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INTEGRAL RESOLVENT FOR VOLTERRA EQUATIONS AND FAVARD SPACES

Abstract. The objective of this work is to give a characterization of the domain $D(A)$ of A in terms of the integral resolvent family of the equation $x(t) = x_0 + \int_0^t a(t-s)\,Ax(s)ds$, $t \geqslant 0$, where A is a linear closed densely defined operator, $a \in L^1_{loc}(\mathbb{R}^+)$ in a general Banach space X and $x_0 \in X$. Furthermore, we give a relationship between the Favard classes (temporal and frequency) for integral resolvents.

Key words: *semigroups, scalar Volterra integral equations, integral resolvent families, Favard spaces*

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1. Introduction, Definitions, and Notations. Let $(X, \|\cdot\|)$ be a Banach space, A a linear closed operator with dense domain $D(A)$, defined in X and $a \in L^1_{loc}(\mathbb{R}^+)$ a scalar kernel. We consider the linear Volterra equation:

$$x(t) = \int_0^t a(t-s)\,Ax(s)ds + f(t), \quad t \geqslant 0, \tag{1}$$

where $f \in \mathcal{C}(\mathbb{R}^+, X)$. Denote by $[D(A)]$ the domain of A equipped with the graph-norm and define the convolution product $*$ of a scalar function a and a vector-valued function f by:

$$(a * f)(t) := \int_0^t a(t-s)\,f(s)\,ds, \quad t \geqslant 0.$$

Definition 1. *[11, Definition 1.1] A function $x \in \mathcal{C}(\mathbb{R}^+, X)$ is called:*

1) *A strong solution of* (1) *if* $x \in \mathcal{C}\left(\mathbb{R}^{+}, [D(A)]\right)$ *and* (1) *is satisfied.*
2) *Mild solution of* (1) *if* $a * x \in \mathcal{C}\left(\mathbb{R}^{+}, [D(A)]\right)$ *and*

$$x = f(t) + A[a * x](t), \quad t \geqslant 0. \tag{2}$$

Obviously, every strong solution of (1) is a mild solution. Conditions under which mild solutions are strong, were studied in [11].

Definition 2. *[7, Definition 2.2] Equation* (1) *is called well-posed if for each* $v \in D(A)$ *there is a unique strong solution* $x(t,v)$ *on* $\mathbb{R}^{+}$ *of*

$$x(t,v) = a(t)v + (a * Ax)(t), \ t \geqslant 0, \tag{3}$$

and for any sequence $(v_n) \subset D(A), v_n \to 0$ *implies* $x(t, v_n) \to 0$ *in* X *uniformly on compact intervals.*

Definition 3. *[7, Definition 2.3] Let* $a \in \mathcal{C}(\mathbb{R}^{+})$ *be a scalar kernel. A strongly continuous family* $(R(t))_{t\geqslant 0} \subset \mathcal{L}(X)$*; (the space of bounded linear operators in* X*) is called an integral resolvent for equation* (1)*, if the following three conditions are satisfied:*

(R1) $R(0) = a(0)I$.

(R2) $R(t)$ *commutes with* A*, which means* $R(t)(D(A)) \subset D(A)$ *for all* $t \geqslant 0$*, and* $AR(t)x = R(t)Ax$ *for all* $x \in D(A)$ *and* $t \geqslant 0$.

(R3) For each $x \in D(A)$ *and all* $t \geqslant 0$ *the resolvent equations holds:*

$$R(t)x = a(t)x + \int_0^t a(t-s)AR(s)x ds.$$

Note that when $a(t) = 1$, $R(t)$ corresponds to a C_0-semigroup.

If there exists an integral resolvent for (1), then a mild solution of (1) may be obtained by the formula

$$x(t) = f(t) + A\int_0^t R(t-s)f(s)ds, \quad t \geqslant 0.$$

In fact, this supposes that $R(t)$ is an integral resolvent for (1); let $f \in \mathcal{C}(\mathbb{R}^{+}, X)$ and $x \in \mathcal{C}(\mathbb{R}^{+}, X)$ be a mild solution for (1). Then $R * f$ is well-defined and continuous and we obtain, from (R3) and (1):

$$a * x = (R - Aa * R) * x = R * x - R * Aa * x = R * f.$$

Hence, $R * f \in \mathcal{C}(\mathbb{R}^+, [D(A)])$, and from (1) we obtain

$$x(t) = f(t) + A \int_0^t R(t-s) f(s) ds, \ t \geqslant 0.$$

The following result establishes the relation between well-posedness and existence of an integral resolvent. In what follows, $\mathcal{R}$ denotes the range of a given operator.

