Probl. Anal. Issues Anal. Vol. 10 (28), No 3, 2021, pp. 31–40
DOI: 10.15393/j3.art.2021.10270

UDC 517.53

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BERNSTEIN-TYPE ESTIMATES FOR THE DERIVATIVES OF TRIGONOMETRIC POLYNOMIALS

Abstract. Using the method of amplitude and phase transformations, we obtain sharp inequalities for the derivatives of real-valued trigonometric polynomials. The inequalities are sharp, as there are the corresponding extremal polynomials, for which they become equalities.

Key words: *amplitude and phase transformations, Bernstein's inequality*

2020 Mathematical Subject Classification: *26D05*

1. Introduction. For a positive integer n, set

$$T_n(t) = \sum_{k=0}^{n} \tau_k(t), \qquad \tau_k(t) := a_k \cos kt + b_k \sin kt, \quad t, a_k, b_k \in \mathbb{R}. \tag{1}$$

An amplitude and phase transformations (AFT) of order at most n is the sum

$$H(T_n, \{X_j\}, \{\lambda_j\}; t) := \sum_{j=1}^{n} X_j T_n\left(t - \lambda_j\right),$$

where λ_j, X_j are free real parameters (some X_j may be zero). The order of an AFT is equal to the number of summands in $H(T_n, \{X_j\}, \{\lambda_j\}; t)$ with pairwise-distinct $\exp(i\lambda_j)$ and $X_j \neq 0$. The AFTs were introduced and used in the papers [7], [18], [19] for obtaining Fejer-type estimates for harmonics and coefficients of trigonometric polynomials. Estimates of this type are well-known in extremal problems for nonnegative trigonometric polynomials (see, e.g., [1], [3], [8], [10], [14], [16]). Algebraic analogues of the AFTs, the so-called amplitude and frequency operators, were used for the Padè interpolation in [5].

Denote by w a set of real weights $w := \{\sigma_1, \sigma_2, \dots, \sigma_n\}$, $\sigma_l \in \mathbb{R}$, and determine the sums of weighted harmonics of the form

$$\mathcal{T}(T_n, w; t) := \sum_{\sigma_l \in w} \sigma_l \tau_l(t).$$

Let us consider the problem of representing the weighted sums in the form of AFTs:

$$\omega\, a_0 + \mathcal{T}(T_n, w; t) = H(T_n, \{X_j\}, \{\lambda_j\}; t), \tag{2}$$

where X_j, λ_j, ω are the required real parameters and $a_0 = a_0(T_n)$ is the constant term of the polynomial T_n. In this case, the equality $\omega = \sum_{j=1}^{n} X_j$ must naturally hold. It is easy to show that the problem (2) is equivalent to the following discrete moments problem:

$$X_1 z_1^l + \ldots + X_n z_n^l = \sigma_l, \qquad l = \overline{1, n}, \quad z_j = e^{-i\lambda_j}, \tag{3}$$

with real unknowns X_j and λ_j, where some X_j may take zero values, and $z_j = e^{-i\lambda_j}$ are distinct. Indeed, given a real-valued T_n, we have $\tau_k(t) = 2 \cdot \mathrm{Re}\left(\alpha_k e^{ikt}\right)$, where $2\alpha_k = a_k - ib_k$ and, therefore,

$$H(T_n, \{X_j\}, \{\lambda_j\}; t) = a_0\omega + 2\mathrm{Re}\Big(\sum_{k=1}^{n} \Big(\sum_{j=1}^{n} X_j z_j^k\Big) \alpha_k e^{ikt}\Big) =$$

$$= a_0\omega + \sum_{k=1}^{n} s_k \tau_k(t), \tag{4}$$

where

$$\omega := \sum_{j=1}^{n} X_j, \qquad s_k := \sum_{j=1}^{n} X_j z_j^k, \qquad s_k \in \mathbb{R}.$$

Hence, when (3) holds, (4) gives (2). The converse is also true: from the equalities (2) and (4) we obtain (3) (for more details, see [7]).

According to the classical Carathèodory theorem (see, e. g., [2], [11], [13]), the system (3) always has a (unique) solution, such that $X_j \geqslant 0$, $z_j = e^{i\lambda_k}$, $\lambda_j \in \mathbb{R}$ (for arbitrary fixed right-hand sides $\sigma_l \in \mathbb{C}$). It follows from the uniqueness of solution, in particular, that in the case under consideration, where the right-hand sides σ_l are real, the roots z_j lie on the unit circle symmetrically with respect to $\mathbb{R}$.

