Probl. Anal. Issues Anal. Vol. 11 (29), No 1, 2022, pp. 81–101
DOI: 10.15393/j3.art.2022.10330

UDC 517.58, 517.518

M. Mejri

q-CHEBYSHEV POLYNOMIALS AND THEIR q-CLASSICAL CHARACTERS

Abstract. In this work, we give some properties of the q-Chebyshev polynomials through the Stieltjes function associated with their regular forms (linear functional). Some connection formulas are highlighted. The integral representation of those forms are given.

Key words: *q-difference equation, H_q-semiclassical polynomials, orthogonality measure*

2020 Mathematical Subject Classification: *Primary: 33C45; Secondary: 42C05*

1. Introduction. In this contribution, we introduce the monic orthogonal polynomial sequence (MOPS) of q-Chebyshev polynomials $\{\hat{T}_n(x,q)\}_{n\geqslant 0}$ of the first kind and $\{\hat{U}_n(x,q)\}_{n\geqslant 0}$ of the second kind, which are orthogonal with respect to the forms (linear functionals) $\mathcal{T}_q, \mathcal{U}_q$, respectively, through a q-difference functional equation similar to that satisfied by the classical Chebyshev forms $\mathcal{T}$ and $\mathcal{U}$ of the first/second kind, respectively [15]. In addition, we preserve the connection property $\mathcal{T}_q^{(1)} = \mathcal{U}_q$, where $\mathcal{T}_q^{(1)}$ is the first associated form of $\mathcal{T}_q$. Note that many authors have been interested in the q-extension of the Chebyshev polynomials and their properties [1], [4], [8]. The normalized sequences associated with those introduced in [1], [4], [8] are equal to $\{\hat{T}_n(x,q)\}_{n\geqslant 0}$ and $\{\hat{U}_n(x,q)\}_{n\geqslant 0}$, respectively, up to a dilation. Furthermore, those polynomials are a particular cases of big q-Jacobi polynomials [1], [9]. Our main aim is to study in detail these polynomials through their q-classical character. The second section is devoted to the preliminaries, some fundamental results useful in the sequel, to the introduction of the MOPSs $\{\hat{T}_n(x,q)\}_{n\geqslant 0}$ and $\{\hat{U}_n(x,q)\}_{n\geqslant 0}$. In the third section, we obtain a connection formula between $\mathcal{T}_q$ and the shifted form $\mathcal{U}_q$, which is a q-extension of the formula given in [10]. As a consequence, we highlight certain formulas connecting the polynomials $\hat{T}_n(x,q)$ and $\hat{U}_n(x,q)$ for $n \geqslant 0$, which are q-extensions of the classical case [3], [16], [17]. In the fourth section, we give the integral representation of the form $\mathcal{T}_q$ using the connection formula given above. This has not been done previously in literature. In the last section, we give explicitly the Stieltjes functions of the q-Chebyshev form of the first and second kind.

2. Preliminaries and fundamental results. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$, and let $\mathcal{P}'$ be its dual. We denote by $\langle w, f\rangle$ the effect of $w \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(w)_n := \langle w, x^n\rangle$, $n \geqslant 0$ the moments of w. Let us introduce some useful operations in $\mathcal{P}'$. For any linear form w, any polynomial g and any $a \in \mathbb{C} - \{0\}$, $c \in \mathbb{C}$, let gw, w', $h_a w$, $(x-c)^{-1}w$, δ_c, and $H_q w$ be the forms (linear functionals) defined by duality

$$\begin{aligned}&\langle gw, f\rangle = \langle w, gf\rangle, \langle w', f\rangle = -\langle w, f'\rangle, \langle h_a w, f\rangle := \langle w, h_a f\rangle,\\ &\langle (x-c)^{-1}w, f\rangle := \langle w, \theta_c f\rangle,\ \langle \delta_c, f\rangle := f(c),\ \langle H_q w, f\rangle = -\langle w, H_q f\rangle, f \in \mathcal{P},\end{aligned}$$

where

$$\begin{aligned}&(\theta_c f)(x) = \frac{f(x)-f(c)}{x-c},\ (h_a f)(x) = f(ax),\\ &H_q(f)(x) = \frac{f(qx)-f(x)}{(q-1)x}, x \neq 0, q \in \mathbb{C} - \Big(\{0\} \cup \bigsqcup_{n\geqslant 0}\{z \in \mathbb{C}, z^n = 1\}\Big),\\ &H_q(f)(0) = f'(0).\end{aligned}$$

We also define the right-multiplication of a form by a polynomial as

$$(wf)(x) := \Big\langle w, \frac{xf(x) - \xi f(\xi)}{x-\xi}\Big\rangle, w \in \mathcal{P}', f \in \mathcal{P}.$$

The Stieltjes function of $w \in \mathcal{P}'$ is defined by

$$S(z,w) = -\sum_{n\geqslant 0} \frac{(w)_n}{z^{n+1}}.$$

A monic polynomial sequence (MPS) $\{P_n\}_{n\geqslant 0}$ is a sequence of monic polynomials P_n, $n \geqslant 0$, with $\deg P_n = n$. Let $\{w_n\}_{n\geqslant 0}$ be its dual sequence, defined by $\langle w_n, P_m\rangle = \delta_{n,m}, n, m \geqslant 0$. The MPS $\{P_n\}_{n\geqslant 0}$ is orthogonal (MOPS) with respect to $w \in \mathcal{P}'$ if the following conditions hold: $\langle w, P_m P_n\rangle = r_n \delta_{n,m}$, $n, m \geqslant 0$, $r_n \neq 0$, $n \geqslant 0$. In this case, the form w

is said to be regular. The form w is called normalized if $(w)_0 = 1$. In this paper, we suppose that all forms are normalized. Thus, $w = w_0$ and $\{P_n\}_{n\geqslant 0}$ satisfies the standard recurrence relation

$$\begin{cases} P_0(x) = 1 \quad , \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \; \gamma_{n+1} \neq 0, \; n \geqslant 0. \end{cases} \tag{1}$$

The regular form w_0 is said to be symmetric if $(w_0)_{2n+1} = 0$, $n \geqslant 0$ or, equivalently, $\beta_n = 0$, $n \geqslant 0$ in (1) [3], [11].