Theorem 1. *[7, Theorem 2.4] Equation (1) is well-posed if and only if (1) admits an integral resolvent $(R(t))_{t\geqslant 0}$. If this is the case, we have, in addition, $\mathcal{R}(a * R(t)) \subset D(A)$ for all $t \geqslant 0$ and*

$$R(t)x = a(t)x + A \int_0^t a(t-s) R(s) x ds, \tag{4}$$

for each $x \in X$, $t \geqslant 0$.

Definition 4. *An integral resolvent $(R(t))_{t\geqslant 0}$ is called exponentially bounded if there exist $M > 0$ and $\omega \in \mathbb{R}$, such that $\|R(t)\| \leqslant Me^{\omega t}$ for all $t \geqslant 0$ and the pair (M, ω) is called the type of $(R(t))_{t\geqslant 0}$.*

The growth bound of $(R(t))_{t\geqslant 0}$ is

$$\omega_0 := \inf\{\omega \in \mathbb{R}, \ \|R(t)\| \leqslant Me^{\omega t}, \ t \geqslant 0, M > 0\};$$

if $\omega_0 < 0$, the integral resolvent is called exponentially stable.

Note that, contrary to the case of the C_0-semigroup, an integral resolvent for (1) does not need to be exponentially bounded (see [3], [11], [7]). However, there are verifiable conditions guaranteeing that (1) possesses an exponentially bounded integral resolvent.

Remark 1. *Note that if $a \in \mathcal{C}^\infty(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ with $a(0) = 1$ and such that $\hat{a}(\lambda)$ admits zeros with arbitrary large real part, the problem (1) cannot admit an exponentially bounded integral resolvent (For more details see [11, Page 45–46]).*

We will use the Laplace transform at times; suppose that $g \colon \mathbb{R}^+ \to X$ is measurable and there exist $M > 0$, $\omega \in \mathbb{R}$, such that $\|g(t)\| \leqslant Me^{\omega t}$ for almost all $t \geqslant 0$; then the Laplace transform

$$\widehat{g}(\lambda) = \int_0^\infty e^{-\lambda t} g(t) dt,$$

exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$.

A function $a \in L^1_{loc}(\mathbb{R}^+)$ is ω (resp. ω^+)-exponentially bounded if $\int_0^\infty e^{-\omega s}|a(s)|\,ds < +\infty$ for some $\omega \in \mathbb{R}$ (resp. $\omega > 0$) (see, e. g [6, p. 113]).

The following proposition, stated in [7, Theorem 3.1], establishes the relation between an integral resolvents and the Laplace transforms:

Proposition 1. *Let $a \in \mathcal{C}(\mathbb{R}^+)$ be ω-exponentially bounded and let $(R(t))_{t\geqslant 0} \subset \mathcal{L}(X)$ be strongly continuous and exponentially bounded, such that the Laplace transform $\widehat{R}(\lambda)$ exists for $\lambda > \omega$. Then $(R(t))_{t\geqslant 0}$ is an integral resolvent of (1) if and only if the following conditions hold:*

1) $\hat{a}(\lambda) \neq 0$ *and* $\dfrac{1}{\hat{a}(\lambda)} \in \rho(A)$, *for all* $\lambda > \omega$, *where* $\rho(A)$ *is the set resolvent of* A.
2) $K(\lambda) := \left(\dfrac{1}{\hat{a}(\lambda)}I - A\right)^{-1}$ *called the resolvent associated to* $R(t)$ *satisfies:*

$$\left(\frac{1}{\hat{a}(\lambda)}I - A\right)^{-1}x = \int_0^\infty e^{-\lambda t}R(t)\,xdt,$$

for all $x \in X$ *and all* $\lambda > \omega$.

