{\phi}_i \right\rangle,$$

and, hence, we obtain

$$\left\langle \bigoplus_{i=1}^{n} D_{\mathfrak{p}^j}T_\lambda\phi_i, \bigoplus_{i=1}^{n} \phi_i \right\rangle =$$

$$= \sum_{i=1}^{n} \int_K \widehat{D_{\mathfrak{p}^j}T_\lambda\phi_i}(\xi)\overline{\widehat{\phi}_i(\xi)}d\xi =$$

$$= (qN)^{-j/2} \sum_{i=1}^{n} \int_{K} \chi_\lambda(-(\mathfrak{p}^{-1}N)^{-j}\xi)\widehat{\phi}_i((\mathfrak{p}^{-1}N)^{-j}\xi)\overline{\widehat{\phi}_i(\xi)}d\xi =$$

$$= (qN)^{j/2} \sum_{i=1}^{n} \int_{\bigcup_{\lambda\in\Lambda}\mathfrak{D}+\lambda} \chi_\lambda(-\xi)\widehat{\phi}_i(\xi)\overline{\widehat{\phi}_i((\mathfrak{p}^{-1}N)^{j}\xi)}d\xi =$$

$$= (qN)^{j/2} \int_{\mathfrak{D}} \Big(\sum_{i=1}^{n}\sum_{\lambda\in\Lambda} \widehat{\phi}_i(\xi+\lambda)\overline{\widehat{\phi}_i((\mathfrak{p}^{-1}N)^{j}(\xi+\lambda)}\Big)\overline{\chi_\lambda(\xi)}d\xi =$$

$$= \overline{(qN)^{j/2} \int_{\mathfrak{D}} \Big(\sum_{i=1}^{n}\sum_{\lambda\in\Lambda} \widehat{\phi}_i((\mathfrak{p}^{-1}N)^{j}(\xi+\lambda)\overline{\widehat{\phi}_i(\xi+\lambda)}\Big)\chi_\lambda(\xi)d\xi}.$$

Comparing the above expression with the Fourier coefficient and Fourier series of a function in $L^1(\mathfrak{D})$, and using the fact that the system $\{\chi_\lambda\}_{\lambda\in\Lambda}$ is an orthonormal basis for $L^2(\mathfrak{D})$, the result follows.

Conversely, suppose that conditions (i)–(iii) hold. In view of the above discussion, to complete the proof it remains only to show that the system $\mathcal{F}(\Phi)$ is dense in $\bigoplus_{n} L^2(\mathbb{K})$. The result follows by writing the following for every $m \in \{1, 2, \ldots, n\}$,

$$\bigoplus_{i=1}^{n} (\delta_{i,m}\times g_m) = \sum_{j'\in\mathbb{Z}}\sum_{\lambda'\in\Lambda} \Big\langle \bigoplus_{i=1}^{n} (\delta_{i,m}\times g_m)\,, D_{\mathfrak{p}^{j'}}T_{\lambda'}\phi_m \Big\rangle D_{\mathfrak{p}^{j'}}T_{\lambda'}\phi_m$$

where $g_m = D_{\mathfrak{p}^{j'}}T_\lambda\phi_m$. This fact is true in view of the following: for $l = 1, 2, \ldots, n$, $j\in\mathbb{Z}$ and $\lambda\in\Lambda$, we can write

$$\bigoplus_{i=1}^{n} D_{\mathfrak{p}^j}T_\lambda\phi_i = \sum_{j'\in\mathbb{Z}}\sum_{\lambda'\in\Lambda} \Big\langle \bigoplus_{i=1}^{n} D_{\mathfrak{p}^j}T_\lambda\phi_i, \bigoplus_{i'=1}^{n} D_{\mathfrak{p}^{j'}}T_{\lambda'}\phi_{i'} \Big\rangle \bigoplus_{i'=1}^{n} D_{\mathfrak{p}^{j'}}T_{\lambda'}\phi_{i'} =$$

$$= \sum_{j'\in\mathbb{Z}}\sum_{\lambda'\in\Lambda}\sum_{i=1}^{n} \big\langle D_{\mathfrak{p}^j}T_\lambda\phi_i, D_{\mathfrak{p}^{j'}}T_{\lambda'}\phi_i \big\rangle \bigoplus_{i'=1}^{n} D_{\mathfrak{p}^{j'}}T_{\lambda'}\phi_{i'},$$

and $D_{\mathfrak{p}^j}T_\lambda\phi_l = \sum_{j'\in\mathbb{Z}}\sum_{\lambda'\in\Lambda} \big\langle D_{\mathfrak{p}^{j'}}T_\lambda\phi_l, D_{\mathfrak{p}^{j'}}T_{\lambda'}\phi_l \big\rangle D_{\mathfrak{p}^{j'}}T_{\lambda'}\phi_l$, and, hence, we have

$$\sum_{j'\in\mathbb{Z}}\sum_{\lambda'\in\Lambda} \big\langle D_{\mathfrak{p}^j}T_\lambda\phi_l, D_{\mathfrak{p}^{j'}}T_{\lambda'}\phi_l \big\rangle D_{\mathfrak{p}^{j'}}T_{\lambda'}\phi_{l'} = 0$$

for $l \neq l'$ and $l, l' \in \{1, 2, \ldots, n\}$. □

The following is an easy consequence of above theorem.