Let $\{\hat{P}_n\}_{n\geqslant 0}$ be the sequence defined by $\hat{P}_n(x) = a^{-n}P_n(ax)$, $n \geqslant 0$, $a \neq 0$. It is a MOPS with respect to $\hat{w}_0 = h_{a^{-1}}w_0$ fulfilling (1) with [10]

$$\hat{\beta}_n = \frac{\beta_n}{a}, \; \hat{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, n \geqslant 0.$$

Given a regular form w and the corresponding MOPS $\{P_n\}_{n\geqslant 0}$ satisfying (1), we define the first associated sequence $\{P_n^{(1)}\}_{n\geqslant 0}$ by [11]

$$P_n^{(1)}(x) = \left\langle w, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x - \xi} \right\rangle = (w\theta_0 P_{n+1})(x).$$

$\{P_n^{(1)}\}_{n\geqslant 0}$ is a MOPS with respect to $w^{(1)}$ satisfying (1) with [3], [11]

$$\beta_n^{(1)} = \beta_{n+1}, \; \gamma_{n+1}^{(1)} = \gamma_{n+2}, n \geqslant 0.$$

The Chebyshev MOPS of the first kind (respectively, of the second kind), orthogonal with respect to $\mathcal{T}$ (respectively, $\mathcal{U}$) are defined by [3], [11], [12]

$$\begin{cases} \hat{T}_0(x) = 1, \quad \hat{T}_1(x) = x, \\ \hat{T}_{n+2}(x) = x\hat{T}_{n+1}(x) - \gamma_{n+1}^{\mathcal{T}}\hat{T}_n(x), \; n \geqslant 0, \end{cases}$$

$$((x^2 - 1)\mathcal{T})' - x\mathcal{T} = 0,$$

$$\gamma_1^{\mathcal{T}} = \frac{1}{2}, \; \gamma_{n+1}^{\mathcal{T}} = \frac{1}{4}, \; n \geqslant 1,$$

$$\langle \mathcal{T}, f \rangle = \frac{1}{\pi} \int_{-1}^{+1} \frac{1}{\sqrt{1 - x^2}} f(x)dx, \;\; f \in \mathcal{P}.$$

$$\begin{cases} \hat{U}_0(x) = 1, \quad \hat{U}_1(x) = x, \\ \hat{U}_{n+2}(x) = x\hat{U}_{n+1}(x) - \gamma_{n+1}^{\mathcal{U}}\hat{U}_n(x), \; n \geqslant 0, \end{cases}$$

$$\gamma_{n+1}^{\mathcal{U}} = \frac{1}{4}, n \geqslant 0,$$

$$((x^2-1)\mathcal{U})' - 3x\mathcal{U} = 0,$$

$$\langle \mathcal{U}, f \rangle = \frac{2}{\pi} \int_{-1}^{+1} \sqrt{1-x^2} f(x) dx \ f \in \mathcal{P}. \tag{2}$$

Let us recall some results:

Lemma 1. [7], [11] *Let $w \in \mathcal{P}'$, $f \in \mathcal{P}$, $a \in \mathbb{C} - \{0\}$. The following formulas hold:*

$$H_q(fw) = (h_{q^{-1}} f) H_q(w) + q^{-1}(H_{q^{-1}} f) w, \tag{3}$$

$$S(z, fw) = f(z) S(z, w) + (w\theta_0 f)(z), \tag{4}$$

$$h_a(fw) = (h_{a^{-1}} f)(h_a w). \tag{5}$$

Definition 1. [7] *A form w is called H_q-semiclassical, if it is regular and if there exist two polynomials ϕ and ψ (ϕ is monic), $\deg \phi = t \geqslant 0$, $\deg \psi = p \geqslant 1$, such that*

$$H_q(\phi w) + \psi w = 0, \tag{6}$$

the corresponding orthogonal sequence $\{P_n\}_{n \geqslant 0}$ is called H_q-semiclassical.

Remark. *When $\deg \phi \leqslant 2$, $\deg \psi = 1$, w is called H_q-classical form [6].*

Lemma 2. [7] *If w is H_q-semiclassical, fulfilling the equation (6), the form $\tilde{w} = h_{a^{-1}} w$, $a \in \mathbb{C} - \{0\}$ is H_q-semiclassical and satisfies*

$$H_q(\tilde{\phi}\tilde{w}) + \tilde{\psi}\tilde{w} = 0,$$

with $\tilde{\phi}(x) = a^{-\deg \phi} \phi(ax)$, $\tilde{\psi}(x) = a^{1-\deg \phi} \psi(ax)$.

The H_q-semiclassical character of a regular form can be described via the formal Stieltjes function, as follows.

Theorem 1. [7] *Let w be a regular form. The following statements are equivalent:*

(a) w is H_q-semiclassical form satisfying (6).

(b) *The Stieltjes function $S(\cdot, w)$ satisfies the q-Riccati equation*

$$(h_{q^{-1}}\phi)(z)H_{q^{-1}}(S(z,w)) = C(z)S(z,w) + D(z),$$

where

$$C = -(H_{q^{-1}}\phi) - q\psi,$$
$$D = -\Big\{(H_{q^{-1}}(w\theta_0\phi) + q(w\theta_0\psi)\Big\}.$$

We are going to use the following notations and results [5]:

$$(a,q)_n := \begin{cases} 1, \;\; n = 0, \\ \prod_{\nu=0}^{n-1}(1 - aq^\nu), \; n \geqslant 1, \; a \in \mathbb{C}. \end{cases}$$

$$\lim_{q\to 1} \frac{(q^a z, q)_\infty}{(z,q)_\infty} = (1-z)^{-a}, \; |z| < 1, \; a \in \mathbb{R}. \tag{7}$$

Let $\{P_n(\alpha)\}_{n\geqslant 0}$ be the symmetric MOPS introduced in the situation (3.25) in [15] with q is replaced by q^2

$$\begin{cases} \gamma_{n+1}(\alpha) = \dfrac{(q^{n+1}-1)(q^{n+1+2\alpha}-1)}{(q^{2n+2\alpha+1}-1)(q^{2n+2\alpha+3}-1)} q^{n+2\alpha+2}, \; n \geqslant 0, \\ H_q((x^2-1)u(\alpha)) + \dfrac{1-q^{-2\alpha-2}}{1-q} x u(\alpha) = 0. \end{cases} \tag{8}$$

If $\alpha = -\frac{1}{2}$, we denote $\mathcal{T}_q := u(-\frac{1}{2})$, $\gamma_{n+1}^{\mathcal{T}_q} := \gamma_{n+1}(-\frac{1}{2})$, $n \geqslant 0$; then

$$\begin{cases} \gamma_1^{\mathcal{T}_q} = \dfrac{q}{q+1}, \; \gamma_{n+1}^{\mathcal{T}_q} = \dfrac{q^{n+1}}{(q^n+1)(q^{n+1}+1)}, \; n \geqslant 1, \\ H_q((x^2-1)\mathcal{T}_q) - q^{-1}x\mathcal{T}_q = 0. \end{cases} \tag{9}$$

Note that if $q \to 1$, we obtain the form $\mathcal{T}$ [3], [12]; then the form $\mathcal{T}_q$ is its q-extension (we say that $\mathcal{T}_q$ is the q-Chebyshev form of the first kind). In the following, $\{\hat{T}_n(x,q)\}_{n\geqslant 0}$ is the MOPS with respect to $\mathcal{T}_q$.