Under these assumptions, the Laplace transform of $R(\cdot)$ is well-defined and it is given by $\widehat{R}(\lambda) = K(\lambda)$ for all $\lambda > \omega$.

2. The Domain of A. We have the following characterization of $D(A)$ given in [11]:

Proposition 2. *Let equation (1) admit an integral resolvent family with growth bound ω (such that the Laplace transform of the resolvent exists for $\lambda > \omega$) for ω-exponentially bounded $a \in L^1_{loc}(\mathbb{R}^+)$. Set, for $0 < \theta < \frac{\pi}{2}$ and $\varepsilon > 0$,*

$$\Omega^\epsilon_\theta := \left\{\frac{1}{\widehat{a}(\lambda)} : \quad \operatorname{Re}\lambda > \omega + \varepsilon,\ |\arg\lambda| \leqslant \theta\right\}.$$

Then the following characterization of $D(A)$ holds:

$$D(A) = \left\{x \in X : \quad \lim_{|\mu|\to\infty,\, \mu\in\Omega^0_\theta} \mu A(\mu I - A)^{-1}x \quad \text{exists}\right\}.$$

For more details concerning the proof of this proposition, see [11, Corollary I.1.6] and the proof of the [11, Theorem 1.4].

Without loss of generality, we may assume that $\int_0^t |a(s)|\, ds \neq 0$ for all $t > 0$. Otherwise, we would have, for some $t_0 > 0$, that $a(t) = 0$ for almost all $t \in [0, t_0]$ and, thus, by definition of a integral resolvent, $R(t) = 0$ for $t \in [0, t_0]$. This implies that A is bounded, which is the trivial case with $X = D(A)$.

In what follows, we will use the following assumption on a, such that $\int_0^t |a(s)|\, ds \neq 0$, for all $t > 0$.

Assumption H: There exist $\varepsilon_a > 0$ and $t_a > 0$, such that for all $0 < t \leqslant t_a$, we have:

$$\left| \int_0^t a(t-s)\, a(s)\, ds \right| \geqslant \varepsilon_a \int_0^t |a(s)|\, ds.$$

This is the case for functions a that satisfy $a(I) \subset (0,1]$ at some interval $I = [0, t_a)$.

In fact, if $a(t) \in (0,1]$ for all $t \in (0,1]$, there exists $0 < \varepsilon_a \leqslant 1$ and $t_a > 0$, such that we have $\left| \int_0^t a(t-s)\, a(s)\, ds \right| \geqslant \varepsilon_a \int_0^t |a(s)|\, ds$ for all $0 < t \leqslant t_a$.

For all $t \in (0,1]$, then we have $\int_0^t a(s)\, ds \leqslant \int_0^1 a(s)\, ds \leqslant 1$. So,

$$\int_0^t a(s)\, ds \leqslant \frac{1}{\varepsilon_a} \int_0^t a(t-s)\, a(s)\, ds.$$

It is necessary and sufficient that

$$1 \leqslant \frac{1}{\varepsilon_a} \int_0^t a(t-s)\, a(s)\, ds.$$

Then we have

$$\int_0^t a(t-s)\, a(s)\, ds \geqslant \varepsilon_a.$$

Thus, $\frac{1}{\varepsilon_a}\int\limits_0^t a(t-s)\,a(s)\,ds \geqslant 1 \geqslant \int\limits_0^t a(s)\,ds$.

Hence, there exist $\varepsilon_a > 0$ and $t_a > 0$, such that we have, for all $0 < t \leqslant t_a$:

$$\Big|\int\limits_0^t a(t-s)\,a(s)\,ds\Big| \geqslant \varepsilon_a \int\limits_0^t |a(s)|\,ds.$$

For almost all reasonable functions in applications, it is easy to see that they satisfy this assumption. There are, nonetheless, examples of functions that do not.