Theorem 5. *Let $\phi_1, \ldots, \phi_n \in L^2(\mathbb{K})$ be such that $|\phi_i| = \chi_{\Gamma_i}$, for $i \in \{1, 2, \ldots, n\}$. Then $(\phi_1, \ldots, \phi_n)$ is a nonuniform super-wavelet of length n if and only if the following equations hold:*

(a) *for each $i \in \{1, 2, \ldots, n\}$, the system $\{(\mathfrak{p}^{-1}N)^{-j}\Gamma_i \colon j \in \mathbb{Z}\}$ is a measurable partition of $\mathbb{K}$,*

(b) *for each $i \in \{1, 2, \ldots, n\}$, the system $\{\Gamma_i + \lambda \colon \lambda \in \Lambda\}$ is a measurable partition of a subset of $\mathbb{K}$,*

(c) *the system $\{\Gamma_i + \lambda \colon \lambda \in \Lambda, 1 \leqslant i \leqslant n\}$ is a measurable partition of $\mathbb{K}$.*

Proof. Suppose $(\phi_1, \ldots, \phi_n)$ is a super-wavelet of length n such that $|\phi_i| = \chi_{W_i}$, for $i \in \{1, 2, \ldots, n\}$. Then, for each $i \in \{1, 2, \ldots, n\}$, the function ϕ_i is a Parseval frame wavelet in $L^2(\mathbb{K})$ and the system $\mathcal{F}(\Phi)$ is an orthonormal basis for $\bigoplus_n L^2(\mathbb{K})$. Hence the conditions (a) and (b) hold in view of Parseval frame wavelet ϕ_i and Theorem 3, and also, the condition (iii) of Theorem 5 is satisfied; that means, for $j \in \mathbb{N}_0$

$$
\begin{aligned}
\delta_{j,0} &= \sum_{\lambda \in \Lambda} \sum_{i=1}^{n} \widehat{\phi}_i \left((\mathfrak{p}^{-1}N)^j (\xi + \lambda) \right) \overline{\widehat{\phi}_i(\xi + \lambda)} = \\
&= \sum_{\lambda \in \Lambda} \sum_{i=1}^{n} \chi_{\Gamma_i} \left((\mathfrak{p}^{-1}N)^j (\xi + \lambda) \right) \chi_{\Gamma_i}(\xi + \lambda) = \\
&= \sum_{\lambda \in \Lambda} \sum_{i=1}^{n} \chi_{((\mathfrak{p}^{-1}N)^{-j}\Gamma_i + \lambda) \cap (\Gamma_i + \lambda)}(\xi),
\end{aligned}
$$

which is true for $j \neq 0$ since

$$
\left| \left((\mathfrak{p}^{-1}N)^{-j} W_i + \lambda \right) \cap (\Gamma_i + \lambda) \right| = 0,
$$

in view of conditions (a) and (b). Now, let $j = 0$. Then, the expression

$$
\sum_{\lambda \in \Lambda} \sum_{i=1}^{n} \chi_{(\Gamma_i + \lambda)}(\xi) = 1
$$

implies that

$$
\left| (\Gamma_l + \lambda) \cap (\Gamma_{l'} + \lambda') \right| = 0
$$

for $\lambda, \lambda' \in \Lambda$; $l, l' \in \{1, 2, \ldots, n\}$ and $(l, \lambda) \neq (l', \lambda')$. Also, we have

$$1 = |\mathfrak{D}| = \int_{\mathfrak{D}} d\xi = \int_{\mathfrak{D}} \sum_{\lambda\in\Lambda} \sum_{i=1}^{n} \chi_{(\Gamma_i+\lambda)}(\xi) d\xi =$$
$$= \int_{K} \chi_{\cup_{i=1}^{n}\Gamma_i}(\xi) d\xi = | \cup_{i=1}^{n} \Gamma_i|,$$

which proves condition (c).

Conversely, let us assume that for each $i \in \{1, 2, \ldots, n\}$, the function ϕ_i satisfies the conditions (a), (b) and (c), where $|\phi_i| = \chi_{\Gamma_i}$. Then, $(\phi_1, \ldots, \phi_n)$ is a super-wavelet of length n. This follows by noting Theorem 3, Theorem 5 and above calculations. □

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