Denote by $\mathcal{U}_q := h_{q^{-\frac{1}{2}}} u(\frac{1}{2})$, $\gamma_{n+1}^{\mathcal{U}_q} := q^{-1}\gamma_{n+1}(\frac{1}{2})$, $n \geqslant 0$ and obtain, due to (8) and Lemma 2:

$$\begin{cases} \gamma_{n+1}^{\mathcal{U}_q} = \dfrac{q^{n+2}}{(q^{n+1}+1)(q^{n+2}+1)}, \; n \geqslant 0, \\ H_q((x^2-q^{-1})\,\mathcal{U}_q) + \dfrac{1-q^{-3}}{1-q} x\,\mathcal{U}_q = 0. \end{cases} \tag{10}$$

If $q \to 1$, we get the form $\mathcal{U}$ [3], [12]; then $\mathcal{U}_q$ is the q-extension of the form $\mathcal{U}$ (it is the q-Chebyshev form of the second kind). Denote by $\{\hat{U}_n(x,q)\}_{n\geqslant 0}$ the MOPS with respect to $\mathcal{U}_q$.

Remark. *We see that, due to (9) and (10), we have*

$$\mathcal{T}_q^{(1)} = \mathcal{U}_q.$$

This is a q-extension of the classical case [10].

3. Connection formulas.

Lemma 3. [14] *Let $(b_n)_{n\geqslant 0}$ with $b_n \neq 0$, $n \geqslant 0$, $(c_n)_{n\geqslant 0}$ be two sequences of complex numbers and $(x_n)_{n\geqslant 0}$ be a sequence satisfying the following recurrence relation*

$$x_{n+1} = b_n x_n + c_n, n \geqslant 0, x_0 = a \in \mathbb{C} - \{0\}.$$

We have $x_{n+1} = \Big(\prod\limits_{k=0}^{n} b_k\Big)\Big\{a + \sum\limits_{k=0}^{n}\Big(\prod\limits_{\mu=0}^{k} b_\mu\Big)^{-1} c_k\Big\}$, $n \geqslant 0$.

Lemma 4. *The following equation holds:*

$$(x^2-1)\mathcal{T}_q = -\frac{1}{q+1} h_{q^{-\frac{1}{2}}} \mathcal{U}_q. \tag{11}$$

Proof. Let w_q be the normalized form defined by

$$(x^2-1)\mathcal{T}_q = -\frac{1}{q+1} w_q. \tag{12}$$

Then $H_q((x^2-1)\mathcal{T}_q) = -\frac{1}{q+1} H_q(w_q)$, and from (9) we get $x\mathcal{T}_q = -\frac{q}{q+1} H_q(w_q)$. Multiplying both sides by x^2-1 and using (12), we deduce that $(x^2-1)H_q(w_q) - q^{-1} x w_q = 0$.

From relation (3), it follows that

$$H_q((x^2-q^{-2})w_q) + \frac{1-q^{-3}}{1-q} x w_q = 0. \tag{13}$$

On the other hand, from (10) and by virtue of Lemma 2, we have

$$H_q\big((x^2-q^{-2}) h_{q^{-\frac{1}{2}}} \mathcal{U}_q\big) + \frac{1-q^{-3}}{1-q} x h_{q^{-\frac{1}{2}}} \mathcal{U}_q = 0. \tag{14}$$

Based on relations (13), (14), and the fact that $(h_{q^{-\frac{1}{2}}}\mathcal{U}_q)_0 = (w_q)_0 = 1$, we get $h_{q^{-\frac{1}{2}}}\mathcal{U}_q = w_q$, which provides (11). □

Remark. *When $q \to 1$ in relation (11), we obtain the connection formula given in [10].*

In the sequel, we denote by $\{\tilde{U}_n(x,q)\}_{n\geqslant 0}$ the MOPS with respect to $\tilde{\mathcal{U}}_q := h_{q^{-\frac{1}{2}}}\mathcal{U}_q$. Then we have

$$\tilde{U}_n(x,q) = q^{-\frac{n}{2}}\hat{U}_n(q^{\frac{1}{2}}x,q),\ n \geqslant 0.$$

$$\begin{cases} \tilde{U}_0(x,q) = 1,\ \tilde{U}_1(x,q) = x, \\ \tilde{U}_{n+2}(x,q)(x,q) = x\tilde{U}_{n+1}(x,q) - \gamma_{n+1}^{\tilde{\mathcal{U}}_q}\tilde{U}_n(x,q),\ \ n \geqslant 0 \end{cases} \tag{15}$$

with

$$\gamma_{n+1}^{\tilde{\mathcal{U}}_q} = \frac{q^{n+1}}{(1+q^{n+1})(1+q^{n+2})},\ \ n \geqslant 0. \tag{16}$$

Lemma 5. *The following formulas hold:*

$$\tilde{U}_n(1,q) = \frac{2}{(-1;q^{-1})_{n+1}}\sum_{k=0}^{n} q^{-\frac{k(k+1)}{2}},\ n \geqslant 0. \tag{17}$$

$$\tilde{U}_n^{(1)}(1,q) = \frac{2(q+1)}{(-1;q^{-1})_{n+2}}\sum_{k=0}^{n} q^{-\frac{(k+1)(k+2)}{2}},\ n \geqslant 0. \tag{18}$$

Proof. From relations (15) and (16), it follows that

$$\tilde{U}_{n+2}(1,q) = \tilde{U}_{n+1}(1,q) - \frac{q^{n+1}}{(1+q^{n+1})(1+q^{n+2})}\tilde{U}_n(1,q),\ n \geqslant 0.$$

Equivalently,

$$\begin{aligned} &(1+q^{n+2})\tilde{U}_{n+2}(1,q) - q^{n+2}\tilde{U}_{n+1}(1,q) = \\ &= \frac{1}{1+q^{n+1}}\Big((1+q^{n+1})\tilde{U}_{n+1}(1,q) - q^{n+1}\tilde{U}_n(1,q)\Big),\ \ n \geqslant 0. \end{aligned}$$