Define the set $\widetilde{D}(A)$ as follows

$$\widetilde{D}(A) = \{x \in X\colon \quad \lim_{t\to 0^+} \frac{R(t)\,x - a(t)\,x}{(a*a)(t)} \quad \text{exists}\}.$$

If $|a(t)|$ is continuous and nondescreasing and

$$\limsup_{t\to 0^+} \frac{\|R(t)\|}{|a(t)|} < \infty,$$

then

$$D(A) = \{x \in X\colon \quad \lim_{t\to 0^+} \frac{R(t)\,x - a(t)\,x}{(a*a)(t)} = Ax\}. \tag{5}$$

This was proved in [8], [7]. Note that in the case of semigroups: $a(t) = 1$ then $(a*a)(t) = t$ and in the case of cosine families: $a(t) = t$ then $(a*a)(t) = \frac{t^2}{2}$, we have the well-known result that $D(A) = \widetilde{D}(A)$ (see [12], [4]). In what follows, we prove that this is the case in general.

Theorem 2. *Under Assumption H, we have $\widetilde{D}(A) = D(A)$ and*

$$\lim_{t\to 0^+} \frac{R(t)\,x - a(t)\,x}{(a*a)(t)} = Ax.$$

Proof. Let $x \in D(A)$ and let there exist $\varepsilon_a > 0$ and $t_a > 0$, such that for all $0 < t \leqslant t_a$ we have

$$\Big|\int\limits_0^t a(t-s)\,a(s)\,ds\Big| \geqslant \varepsilon_a \int\limits_0^t |a(s)|\,ds.$$

For an arbitrary $\varepsilon > 0$, let $t \in (0, t_a]$, such that $\|R(s)Ax - a(s)Ax\| < \varepsilon$ for all $s \in [0, t]$ due to the fact

$$\lim_{s \to 0^+} \|R(s) - a(s) I\| = 0$$

by the strong continuity of $R(\cdot)$ and $a(\cdot)$ and the property (R1). Then

$$\begin{aligned}
&\left\| \frac{R(t)x - a(t)x}{(a*a)(t)} - Ax \right\| = \\
&= \left\| \frac{1}{(a*a)(t)} \left(R(t)x - a(t)x - (a*a)(t)Ax\right) \right\| = \\
&= \left\| \frac{1}{(a*a)(t)} \Big(\int_0^t a(t-s)AR(s)xds - \int_0^t a(t-s)a(s)Axds \Big) \right\| = \\
&= \left\| \frac{1}{(a*a)(t)} \Big(\int_0^t a(t-s)R(s)Axds - \int_0^t a(t-s)a(s)Axds \Big) \right\| = \\
&= \left\| \frac{1}{(a*a)(t)} \Big(\int_0^t a(t-s)(R(s) - a(s))Axds \Big) \right\| \leqslant \\
&\leqslant \frac{1}{|(a*a)(t)|} \int_0^t |a(t-s)| \, \|(R(s) - a(s))Ax\| \, ds \leqslant \\
&\leqslant \frac{\varepsilon}{|(a*a)(t)|} \int_0^t |a(t-s)| \, ds = \\
&= \frac{\varepsilon}{|(a*a)(t)|} \int_0^t |a(s)| \, ds \leqslant \frac{\varepsilon}{\varepsilon_a} < \infty,
\end{aligned}$$

due to the properties (R2), (R3) and Assumption H. This shows that $D(A) \subset \widetilde{D}(A)$.

Conversely, let $x \in \widetilde{D}(A)$ and let

$$\lim_{t \to 0^+} \frac{R(t)x - a(t)x}{(a*a)(t)} = y.$$

Under Assumption H, we have

$$\left\|\frac{1}{(a*a)\,(t)}\int\limits_0^t a\,(t-s)\,R(s)xds - x\right\| =$$

$$= \left\|\frac{1}{(a*a)\,(t)}\Big(\int\limits_0^t a\,(t-s)\,R(s)xds - \int\limits_0^t a\,(t-s)\,a\,(s)\,xds\Big)\right\| \leqslant$$

$$\leqslant \frac{1}{|(a*a)\,(t)|}\int\limits_0^t |a\,(t-s)|\,\|(R(s)-a\,(s))\,x\|\,ds \leqslant$$

$$\leqslant \frac{1}{|(a*a)\,(t)|}\int\limits_0^t |a\,(s)|\,\varepsilon ds \leqslant \frac{\varepsilon}{\varepsilon_a}.$$