An important role during the analysis of systems (3) is played by the Toeplitz matrix $G_{n+1}(w; \omega) = \{g_{i,j}\}$ of order $n+1$ with the following structure: the main diagonal contains the parameter $\omega = \sum_{j=1}^{n} X_j$, and the weights σ_l are symmetrically located on the parallel diagonals, and $g_{i,j} = \sigma_l$ for $|i - j| = l$, $l = 1, \dots, n$ (see, e. g., [19]).

It is known (see, e. g., [2], [11], [13]) that for the solution to the Carathèodory problem (i.e., the problem (3) for $X_j \geqslant 0$ and real $\{\lambda_j\}$), the value of parameter $\omega = \sum_{j=1}^{n} X_j$ is equal to the largest root ω_w^+ ($\omega_w^+ = \omega_w^+(n) > 0$) of the polynomial $\det G_{n+1}(w; \omega)$, the determinant of the Toeplitz matrix. It obviously implies that for the solution of the problem (3) with *non-positive* $\{X_j\}$ and real $\{\lambda_j\}$, the value $\omega = \sum_{j=1}^{n} X_j$ is equal to the smallest root ω_w^- ($\omega_w^- = \omega_w^-(n) < 0$) of the same Toeplitz matrix determinant $G_{n+1}(w; \omega)$ (it is sufficient to consider the problem (3) with X_j replaced by $X_j' = -X_j$ and σ_l by $\sigma_l' = -\sigma_l$).

Consequently, the equalities (2) are valid for $\omega = \omega_w^+ > 0$ (and then all $X_j \geqslant 0$) and for $\omega = \omega_w^- < 0$ (and then all $X_j \leqslant 0$). The representation (2) in each of these cases we call *regular* if the order of the corresponding AFT is exactly n (then all $X_j \neq 0$).

If $\omega = \omega_w^\pm$ are known, the *Prony polynomial* $P_n(z; w, \omega_w^\pm)$ is used to determine the unknowns $z_j = e^{-i\lambda_j}$; it is the determinant obtained from the above-mentioned Toeplitz determinant by replacing its first row with a row $(z^n, z^{n-1}, \dots, 1)$ (see the examples below).

In a *regular case*, all roots $z_j = e^{-i\lambda_j}$, $j = 1, \dots, n$, of the polynomial $P_n(z; w, \omega_w^\pm)$ are distinct and lie on the unit circle symmetrically with respect to $\mathbb{R}$. Their arguments give the required values λ_j in (2) and (3). In this case, the unknowns X_j are found from the system (3), linear with respect to X_j with a nonzero Vandermonde determinant (for more details, see, e. g., [7], [18], [19]).

In an *irregular case*, when the order of AFT in (2) is less than n, the Prony polynomial is identically zero, $P_n(z; w, \omega_w^\pm) \equiv 0$. For this case, several methods have been developed to regularize the problem (3) by certain variations of the right-hand sides $\{\sigma_k\}$ ([18], [19]). For the regularized system (3), the corresponding Prony polynomial is nonzero and its nonzero roots are the desired solution to the system (3) $z_j = e^{-i\lambda_j}$, $j = 1, \dots, m$, $m < n$.

2. Extremal polynomials in the representation (2). Let us briefly describe a method of construction extremal nonnegative polynomials $S_\pm$ for which the following equalities hold (for more details, see [18], [19]):

$$\omega_w^\pm\, a_0(S_\pm) + \mathcal{T}(S_\pm, w; 0) = 0, \qquad w := \{\sigma_1, \dots, \sigma_n\}. \tag{5}$$

Let the representation (2) (regular or irregular) be obtained, for a given set of real weights $w := \{\sigma_1, \dots, \sigma_n\}$, in each of the cases $\omega = \omega_w^{\pm}$, where the AFT has order $m^{\pm} \leqslant n$. Using the obtained solutions $\lambda_j^{\pm}$, $j = 1, \dots, m^{\pm}$, we define the following even non-negative polynomials of order $m^{\pm}$:

$$S_{\pm}(t) = S_{\pm}(\omega_w^{\pm}; t) := \prod_{j=1}^{m^{\pm}} \sin^2 \Big(\frac{t + \lambda_j^{\pm}}{2}\Big). \tag{6}$$

Note that polynomials of the form (6) have already been used in [6], [12], [19]. Since $S_{\pm}(-\lambda_j^{\pm}) = 0$ for all $j = 1, \dots, m^{\pm}$, we have equality $H(S_{\pm}, \{X_j^{\pm}\}, \{\lambda_j^{\pm}\}; 0) = 0$ and, therefore, we obtain (5) from (2).