Therefore,

$$(1+q^{n+2})\tilde{U}_{n+2}(1,q) - q^{n+2}\tilde{U}_{n+1}(1,q) = \frac{(1+q)\tilde{U}_1(1,q) - q\tilde{U}_0(1,q)}{\prod_{k=0}^{n}(1+q^{k+1})},\ \ n \geqslant 0.$$

Since $\tilde{U}_0(1,q)=\tilde{U}_1(1,q)=1$, we get

$$\tilde{U}_{n+1}(1,q)=\frac{1}{1+q^{-n-1}}\tilde{U}_n(1,q)+\frac{q^{-\frac{(n+1)(n+2)}{2}}}{\prod\limits_{k=0}^{n+1}(1+q^{-k})},\quad n\geqslant 1,$$

but the previous relation is valid for $n=0$. Using Lemma 3, we obtain

$$\begin{aligned}\tilde{U}_{n+1}(1,q)&=\frac{1}{\prod\limits_{k=1}^{n+2}(1+q^{-k})}\left(1+\sum_{k=0}^{n}q^{-\frac{(k+1)(k+2)}{2}}\right)=\\&=\frac{2}{(-1;q^{-1})_{n+2}}\sum_{k=0}^{n+1}q^{-\frac{k(k+1)}{2}},\quad n\geqslant 0.\end{aligned}$$

Thus, $\tilde{U}_n(1,q)=\dfrac{2}{(-1;\,q^{-1})_{n+1}}\sum\limits_{k=0}^{n}q^{-\frac{k(k+1)}{2}}$, $n\geqslant 1$. But the previous relation is valid for $n=0$: this provides (17).

Based on relations (15) and (16), we get

$$\begin{aligned}&(1+q^{n+3})\tilde{U}^{(1)}_{n+2}(1,q)-q^{n+3}\tilde{U}^{(1)}_{n+1}(1,q)=\\&=\frac{1}{1+q^{n+2}}\Big((1+q^{n+2})\tilde{U}^{(1)}_{n+1}(1,q)-q^{n+2}\tilde{U}^{(1)}_{n}(1,q)\Big),\quad n\geqslant 0.\end{aligned}$$

Then

$$\begin{aligned}&(1+q^{n+3})\tilde{U}^{(1)}_{n+2}(1,q)-q^{n+3}\tilde{U}^{(1)}_{n+1}(1,q)=\\&\qquad=\frac{(1+q^2)\tilde{U}^{(1)}_{1}(1,q)-q^2\tilde{U}^{(1)}_{0}(1,q)}{\prod\limits_{k=0}^{n}(1+q^{k+2})}=\frac{2(1+q)}{\prod\limits_{k=0}^{n+2}(1+q^{k})},\quad n\geqslant 0.\end{aligned}$$

So,

$$\tilde{U}^{(1)}_{n+1}(1,q)=\frac{1}{1+q^{-n-2}}\tilde{U}^{(1)}_{n}(1,q)+\frac{2(1+q)q^{-\frac{(n+2)(n+3)}{2}}}{\prod\limits_{k=0}^{n+2}(1+q^{-k})},\quad n\geqslant 1,$$

and, by Lemma 3, since the last relation is valid for $n=0$, we get

$$\tilde{U}^{(1)}_{n+1}(1,q)=\frac{2(1+q)}{(-1;q^{-1})_{n+3}}\sum_{k=0}^{n+1}q^{-\frac{(k+1)(n+2)}{2}},\quad n\geqslant 0.$$

By virtue of the previous relation and the fact that $\tilde{U}_0^{(1)}(1,q) = 1$, we obtain relation (18). □

Theorem 2. *We have the following connection formulas:*

$$\hat{T}_{n+2}(x,q) = \tilde{U}_{n+2}(x,q) - \frac{q^{2n+3}}{(1+q^{n+1})(1+q^{n+2})}\tilde{U}_n(x,q),\ n \geqslant 0, \tag{19}$$

$$H_q(\hat{T}_{n+1}(x,q)) = \frac{q^{n+1}-1}{q-1}\tilde{U}_n(x,q),\ \ n \geqslant 0, \tag{20}$$

$$(x^2-1)\tilde{U}_n(x,q) = \hat{T}_{n+2}(x,q) + b_n\hat{T}_n(x,q),\ \ n \geqslant 0, \tag{21}$$

$$(x^2-1)H_q(\hat{T}_{n+1}(x,q)) = \frac{q^{n+1}-1}{q-1}\Big\{\hat{T}_{n+2}(x,q) + b_n\hat{T}_n(x,q)\Big\},\ n \geqslant 0, \tag{22}$$

where

$$b_0 = -\frac{1}{1+q},\ b_n = -\frac{1}{(1+q^n)(1+q^{n+1})},\ \ n \geqslant 1.$$

Proof. Based on relation (11), we learn that [2]

$$\hat{T}_{n+2}(x,q) = \tilde{U}_{n+2}(x,q) + a_n\tilde{U}_n(x,q),\ n \geqslant 0,$$

where

$$a_n = -\frac{\tilde{U}_{n+2}(1,q) - \frac{1}{1+q}\tilde{U}_{n+1}^{(1)}(1,q)}{\tilde{U}_n(1,q) - \frac{1}{1+q}\tilde{U}_{n-1}^{(1)}(1,q)},\ \ n \geqslant 0,\ \ \tilde{U}_{-1}(1,q) = 0.$$

By virtue of Lemma 5, we obtain

$$a_n = -\frac{1}{(1+q^{-n-1})(1+q^{-n-2})},\ \ n \geqslant 0.$$

This provides (19).

We know that the sequence $\left\{\dfrac{q-1}{q^{n+1}-1}H_q\hat{T}_{n+1}(x,q)\right\}_{n\geqslant 0}$ is a H_q-classical orthogonal sequence with respect to $\mathcal{T}_q^{[1]}$ [6]. Moreover, by (9) and formula (2.9) in [6, p.58], we get

$$H_q((x^2-q^{-2})\mathcal{T}_q^{[1]}) + \frac{1-q^{-3}}{1-q}x\mathcal{T}_q^{[1]} = 0.$$

Comparing the previous equation with the equation (14), we obtain

$$\mathcal{T}_q^{[1]} = h_{q^{-\frac{1}{2}}}\mathcal{U}_q.$$

Whence, $\tilde{U}_n(x,q) = \frac{q-1}{q^{n+1}-1} H_q \hat{T}_{n+1}(x,q)$, $n \geqslant 0$. Thus, we get (20).