Then $\frac{1}{(a*a)(t)}\int\limits_0^t a\,(t-s)\,R(s)xds \to x$ as $t \to 0^+$. On the other hand, by Theorem 1, we have $\int\limits_0^t a\,(t-s)\,R(s)xds \in D\,(A)$. Since A is closed, we obtain, by

$$\lim_{t\to 0^+}\frac{1}{(a*a)\,(t)}\int\limits_0^t a\,(t-s)\,R(s)xds = x,$$

the following:

$$\lim_{t\to 0^+}\frac{1}{(a*a)(t)}A\int\limits_0^t a(t-s)R(s)xds = \lim_{t\to 0^+}\frac{1}{(a*a)(t)}(R(t)x-a(t)x) = y.$$

Finally, $y = Ax$. Hence, $x \in D\,(A)$ and

$$\lim_{t\to 0^+}\frac{R\,(t)\,x - a\,(t)\,x}{(a*a)\,(t)} = Ax.$$

□

From now on and in view of this result, we say that the pair (A, a) is a generator of an integral resolvent $(R\,(t))_{t\geqslant 0}$.

3. The Favard class of A. The following definition, which corresponds to a natural extension, in our context, of the Favard class, is frequently used in the approximation theory for semigroups and resolvent families (see e. g., [10], [4], [7], [1], [2], [9]).

Definition 5. *Let equation (1) admit a bounded integral resolvent $(R(t))_{t\geqslant 0}$ on X, for an ω^+-exponentially bounded $a \in L^1_{loc}(\mathbb{R}^+)$. We define the Favard spaces (frequency and temporal) associated to (A, a) as follows:*

$$F_{a,A} := \left\{ x \in X / \sup_{\lambda>\omega} \left\| \frac{1}{\hat{a}(\lambda)} A \Big(\frac{1}{\hat{a}(\lambda)} I - A \Big)^{-1} x \right\| < \infty \right\} = \\ = \left\{ x \in X / \sup_{\lambda>\omega} \left\| \frac{1}{\hat{a}(\lambda)} AK(\lambda)\, x \right\| < \infty \right\}.$$

and

$$\widetilde{F}_{a,A} := \left\{ x \in X / \sup_{0<t\leqslant 1} \frac{\| R(t)\, x - a(t)\, x \|}{|(a * a)(t)|} < \infty \right\}.$$

Remark 2. *The Favard class of A with kernel $a(t)$ can be alternatively defined as the subspace of X given by*

$$\left\{ x \in X / \limsup_{\lambda\to\infty} \left\| \frac{1}{\hat{a}(\lambda)} A \left(\frac{1}{\hat{a}(\lambda)} I - A \right)^{-1} x \right\| < \infty \right\}.$$

We prove that $F_{a,A}$ is stable by $R(t)$ for any scalar kernel a.

Proposition 3. *Let equation (1) admit a bounded integral resolvent $(R(t))_{t\geqslant 0}$ on X, for an ω^+-exponentially bounded $a \in L^1_{loc}(\mathbb{R}^+)$. We have $R(t)(F_{a,A}) \subset F_{a,A}$, for all $t \geqslant 0$.*

Proof. For all $x \in D(A)$ and $t \geqslant 0$, from $(R2)$ we have:

$$AR(t)x = R(t)Ax,$$

and, by [5, Theorem 7], $((\mu I - A)^{-1}$ commutes with $R(t)$ for all $\mu \in \rho(A))$.