This equality will be used below to prove the sharpness of Bernstein-type inequalities. Note that the parity of polynomials $S_{m^{\pm}}(\omega_w^{\pm}; t)$ obviously follows from the symmetry of roots $z_j^{\pm} = e^{-i\lambda_j^{\pm}}$ in (3) with respect to $\mathbb{R}$.

3. The application of AFT for trigonometric Bernstein-type inequalities. Let us take a set of weights $w = w(n, 2m)$ for positive integer n and m: $w(n, 2m) := \{1^{2m}, 2^{2m}, \dots, n^{2m}\}$. For the derivative of order $2m$ of the polynomial (1), we have

$$(-1)^m T_n^{(2m)}(t) = \sum_{k=1}^{n} k^{2m} \tau_k(t) = \mathcal{T}(T_n, w; t).$$

Therefore, for the largest and smallest roots $\omega_w^{\pm} = \omega_{w(n,2m)}^{\pm}$ of the Toeplitz determinant $\det G_{n+1}(w; \omega)$ from (2), we get

$$\omega_w^{\pm} a_0 + (-1)^m T_n^{(2m)}(t) = \omega_w^{\pm} a_0 + \mathcal{T}(T_n, w; t) = \sum_{j=1}^{n} X_j^{\pm} T_n(t - \lambda_j^{\pm}),$$
$$\sum_{j=1}^{n} X_j^{\pm} = \omega_w^{\pm}, \tag{7}$$

where $\omega_w^{+} > 0$, $\omega_w^{-} < 0$, all $X_j^{+} \geqslant 0$, $X_j^{-} \leqslant 0$, $\lambda_j^{\pm} \in \mathbb{R}$.

Theorem 1. *For positive integer n, m, weights $w = w(n, 2m) := \{1^{2m}, 2^{2m}, \dots, n^{2m}\}$, $a_0 = a_0(T_n)$, and all $t \in \mathbb{R}$, the following two-sided inequalities hold:*

$$\omega_w^{+}(\min_x T_n(x) - a_0) \leqslant (-1)^m T_n^{(2m)}(t) \leqslant \omega_w^{+}(\max_x T_n(x) - a_0), \tag{8}$$

$$\omega_w^-(\max_x T_n(x) - a_0) \leqslant (-1)^m T_n^{(2m)}(t) \leqslant \omega_w^-(\min_x T_n(x) - a_0). \qquad (9)$$

In particular, if T_n is a positive-valued polynomial, then the first inequality in (8) and the second inequality in (9) imply

$$-\omega_w^+ a_0 \leqslant (-1)^m T_n^{(2m)}(t) \leqslant -\omega_w^- a_0, \qquad a_0 > 0.$$

The latter two-sided inequality is sharp, as there are extremal nonnegative polynomials $S_\pm(t)$, $\deg S_\pm \leqslant n$, for which the following equalities hold:

$$\omega_w^\pm a_0^\pm(S_\pm) + (-1)^m S_\pm^{(2m)}(0) = 0, \qquad a_0^\pm(S_\pm) > 0. \qquad (10)$$

Proof. The upper estimate in (8) is obtained from (7) taking into account that all $X_j^+ \geqslant 0$ and $\omega_w^+ = \sum_{j=1}^n X_j^+ > 0$:

$$\omega_w^+ a_0 + (-1)^m T_n^{(2m)}(t) = \sum_{j=1}^n X_j^+ T_n(t - \lambda_j^+) \leqslant \omega_w^+ \max_x T_n(x);$$

similarly, we obtain the lower estimate in (8). The lower estimate (9) is obtained from (7) taking into account that all $X_j^- \leqslant 0$ and $\omega_w^- = \sum_{j=1}^n X_j^- < 0$:

$$-\omega_w^- a_0 + (-1)^{m+1} T_n^{(2m)}(t) = \sum_{j=1}^n (-X_j^-) T_n(t - \lambda_j^-) \leqslant -\omega_w^- \max_x T_n(x);$$

similarly, we obtain the upper estimate in (9). The statement (10) follows from (5). □

Examples. Let us give examples of constructing several extremal polynomials in (10) for the second derivative.

For $n = 2$, $m = 1$ and weights $w = w(2,2) = \{1^2, 2^2\}$ we have

$$G_3(w,\omega) = \begin{bmatrix} \omega & 1 & 4 \\ 1 & \omega & 1 \\ 4 & 1 & \omega \end{bmatrix}, \quad P_2(z; w, \omega) = \begin{bmatrix} z^2 & z & 1 \\ 1 & \omega & 1 \\ 4 & 1 & \omega \end{bmatrix},$$

where $\det G_3(w,\omega) = (\omega - 4)(\omega^2 + 4\omega - 2)$, and for the second derivatives the representations (7) and the inequalities (8), (9) are obtained with the parameters $\omega_w^+ = 4$, $\omega_w^- = -2 - \sqrt{6} \approx -4.449\ldots$ (here and everywhere below, numerical values are displayed with three decimal places).