From the functional equation (11), we have [13]

$$(x^2-1)\tilde{U}_n(x,q) = \hat{T}_{n+2}(x,q) + b_n \hat{T}_n(x,q), \quad n \geqslant 0, \tag{23}$$

with $b_n = \dfrac{\langle \mathcal{T}_q, (x^2-1)\tilde{U}_n(x,q)\hat{T}_n(x,q)\rangle}{\langle \mathcal{T}_q, \hat{T}_n^2(x,q)\rangle}, n \geqslant 0.$

Equivalently,

$$b_n = \frac{\langle (x^2-1)\mathcal{T}_q, \tilde{U}_n(x,q)\hat{T}_n(x,q)\rangle}{\langle \mathcal{T}_q, \hat{T}_n^2(x,q)\rangle}, \quad n \geqslant 0,$$

and, by the formula (11), we obtain $b_n = -\dfrac{1}{1+q} \dfrac{\langle \tilde{\mathcal{U}}_q, \tilde{U}_n(x,q)\hat{T}_n(x,q)\rangle}{\langle \mathcal{T}_q, \hat{T}_n^2(x,q)\rangle}$, $n \geqslant 0$. Therefore,

$$b_n = -\frac{1}{1+q} \frac{\langle \tilde{\mathcal{U}}_q, \tilde{U}_n^2(x,q)\rangle}{\langle \mathcal{T}_q, \hat{T}_n^2(x,q)\rangle}, n \geqslant 0. \tag{24}$$

On one hand, we have

$$\langle \mathcal{T}_q, T_n^2(x,q)\rangle = \gamma_n^{\mathcal{T}_q} \langle \mathcal{T}_q, \hat{T}_{n-1}^2(x,q)\rangle, \quad n \geqslant 1.$$

So, $\langle \mathcal{T}_q, \hat{T}_n^2(x,q)\rangle = \prod_{k=1}^{n} \gamma_k^{\mathcal{T}_q}, n \geqslant 1$, and, by (9), we obtain

$$\langle \mathcal{T}_q, \hat{T}_n^2(x,q)\rangle = 4\frac{(1+q^n)q^{\frac{n(n+1)}{2}}}{\left(\prod_{k=0}^{n}(1+q^k)\right)^2}, \quad n \geqslant 1. \tag{25}$$

On the other hand, we may write

$$\langle \tilde{\mathcal{U}}_q, \tilde{U}_n^2(x,q)\rangle = \prod_{k=1}^{n} \gamma_k^{\mathcal{U}_q}, n \geqslant 1.$$

Using relation (10), we get

$$\langle \tilde{\mathcal{U}}_q, \tilde{U}_n^2(x,q)\rangle = 4(1+q)\frac{(1+q^{n+1})q^{\frac{n(n+1)}{2}}}{\left(\prod_{k=0}^{n+1}(1+q^k)\right)^2}, \quad n \geqslant 0. \tag{26}$$

Relations (23), (24), (25), (26) and the fact that $b_0 = -\frac{1}{1+q}$, give (21).

After multiplying both sides of equation (20) by $x^2 - 1$ and using relation (21), we deduce (22). □

Remark. *When $q \to 1$ in equations (19), (20), (21) and, (22) respectively, we meet, again, the formulas given in [10], [16], [17] concerning the classical monic Chebyshev polynomials.*

4. Integral representation of $\mathcal{T}_q$ and $\mathcal{U}_q$.

Theorem 3. *For $f \in \mathcal{P}$, we have*

$$\langle \mathcal{U}_q, f\rangle = K_1^q \int_{-1}^{+1} \frac{(x^2;q^2)_\infty}{(qx^2;q^2)_\infty} f(x)\,dx,\ 0<q<1, \tag{27}$$

$$\langle \mathcal{U}_q, f\rangle = K_2^q \int_{-q^{\frac{1}{2}}}^{+q^{\frac{1}{2}}} \frac{(q^{-1}x^2;q^{-2})_\infty}{(q^{-2}x^2;q^{-2})_\infty} f(x)\,dx,\ q>1, \tag{28}$$

$$\begin{aligned}\langle \mathcal{T}_q, f\rangle = &\frac{1}{2}\Big\{1 - \frac{q^{\frac{1}{2}}K_1^q}{q+1}\int_{-q^{-\frac{1}{2}}}^{+q^{-\frac{1}{2}}} \frac{(qx^2;q^2)_\infty}{(x+1)(q^2x^2;q^2)_\infty}\,dx\Big\} f(-1) +\\ &+\frac{1}{2}\Big\{1 + \frac{q^{\frac{1}{2}}K_1^q}{q+1}\int_{-q^{-\frac{1}{2}}}^{+q^{-\frac{1}{2}}} \frac{(qx^2;q^2)_\infty}{(x-1)(q^2x^2;q^2)_\infty}\,dx\Big\} f(1) +\\ &+\frac{q^{\frac{1}{2}}K_1^q}{q+1}\int_{-q^{-\frac{1}{2}}}^{+q^{-\frac{1}{2}}} \frac{(qx^2;q^2)_\infty}{(1-x^2)(q^2x^2;q^2)_\infty} f(x)\,dx,\ 0<q<1, \end{aligned}\tag{29}$$

$$\begin{aligned}\langle \mathcal{T}_q, f\rangle = &\frac{1}{2}\Big\{1 - \frac{q^{\frac{1}{2}}K_2^q}{q+1}\int_{-1}^{+1} \frac{(x^2;q^{-2})_\infty}{(x+1)(q^{-1}x^2;q_\infty^{-2)}}\,dx\Big\} f(-1) +\\ &+\frac{1}{2}\Big\{1 + \frac{q^{\frac{1}{2}}K_2^q}{q+1}\int_{-1}^{+1} \frac{(x^2;q^{-2})_\infty}{(x-1)(q^{-1}x^2;q^{-2})_\infty}\,dx\Big\} f(1) + \end{aligned}$$

$$+\frac{q^{\frac{1}{2}}K_2^q}{q+1}\int_{-1}^{+1}\frac{(x^2;q^{-2})_\infty}{(1-x^2)(q^{-1}x^2;q^{-2})_\infty}f(x)dx,\ q>1, \quad (30)$$

where

$$(K_1^q)^{-1}=\int_{-1}^{+1}\frac{(x^2;q^2)_\infty}{(qx^2;q^2)_\infty}\,dx,\ (K_2^q)^{-1}=\int_{-q^{\frac{1}{2}}}^{+q^{\frac{1}{2}}}\frac{(q^{-1}x^2;q^{-2})_\infty}{(q^{-2}x^2;q^{-2})_\infty}\,dx. \quad (31)$$

Proof. We need the following formula [11]:

$$(x-a)^{-1}(x-a)w=w-(w)_0\delta_a,\ a\in\mathbb{C},w\in\mathcal{P}'. \quad (32)$$

From the definition of the form $\mathcal{U}_q$, we have

$$\langle\mathcal{U}_q,f\rangle=\big\langle u(\frac{1}{2}),f(q^{-\frac{1}{2}}x)\big\rangle,\ f\in\mathcal{P}.$$

By virtue of Proposition 4.3 in [15], we get

$$\langle\mathcal{U}_q,f\rangle=K_1\int_{-q^{\frac{1}{2}}}^{+q^{\frac{1}{2}}}\frac{(q^{-1}x^2;q^2)_\infty}{(x^2;q^2)_\infty}f(q^{-\frac{1}{2}}x)\,dx,\ 0<q<1, \quad (33)$$

$$\langle\mathcal{U}_q,f\rangle=K_2\int_{-q}^{+q}\frac{(q^{-2}x^2;q^{-2})_\infty}{(q^{-3}x^2;q^{-2})_\infty}f(q^{-\frac{1}{2}}x)\,dx,\ q>1, \quad (34)$$

where K_1 and K_2 are normalization constants.