Since $R(t)$ is bounded, then, under Proposition 1, the following holds for all $\lambda > \omega$:

$$\hat{a}(\lambda) \neq 0 \text{ and } \frac{1}{\hat{a}(\lambda)} \in \rho(A).$$

Hence, we have

$$\left(\frac{1}{\hat{a}(\lambda)} I - A \right)^{-1} R(t) = R(t) \left(\frac{1}{\hat{a}(\lambda)} I - A \right)^{-1}. \tag{6}$$

Now, if $x \in F_{a,A}$, then

$$\sup_{\lambda>\omega}\left\|\frac{1}{\hat{a}(\lambda)}A\Big(\frac{1}{\widehat{a}(\lambda)}I-A\Big)^{-1}x\right\|<\infty,$$

by (6), (R2) and the boundedness of $R(t)$. We have:

$$\sup_{\lambda>\omega}\left\|\frac{1}{\hat{a}(\lambda)}A\Big(\frac{1}{\widehat{a}(\lambda)}I-A\Big)^{-1}R(t)\,x\right\|\leqslant$$
$$\leqslant\|R(t)\|\times\sup_{\lambda>\omega}\left\|\frac{1}{\hat{a}(\lambda)}A\Big(\frac{1}{\widehat{a}(\lambda)}I-A\Big)^{-1}x\right\|<+\infty.$$

Then $R(t)\,x \in F_{a,A}$ for all $t \geqslant 0$; hence, we deduce that $R(t)\,(F_{a,A}) \subset F_{a,A}$ for all $t \geqslant 0$. □

The proof of the following proposition is immediate.

Proposition 4. *The Favard classes of A with kernel $a(t)$, $F_{a,A}$, and $\widetilde{F}_{a,A}$ are Banach spaces with respect to the norms*

$$\|x\|_{F_{a,A}}:=\|x\|+\sup_{\lambda>\omega}\left\|\frac{1}{\hat{a}(\lambda)}A\left(\frac{1}{\hat{a}(\lambda)}I-A\right)^{-1}x\right\|$$

and

$$\|x\|_{\widetilde{F}_{a,A}}:=\|x\|+\sup_{0<t\leqslant 1}\frac{\|R(t)\,x-a(t)\,x\|}{|(a*a)(t)|},$$

respectively.

Now we will prove that $F_{a,A}=\widetilde{F}_{a,A}$ holds for all $a \in L^1_{loc}(\mathbb{R}^+)$ satisfying Assumption H.

Theorem 3. *Let equation (1) admit a bounded integral resolvent family $(R(t))_{t\geqslant 0}$ on X, for an ω^+-exponentially bounded $a \in L^1_{loc}(\mathbb{R}^+)$, and suppose that Assumption H holds. Then*

$$F_{a,A}=\widetilde{F}_{a,A}.$$

Proof. Let Assumption H hold: there exist $\varepsilon_a > 0$ and $t_a > 0$, such that for all $0 < t \leqslant t_a$:

$$\Big|\int_0^t a(t-s)\,a(s)\,ds\Big|\geqslant\varepsilon_a\int_0^t|a(s)|\,ds.$$

Take $x \in F_{a,A}$; let $\|R(s)\| \leqslant M$ for some $M > 0$ and all $s \in [0,t]$, where $0 < t \leqslant t_a$ and $t_a \geqslant 1$. We have

$$|(a * a)(t)| \geqslant \varepsilon_a (1 * |a|)(t).$$

Then $\dfrac{(1 * |a|)(t)}{|(a * a)(t)|} \leqslant \dfrac{1}{\varepsilon_a}$. By Theorem 1, Remark 2, under the stability of $F_{a,A}$ by $R(t)$:

$$\frac{\|R(t)x - a(t)x\|}{|(a*a)(t)|} = \frac{1}{|(a*a)(t)|}\left\|A\int_0^t a(t-s)R(s)x\,ds\right\| \leqslant$$

$$\leqslant \frac{1}{|(a*a)(t)|}\limsup_{\lambda\to+\infty}\left\|\int_0^t a(t-s)R(s)\frac{1}{\hat{a}(\lambda)}A\Big(\frac{1}{\hat{a}(\lambda)}I - A\Big)^{-1}x\,ds\right\| \leqslant$$

$$\leqslant \frac{1}{|(a*a)(t)|}\left|\int_0^t a(s)\,ds\right|\limsup_{\lambda\to+\infty}\left\|\frac{1}{\hat{a}(\lambda)}A\Big(\frac{1}{\hat{a}(\lambda)}I - A\Big)^{-1}R(s)x\right\| \leqslant$$

$$\leqslant \frac{(1*|a|)(t)}{|(a*a)(t)|}\|R(s)x\|_{F_{a,A}} \leqslant \frac{M}{\varepsilon_a}\|x\|_{F_{a,A}}.$$

Hence, we obtain $x \in \widetilde{F}_{a,A}$.