Thus, we get the two regular cases $\omega = \omega^{\pm}$; the Prony polynomials $P_2(z; w, \omega_w^{\pm})$ have the pairs of roots ± 1 and $-0.224 \pm 0.974i$ with the arguments $\{0, \pi\}$ and $\{1.797, -1.797\}$. Hence, according to the formula (6), we find an extremal nonnegative polynomials, for which the equalities (10) hold for the second derivative:

$$S_+(t) \approx 0.125 - 0.125 \cos(2\,t)\,, \qquad \omega_w^+ = 4;$$

$$S_-(t) \approx 0.137 + 0.125 \cos(2\,t) + 0.112 \cos(t)\,, \qquad \omega_w^- \approx -4.449.$$

When $n = 3$ and $w = w(3,2) = \{1^2, 2^2, 3^2\}$, according to the same scheme, we have

$$G_4(w,\omega) = \begin{pmatrix} \omega & 1 & 4 & 9 \\ 1 & \omega & 1 & 4 \\ 4 & 1 & \omega & 1 \\ 9 & 4 & 1 & \omega \end{pmatrix}, \quad P_3(z; w, \omega) = \begin{vmatrix} z^3 & z^2 & z & 1 \\ 1 & \omega & 1 & 4 \\ 4 & 1 & \omega & 1 \\ 9 & 4 & 1 & \omega \end{vmatrix},$$

therefore, $\det G_4(w,\omega) = \omega\,(\omega - 10)\,(\omega^2 + 10\omega - 16)$ and $\omega_w^+ = 10$, $\omega_w^- = -5 - \sqrt{41} \approx -11.403$. This implies representations (7) and inequalities (8), (9). Here, for $\omega = \omega^{\pm}$, regular cases are obtained and the Prony polynomials $P_3(z; w, \omega_w^{\pm})$ have three different roots with unit moduli:

$$\{1,\ -0.666 - 0.745i,\ -0.666 + 0.745i\}\,,$$

$$\{-1,\ 0.259 - 0.965i,\ 0.259 + 0.965i\}\,.$$

Calculating their arguments $\lambda_j^{\pm}$, by the formula (6), we find extremal nonnegative polynomials, for which the equalities (10) hold for $m = 1$:

$$S_+(t) \approx 0.034 - 0.031 \cos(3t) - 0.020 \cos(2t) + 0.017 \cos(t)\,;$$

$$S_-(t) \approx 0.038 + 0.031 \cos(3t) + 0.030 \cos(2t) + 0.037 \cos(t)\,;$$

$$\omega_w^+ = 10, \qquad \omega_w^- \approx -11.403.$$

Remark 1. *One can see from Theorem 1 that the main role is played by the module estimates $\omega^{\pm}_{w(n,2m)}$ for the derivative estimates based on (7). They can be estimated using the well-known Hadamard theorem about the non-degeneracy of matrices with strictly dominant diagonal (see, e.g., [9]). Namely, if the modulus of the diagonal element $|\omega|$ in each row of the* *matrix $G_{n+1}(w;\omega)$ is greater than the sum of remaining elements moduli of the same row, then $\det G_{n+1}(w;\omega) \neq 0$. This implies the estimate*

$$|\omega^{\pm}_{w(n,2m)}| < \Omega(n,2m) := \sum_{k=1}^{n} k^{2m}.$$

For example, for the second derivative ($m = 1$) we have

$$|\omega^{\pm}_{w(n,2)}| < \Omega(n,2) = \frac{1}{6}\, n\,(n+1)\,(2n+1)\,,$$

while the calculations for $n = 10$ give

$$0.571 \cdot \Omega(10,2) \approx \omega^{+}_{w(10,2)} < |\omega^{-}_{w(10,2)}| \approx 0.666 \cdot \Omega(10,2).$$

As it can be easily checked, the growth order of $\Omega(n,2m)$ for $n \to \infty$ and each fixed m has the form

$$\Omega(n,2m) \asymp \frac{1}{2m+1}\, n^{2m+1}, \qquad n \to \infty.$$

The calculations up to $n = 30$ show that this growth order is rather precise for $|\omega^{\pm}_{w}(n,2m)|$. In particular, this is a distinctive property of the inequalities considered in Theorem 1 compared to the (sharp) classical inequalities $\|T_n^{(2m)}\| \leqslant n^{2m}\|T_n\|$ (S. N. Bernstein, M. Riesz [4], [15]).