The change of variable $t=q^{-\frac{1}{2}}x$ in (33) and (34) gives relations (27) and (28), respectively.

Taking into account the functional equation (11), we may write

$$(x-1)^{-1}(x-1)(x+1)\mathcal{T}_q=-\frac{1}{q+1}(x-1)^{-1}h_{q^{-\frac{1}{2}}}\mathcal{U}_q,$$

and, by formula (32) and the fact that $((x+1)\mathcal{T}_q)_0=1$, we get

$$(x+1)\mathcal{T}_q=\delta_1-\frac{1}{q+1}(x-1)^{-1}h_{q^{-\frac{1}{2}}}\mathcal{U}_q.$$

Always by (32), it follows that

$$\mathcal{T}_q = \delta_{-1} + (x+1)^{-1}\delta_1 - \frac{1}{q+1}(x+1)^{-1}(x-1)^{-1}h_{q^{-\frac{1}{2}}}\mathcal{U}_q.$$

But $(x+1)^{-1}\delta_1 = \frac{1}{2}(\delta_1 - \delta_{-1})$; then

$$\mathcal{T}_q = \frac{1}{2}(\delta_{-1} + \delta_1) - \frac{1}{q+1}(x+1)^{-1}(x-1)^{-1}h_{q^{-\frac{1}{2}}}\mathcal{U}_q.$$

This implies for $f \in \mathcal{P}$

$$\langle \mathcal{T}_q, f\rangle = \frac{1}{2}(f(-1) + f(1)) - \frac{1}{q+1}\Lambda(f). \tag{35}$$

where

$$\Lambda(f) = \langle (x+1)^{-1}(x-1)^{-1}h_{q^{-\frac{1}{2}}}\mathcal{U}_q, f\rangle.$$

We may write

$$\begin{aligned}\Lambda(f) &= \langle h_{q^{-\frac{1}{2}}}\mathcal{U}_q, \theta_1\theta_{-1}f\rangle = \\ &= \frac{1}{2}\Big\langle h_{q^{-\frac{1}{2}}}\mathcal{U}_q, \frac{2f(x) - x(f(1) - f(-1)) - f(-1) - f(1)}{x^2-1}\Big\rangle.\end{aligned}$$

Therefore,

$$\Lambda(f) = \frac{1}{2}\Big\langle \mathcal{U}_q, \frac{2f(q^{-\frac{1}{2}}x) - (q^{-\frac{1}{2}}x+1)f(1) + (q^{-\frac{1}{2}}x-1)f(-1)}{q^{-1}x^2-1}\Big\rangle. \tag{36}$$

When $0 < q < 1$, we get, by (27),

$$\begin{aligned}\Lambda(f) &= \frac{1}{2}K_1^q \times \\ &\times \int_{-1}^{+1} \frac{(x^2;q^2)_\infty}{(qx^2;q^2)_\infty} \frac{2f(q^{-\frac{1}{2}}x) - (q^{-\frac{1}{2}}x+1)f(1) + (q^{-\frac{1}{2}}x-1)f(-1))}{q^{-1}x^2-1}\,dx.\end{aligned}$$

Let $t = q^{-\frac{1}{2}}x$, it follows that

$$\Lambda(f) = q^{\frac{1}{2}}K_1^q \int_{-q^{-\frac{1}{2}}}^{+q^{-\frac{1}{2}}} \frac{(qt^2;q^2)_\infty f(x)}{(t^2-1)(q^2t^2;q^2)_\infty}\,dt +$$

$$+\frac{1}{2}q^{\frac{1}{2}}K_1^q\Big(\int_{-q^{-\frac{1}{2}}}^{+q^{-\frac{1}{2}}}\frac{(qt^2;q^2)_\infty}{(t+1)(q^2t^2;q^2)_\infty}\,dt\Big)f(-1)\times$$

$$\times\frac{1}{2}q^{\frac{1}{2}}K_1^q\Big(\int_{-q^{-\frac{1}{2}}}^{+q^{-\frac{1}{2}}}\frac{(qt^2;q^2)_\infty}{(1-t)(q^2t^2;q^2)_\infty}\,dt\Big)f(1).$$

By virtue of the previous relation and (35), we deduce (29).

In the case of $q>1$, we get, by (28) and (36),

$$\Lambda(f)=\frac{1}{2}K_2^q\times$$

$$\times\int_{-q^{\frac{1}{2}}}^{+q^{\frac{1}{2}}}\frac{(q^{-1}x^2;q^{-2})_\infty}{(q^{-2}x^2;q^{-2})_\infty}\frac{2f(q^{-\frac{1}{2}}x)-(q^{-\frac{1}{2}}x+1)f(1)+(q^{-\frac{1}{2}}x-1)f(-1)}{q^{-1}x^2-1}\,dx,$$

Using the change of variable $y=q^{-\frac{1}{2}}x$, we obtain

$$\Lambda(f)=q^{\frac{1}{2}}K_2^q\int_{-1}^{+1}\frac{(y^2;q^{-2})_\infty f(y)}{(y^2-1)(q^{-1}y^2;q^{-2})_\infty}dy+$$

$$+\frac{1}{2}q^{\frac{1}{2}}K_2^q\Big(\int_{-1}^{+1}\frac{(y^2;q^{-2})_\infty}{(y+1)(q^{-1}y^2;q^{-2})_\infty}dy\Big)f(-1)+$$

$$+\frac{1}{2}q^{\frac{1}{2}}K_2^q\Big(\int_{-1}^{+1}\frac{(y^2;q^{-2})_\infty}{(1-y)(q^{-1}y^2;q^{-2})_\infty}dy\Big)f(1).$$