Conversely, let $x \in \widetilde{F}_{a,A}$ and set

$$\sup_{0<t\leqslant 1}\frac{\|R(t)x - a(t)x\|}{|(a*a)(t)|} := J_x < \infty.$$

Write

$$\frac{1}{\hat{a}(\lambda)}A\Big(\frac{1}{\hat{a}(\lambda)}I - A\Big)^{-1} = \frac{1}{\hat{a}(\lambda)}AK(\lambda),$$

for all $\lambda > \omega$.

Using the integral representation of the resolvent (see Proposition 1), we obtain:

$$\frac{1}{\hat{a}(\lambda)}AK(\lambda)x = \frac{1}{(\widehat{a}(\lambda))^2}K(\lambda)x - \frac{1}{\hat{a}(\lambda)}x =$$

$$= \frac{K(\lambda)x - \hat{a}(\lambda)x}{\widehat{a*a}(\lambda)} = \int_0^\infty e^{-\lambda s}\frac{R(s)x - a(s)x}{(a*a)(s)}ds.$$

Since $R(t)$ is bounded, by Theorem 1, and under Assumption H, we have:

$$\left\|\frac{1}{\hat{a}(\lambda)}AK(\lambda)x\right\| \leqslant \left|\int_0^\infty e^{-\lambda s}ds\right| \sup_{t>0}\frac{\|R(t)x - a(t)x\|}{|(a*a)(t)|} \leqslant$$

$$\leqslant \frac{1}{\lambda}\Big(\sup_{t\geqslant 1}\frac{\left\|A\int_0^t a(t-s)R(s)x\right\|}{|(a*a)(t)|} + J_x\Big) \leqslant \frac{1}{\omega}\Big(\frac{L\|x\|}{\varepsilon_a} + J_x\Big) < \infty$$

with $L = \sup_{t\geqslant 1}\|AR(t)\|$. This implies $\sup_{\lambda>\omega}\left\|\frac{1}{\hat{a}(\lambda)}AK(\lambda)x\right\| < \infty$, which ends the proof. $\square$

Example 1. When $a(t) = 1$, we recall that $(R(t))_{t\geqslant 0}$ corresponds to a bounded C_0-semigroup generated by A. In this situation, we obtain

$$F_{1,A} = \Big\{x \in X/ \sup_{\lambda>0}\left\|\lambda A(\lambda I - A)^{-1}x\right\| < \infty\Big\}$$

and

$$\widetilde{F}_{1,A} = \Big\{x \in X/ \sup_{t>0}\frac{\|\mathbb{T}(t)x - x\|}{t} < \infty\Big\}$$

and we have $F_{1,A} = \widetilde{F}_{1,A}$. This case is well-known (see e. g., [4]).

Example 2. Let $a(t) = b + ct$ with $b > 0$ and $c > 0$ with $b + c < 1$, satisfying Assumption H; let equation (1) correspond to a solid in the Kelvin-Voigt model (see [11, Page 131]). Since $\hat{a}(\lambda) = \frac{b}{\lambda} + \frac{c\Gamma(2)}{\lambda^2}$, where Γ denotes the Gamma function, and since $(a*a)(t) = b^2t + bt^2 + \frac{c^2}{6}t^3$; then in this situation we obtain

$$F_{a,A} = \Big\{x \in X/ \sup_{\lambda>0}\left\|A\Big(I - \Big(\frac{b}{\lambda} + \frac{c\Gamma(2)}{\lambda^2}\Big)A\Big)^{-1}x\right\| < \infty\Big\}$$

and

$$\widetilde{F}_{a,A} := \Big\{x \in X/ \sup_{0<t\leqslant 1}\frac{\|R(t)x - (b+ct)x\|}{b^2t + bt^2 + \frac{c^2}{6}t^3} < \infty\Big\}.$$

And we have $F_{a,A} = \widetilde{F}_{a,A}$ thanks to Theorem 3.

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