4. The application of AFT for estimating the derivative of conjugate polynomials. The polynomial (1) with real coefficients can be represented as

$$T_n(t) = a_0 + \sum_{k=1}^{n} \tau_k(t) = a_0 + \operatorname{Re} \sum_{k=1}^{n} (a_k - i b_k) e^{it\,k},$$

$$\tau_k(t) := a_k \cos kt + b_k \sin kt.$$

The polynomial

$$\tilde{T}_n(t) = \sum_{k=1}^{n} a_k \sin kt - b_k \cos kt = \operatorname{Im} \sum_{k=1}^{n} (a_k - i b_k) e^{it\,k}$$

is called conjugate to the polynomial (1). It can be easily checked that $\tilde{T}_n(t)' = \sum_{k=1}^{n} k\tau_k(t)$. Thus, for weights $w = w(n,1) = \{1,\ldots,n\}$ we have from (2):

$$\omega^{\pm}_{w}\, a_0 + \tilde{T}_n(t)' = \omega^{\pm}_{w}\, a_0 + \mathcal{T}(T_n,w;t) = \sum_{j=1}^{n} X^{\pm}_{j} T_n(t - \lambda^{\pm}_{j}), \qquad (11)$$

where $\omega_w^\pm$ are the largest and smallest roots of the Toeplitz determinant $\det G_{n+1}(w(n,1);\omega)$, with $\sum_{j=1}^n X_j^\pm = \omega_w^\pm$ ($\omega_w^+ > 0$, $\omega_w^- < 0$).

Similarly to Theorem 1, from the representation (11) we obtain

Theorem 2. *For positive integer n, weights $w(n,1) := \{1,2,\dots,n\}$, $a_0 = a_0(T_n)$ and all $t \in \mathbb{R}$, the following two-sided inequalities hold:*

$$\omega_w^+(\min_x T_n(x) - a_0) \leqslant \tilde{T}_n(t)' \leqslant \omega_w^+(\max_x T_n(x) - a_0), \tag{12}$$

$$\omega_w^-(\max_x T_n(x) - a_0) \leqslant \tilde{T}_n(t)' \leqslant \omega_w^-(\min_x T_n(x) - a_0). \tag{13}$$

In particular, if T_n is a positive-valued polynomial, then the first inequality in (12) and the second inequality in (13) imply

$$-\omega_w^+ a_0 \leqslant \tilde{T}_n(t)' \leqslant -\omega_w^- a_0, \qquad a_0 > 0.$$

These inequalities are sharp, as there are extremal nonnegative polynomials $S_\pm(t)$ of order $\leqslant n$ with the constant term $a_0(S_\pm) > 0$, for which the equalities hold:

$$\omega_w^\pm a_0(S_\pm) + \tilde{S}_\pm'(0) = 0. \tag{14}$$

Examples. Extremal polynomials are constructed as in the previous section, so we do not dwell on the details. For example, let $n = 2$ and weights $w = \{1,2\}$; according to the formula (6), the following polynomials, for which the equalities (14) hold, are obtained:

$$S_+(t) \approx 0.125 - 0.125\cos(2t), \qquad \omega_w^+ = 2;$$

$$S_-(t) \approx 0.158 + 0.125\cos(2t) + 0.183\cos(t), \qquad \omega_w^- \approx -2.732.$$

For $n = 3$ and $w = \{1,2,3\}$ we have

$$S_+(t) \approx 0.036 - 0.031\cos(3t) - 0.025\cos(2t) + 0.020\cos(t);$$

$$S_-(t) \approx 0.047 + 0.031\cos(3t) + 0.045\cos(2t) + 0.061\cos(t);$$

$$\omega_w^+ \approx 3.414, \qquad \omega_w^- \approx -5.162.$$

Remark 2. *The estimate for $\omega_{w(n,1)}^\pm$ by the above-mentioned Hadamard theorem gives:*

$$|\omega_{w(n,1)}^\pm| < \sum_{k=1}^n k = \frac{1}{2}n(n+1).$$

The calculations up to $n = 30$ show that this growth order is rather precise. In particular, this is a distinctive property of the inequalities (12), (13) *considered in Theorem* 2 *compared to the* (*sharp*) *classical Szegö inequalities* $\|\tilde{T}_n'\| \leqslant n\|T_n\|$, *see* [17].

Acknowledgments. The reported study was funded by RFBR according to the research project 20-31-90010.

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DOI: https://doi.org/10.1134/S0081543820010083

Received April 25, 2021.
In revised form, August 03, 2021.
Accepted August 08, 2021.
Published online August 31, 2021.

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