Taking into account the last relation and (35), we get (30). □

Corollary 1. *When $q\to 1$ in representation (29) (respectively, (30)), we obtain the integral representation of $\mathcal{T}$.*

Proof. We need the following relations [2]:

$$\int_{-1}^{+1}\frac{\sqrt{1-x}}{\sqrt{1+x}}dx=\int_{-1}^{+1}\frac{\sqrt{1+x}}{\sqrt{1-x}}dx=\pi. \tag{37}$$

Using relation (7), we obtain, successively:

$$\lim_{q\to 1}\frac{(x^2,q^2)_\infty}{(qx^2,q^2)_\infty}=\lim_{q\to 1}\frac{(qx^2,q^2)_\infty}{(q^2x^2,q^2)_\infty}=\sqrt{1-x^2},\ |x|<1. \tag{38}$$

$$\lim_{q\to 1}\frac{(x^2,q^{-2})_\infty}{(q^{-1}x^2,q^{-2})_\infty}=\lim_{q\to 1}\frac{(q^{-1}x^2,q^{-2})_\infty}{(q^{-2}x^2,q^{-2})_\infty}=\sqrt{1-x^2},\ |x|<1 \tag{39}$$

Based on relations (38) and (2), we get

$$\lim_{q\to 1}K_1^q=\lim_{q\to 1}\Big(\int_{-1}^{+1}\frac{(x^2;q^2)_\infty}{(qx^2;q^2)_\infty}dx\Big)^{-1}=\frac{1}{\int_{-1}^{+1}\sqrt{1-x^2}\,dx}=\frac{2}{\pi}. \tag{40}$$

On one hand, by (38) and (37) we have

$$\lim_{q\to 1}\int_{-q^{-\frac12}}^{+q^{-\frac12}}\frac{(qx^2;q^2)_\infty}{(x+1)(q^2x^2;q^2)_\infty}dx=\int_{-1}^{+1}\frac{\sqrt{1-x}}{\sqrt{1+x}}dx=\pi. \tag{41}$$

On the other hand, by relations (38) and (37), we obtain

$$\lim_{q\to 1}\int_{-q^{-\frac12}}^{+q^{-\frac12}}\frac{(qx^2;q^2)_\infty}{(x-1)(q^2x^2;q^2)_\infty}dx=-\int_{-1}^{+1}\frac{\sqrt{1+x}}{\sqrt{1-x}}dx=-\pi. \tag{42}$$

Using relation (38), we see that

$$\lim_{q\to 1}\int_{-q^{-\frac12}}^{+q^{-\frac12}}\frac{(qx^2;q^2)_\infty f(x)}{(1-x^2)(q^2x^2;q^2)_\infty}\,dx=\int_{-1}^{+1}\frac{f(x)}{\sqrt{1-x^2}}\,dx,\ f\in\mathcal{P}. \tag{43}$$

Taking into account relations (38)–(43) and (30), we get

$$\lim_{q\to 1}\langle\mathcal{T}_q,f\rangle=\frac{1}{\pi}\int_{-1}^{+1}\frac{1}{\sqrt{1-x^2}}f(x)\,dx=\langle\mathcal{T},f\rangle,\ f\in\mathcal{P}.$$

In a similar way, we obtain, from (39):

$$\lim_{q\to 1} K_2^q = \frac{2}{\pi}.$$

Also, by relations (39) and (38), we can deduce, successively,

$$\lim_{q\to 1}\int_{-1}^{+1}\frac{(x^2;q^{-2})_\infty}{(x+1)(q^{-1}x^2;q_\infty^{-2)}}dx=\int_{-1}^{+1}\frac{\sqrt{1-x}}{\sqrt{x+1}}\,dx=\pi. \tag{44}$$

$$\lim_{q\to 1}\int_{-1}^{+1}\frac{(x^2;q^{-2})_\infty}{(x-1)(q^{-1}x^2;q^{-2})_\infty}dx=-\int_{-1}^{+1}\frac{\sqrt{1+x}}{\sqrt{1-x}}dx. \tag{45}$$

By (39), we get

$$\lim_{q\to 1}\int_{-1}^{+1}\frac{(x^2;q^{-2})_\infty}{(1-x^2)(q^{-1}x^2;q^{-2})_\infty}f(x)\,dx=\int_{-1}^{+1}\frac{f(x)}{\sqrt{1-x^2}}dx,\ f\in\mathcal{P}. \tag{46}$$

Based on relations (44), (45), (46), and (31), we obtain

$$\lim_{q\to 1}\langle\mathcal{T}_q,f\rangle=\frac{1}{\pi}\int_{-1}^{+1}\frac{1}{\sqrt{1-x^2}}f(x)\,dx=\langle\mathcal{T},f\rangle,\ f\in\mathcal{P}.$$

Hence, the desired results. □

In the following section, we give explicitly the expression of the Stieltjes functions of the forms $\mathcal{T}_q$ and $\mathcal{U}_q$.

5. The Stieltjes functions of the forms $\mathcal{T}_q$ and $\mathcal{U}_q$.

Lemma 6. *We have*

$$(z^2-q^2)H_{q^{-1}}(S(z,\mathcal{T}_q))=-qzS(z,\mathcal{T}_q), \tag{47}$$

$$(z^2-q)H_{q^{-1}}(S(z,\mathcal{U}_q))=zS(z,\mathcal{U}_q)+q+1. \tag{48}$$

Proof. We need the following formulas [2]:

$$w(1)(x)=1,\ w(\xi)(x)=x,\ w\in\mathcal{P}'\ \text{(symmetric form)}. \tag{49}$$

From the functional equation in (9) and Theorem 2, we have

$$\big(q^{-2}z^2-1\big)H_{q^{-1}}(S(z,\mathcal{T}_q)=-\Big(H_{q^{-1}}(z^2-1))-z\Big)(S(z,\mathcal{T}_q)(z)-$$
$$-H_{q^{-1}}(\mathcal{T}_q\theta_0(x^2-1))(z)-q(\mathcal{T}_q\theta_0)(-q^{-1}x)(z).$$

However, $H_{q^{-1}}(z^2-1))-z=-q^{-1}z$, and with (49), we have

$$-H_{q^{-1}}(\mathcal{T}_q\xi)(z)+(\mathcal{T}_q1)(z)=-H_{q^{-1}}(z)+(\mathcal{T}_q1)(z)=0.$$

Thus, we conclude the relation (47).
By the functional equation in (10) and Theorem 2, we may write

$$\big(q^{-2}z^2-q^{-1}\big)H_{q^{-1}}(S(z,\mathcal{U}_q))=-\Big(H_{q^{-1}}(z^2-q^{-1})+\frac{q-q^{-2}}{1-q}z\Big)(S(z,\mathcal{T}_q)(z)-$$
$$-H_{q^{-1}}(\mathcal{U}_q\theta_0(x^2-q^{-1}))(z)-q(\mathcal{U}_q\theta_0)\Big(\frac{1-q^{-3}}{1-q}x\Big)(z),$$

since $H_{q^{-1}}(z^2-q^{-1})+\frac{q-q^{-2}}{1-q}z=-q^{-2}z$. By (47),

$$-H_{q^{-1}}(\mathcal{U}_q\theta_0(x^2-q^{-1}))(z)-q(\mathcal{U}_q\theta_0)(\frac{1-q^{-3}}{1-q}x)(z)=$$
$$=-H_{q^{-1}}(x)(z)-\frac{q-q^{-2}}{1-q}\mathcal{U}_q(1)(z)=q^{-1}+q^{-2}.$$

Which proves relation (48). □

Theorem 4. *The following formulas hold:*

$$S(z,\mathcal{T}_q)=-\frac{1}{z}\frac{(q^2z^{-2};q^2)_\infty}{(qz^{-2};q^2)_\infty},\ |q|<1,\ z\neq 0, \tag{50}$$

$$S(z,\mathcal{T}_q)=-\frac{1}{z}\frac{(q^{-1}z^{-2};q^{-2})_\infty}{(z^{-2};q^{-2})_\infty},\ |q|>1,\ z\neq 0, \tag{51}$$

$$S(z,\mathcal{U}_q)=(1+q^{-1})\Big\{\frac{z^2-q}{z}\frac{(q^3z^{-2};q^2)_\infty}{(q^2z^{-2};q^2)_\infty}-z\Big\},\ |q|<1,\ z\neq 0, \tag{52}$$

$$S(z,\mathcal{U}_q)=(1+q^{-1})\Big\{\frac{z^2-q}{z}\frac{(z^{-2};q^{-2})_\infty}{(qz^{-2};q^{-2})_\infty}-z\Big\},\ |q|>1,\ z\neq 0. \tag{53}$$

Proof. Equation (47) can be written as follows:

$$S(q^{-1}z,\mathcal{T}_q)=\frac{qz^2-q^2}{z^2-q^2}S(z,\mathcal{T}_q).$$

Therefore, $S\Big(\frac{1}{qz},\mathcal{T}_q\Big)=q\frac{1-qz^2}{1-q^2z^2}S\Big(\frac{1}{z},\mathcal{T}_q\Big),\ \ z\neq 0.$

Let

$$S\Big(\frac{1}{z},\mathcal{T}_q\Big)=zA(z),\ \ z\neq 0. \tag{54}$$

Then

$$A(qz)=\frac{1-qz^2}{1-q^2z^2}A(z),\ z\neq 0. \tag{55}$$

This implies

$$A(z)=\alpha\frac{(q^2z^2,q^2)_\infty}{(qz^2,q^2)_\infty},\ |q|<1,\ z\neq 0,\ \alpha\in\mathbb{C}.$$

By (54), we get

$$S(z,\mathcal{T}_q)=\frac{\alpha}{z}\frac{(q^2z^{-2},q^2)_\infty}{(qz^{-2},q^2)_\infty},\ |q|<1,\ z\neq 0.$$

But $\frac{1}{z}S(\frac{1}{z},\mathcal{T}_q)=\alpha\frac{(q^2z^2,q^2)_\infty}{(qz^2,q^2)_\infty}$, and $\lim_{z\to 0}\frac{1}{z}S(\frac{1}{z},\mathcal{T}_q)=-1$. Then $\alpha=-1$, which provides (50).

From relation (55), we get

$$A(q^{-1}z)=\frac{1-z^2}{1-q^{-1}z^2}A(z),\ z\neq 0.$$

Thus,

$$A(z)=\beta\frac{(q^{-1}z^2;q^{-2})_\infty}{(z^2;q^{-2})_\infty},\ z\neq 0,\ |q|>1,\ \beta\in\mathbb{C},$$

and, by relation (54), it follows that

$$S(z,\mathcal{T}_q)=\frac{\beta}{z}\frac{(q^{-1}z^{-2},q^2)_\infty}{(z^{-2},q^{-2})_\infty},\ |q|>1,\ z\neq 0.$$

Since $\lim_{z\to 0}\frac{1}{z}S(\frac{1}{z},\mathcal{T}_q)=-1$, we obtain $\beta=-1$. So, we get (51).

From the functional equation (11), we get

$$\mathcal{U}_q = -(q+1)h_{q^{\frac{1}{2}}}((x^2-1)\mathcal{T}_q),$$

and by formula (5), it follows that

$$\mathcal{U}_q = -(q+1)((q^{-1}x^2-1)h_{q^{\frac{1}{2}}}\mathcal{T}_q).$$

Then

$$S(z,\mathcal{U}_q) = -(q+1)S(z,(q^{-1}x^2-1)h_{q^{\frac{1}{2}}}\mathcal{T}_q)),$$

and, by (4), we get

$$S(z,\mathcal{U}_q) = -(q+1)(q^{-1}z^2-1)S(z,h_{q^{\frac{1}{2}}}\mathcal{T}_q))- \\ -(q+1)(h_{q^{\frac{1}{2}}}\mathcal{T}_q)\theta_0(q^{-1}x^2-1))(z).$$

But

$$(h_{q^{\frac{1}{2}}}\mathcal{T}_q)\theta_0(q^{-1}x^2-1))(z) = q^{-1}(h_{q^{\frac{1}{2}}}\mathcal{T}_q)(\xi)(x) = q^{-1}x,$$
$$S(z,h_{q^{\frac{1}{2}}}\mathcal{T}_q)) = q^{-\frac{1}{2}}S(q^{-\frac{1}{2}}z,\mathcal{T}_q).$$

Then

$$S(z,\mathcal{U}_q) = -(q^{-1}+1)\Big(q^{\frac{1}{2}}(q^{-1}z^2-1)S(q^{-\frac{1}{2}}z,\mathcal{T}_q)+z\Big).$$

Using relations (50) and (51), we obtain formulas (52) and (53). □

Acknowledgments. I am very grateful to the editors and the referee for the constructive and valuable comments and recommendations.

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Received May 22, 2021.
In revised form, October 12, 2021.
Accepted October 15, 2021.
Published online December 5, 2021.

University Tunis El Manar,
Preparatory Institute for Engineering Studies,
Campus universitaire El Manar-B.P 244,
El Manar II, Tunis 2092, Tunisia
E-mail: manoubi.mejri@issatgb.rnu.tn