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ABOUT ONE PROBLEM ON EXTREMAL DECOMPOSITION

Abstract. In the paper, we consider an open problem of finding the maximum of product of inner radii of mutually non-overlapping domains with respect to the points of the unit circle on a certain positive degree γ of the inner radius of the domain with respect to the origin, moreover, the domain containing origin does not intersect with other domains.

Key words: *inner radius of the domain, mutually non-overlapping domains, the Green function, quadratic differential, the Dubinin problem*

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1. Preliminaries. Let $\mathbb{N}$, $\mathbb{C}$ be the sets of natural and complex numbers, respectively, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be its one-point compactification, $\mathbb{R}^+ = (0, \infty)$. Let B be a domain in $\overline{\mathbb{C}}$. Let

$$g_B(z, a) = h_{B,a} + \log \frac{1}{|z-a|} + o(1)$$

be the Green function of the domain B with respect to a point $a \in B$. Quantity $r(B, a) := \exp(h_{B,a})$ is called an inner radius of the domain $B \subset \overline{\mathbb{C}}$ with respect to a point $a \in B$ (see, for example, [6], [9], [11], [15]).

Let $n \in \mathbb{N}$, $n \geqslant 2$. A set of points $A_n := \{a_k \in \mathbb{C} : k = \overline{1, n}\}$ is called an n-radial system if $|a_k| \in \mathbb{R}^+$ and $0 = \arg a_1 < \ldots < \arg a_n < 2\pi$. Denote the numbers α_k, $k = \overline{1, n}$ as follows: $\alpha_1 := \frac{1}{\pi}(\arg a_2 - \arg a_1)$, $\alpha_2 := \frac{1}{\pi}(\arg a_3 - \arg a_2)$, ..., $\alpha_n := \frac{1}{\pi}(2\pi - \arg a_n)$. Obviously, $\sum\limits_{k=1}^{n} \alpha_k = 2$. Let $\alpha_0 = \max\limits_k \alpha_k$.

Consider the following problem, which was formulated in [9], [10] in the list of unsolved problems.

The Dubinin Problem. Prove that the maximum of the functional

$$I_n(\gamma) = r^{\gamma}(B_0, a_0) \prod_{k=1}^{n} r(B_k, a_k),$$

where B_0, B_1, B_2, ..., B_n, $(n \geqslant 2)$ are pairwise non-overlapping domains in $\overline{\mathbb{C}}$, $a_0 = 0$, $|a_k| = 1$, $k = \overline{1,n}$, $a_k \in B_k$, $k = \overline{0,n}$ and $\gamma \leqslant n$, is attained at a configuration of domains B_k and points a_k possessing rotational n-symmetry.

In the paper [9] the above-formulated problem was solved for $\gamma = 1$ and all values of the natural parameter $n \geqslant 2$. Namely, it was shown that the following inequality holds:

$$r(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k) \leqslant r\left(D_0, 0\right) \prod_{k=1}^{n} r\left(D_k, d_k\right),$$

where d_k, D_k, $k = \overline{0,n}$, are the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-1)w^n + 1}{w^2(w^n-1)^2}\, dw^2.$$

In the work [14], Kovalev got the solution for systems of points for which the following inequalities hold:

$$|a_k| = 1, \quad 0 < \alpha_k \leqslant 2/\sqrt{\gamma}, \quad k = \overline{1,n}, \quad n \geqslant 5.$$

In the work [3], it was shown that the result by Kovalev is also true for $n = 4$. In the monograph [2] the problem was solved for an arbitrary $\gamma > 1$ but starting from some previously unknown number n. The next step was to study this problem at the restrictions $1 < \gamma \leqslant n^{\delta}$, where $0 < \delta < 1$ (see, for example [7], [8], [18–20]).

In the paper [5], the authors obtained its complete solution for $n = 2$. Note that the result of [5] is a consequence of the well-known theorem of Kolbina [13].

Let

$$I_n^0(\gamma) := r^{\gamma}(D_0, 0) \prod_{k=1}^{n} r(D_k, d_k),$$

where d_k, D_k, $k = \overline{0,n}$, $d_0 = 0$, be, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-\gamma)w^n + \gamma}{w^2(w^n-1)^2}\, dw^2. \tag{1}$$

As has been shown in Theorem 5.2.3 [2], the quantity $I_n^0(\gamma)$ takes the form

$$I_n^0(\gamma) = \left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \left(\frac{1-\frac{\sqrt{\gamma}}{n}}{1+\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}. \tag{2}$$

Theorem 1. [5] *Let $\gamma \in (1, 2]$. Then, for any different points a_1 and a_2 of the unit circle and any pairwise non-overlapping domains B_0, B_1, B_2, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_1 \in B_1 \subset \overline{\mathbb{C}}$, $a_2 \in B_2 \subset \overline{\mathbb{C}}$, the following inequality holds:*

$$r^\gamma\left(B_0, 0\right) r\left(B_1, a_1\right) r\left(B_2, a_2\right) \leqslant I_2^0(\gamma)\left(\frac{1}{2}\left|a_1 - a_2\right|\right)^{2-\gamma}.$$

The sign of equality is attained when the points a_0, a_1, a_2 and the domains B_0, B_1, B_2 are, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(4-\gamma)w^2+\gamma}{w^2(w^2-1)^2}dw^2.$$

Besides, in the paper [5] the solution for $n \geqslant 3$ and $\gamma \in (1, \sqrt{n}\,]$ is obtained using an upper estimate for the functional $I_n(\gamma)$ [4].

Theorem 2. [5] *Let $n \in \mathbb{N}$, $n \geqslant 3$, $\gamma \in (1, \sqrt{n}\,]$. Then, for any system of different points $A_n = \{a_k\}_{k=1}^n$ of a unit circle and for any collection of mutually non-overlapping domains B_0, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1, n}$, the following inequality holds:*

$$r^\gamma\left(B_0, 0\right) \prod_{k=1}^n r\left(B_k, a_k\right) \leqslant r^\gamma\left(D_0, 0\right) \prod_{k=1}^n r\left(D_k, d_k\right),$$

where d_k, D_k, $k = \overline{0, n}$, $d_0 = 0$, are, respectively, the poles and circular domains of the quadratic differential (1).

Also, in [5] the following upper estimate for $I_n(\gamma)$ is proved for $n \geqslant 3$ and $\gamma \in (1, n]$:

Theorem 3. [5] *Let $n \in \mathbb{N}$, $n \geqslant 3$, $\gamma \in (1, n]$. Then, for any system of different points $A_n = \{a_k\}_{k=1}^n$ of a unit circle and for any collection of mutually non-overlapping domains B_0, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1, n}$, the inequality*

$$r^\gamma(B_0, 0) \prod_{k=1}^n r(B_k, a_k) \leqslant \left(\sin\frac{\pi}{n}\right)^{n-\gamma} \left(I_2^0\left(\frac{2\gamma}{n}\right)\right)^{\frac{n}{2}}$$

holds.

In deriving our results, the following estimates specified below are required:

Lemma 1. [1] *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma > 0$. Let $\{B_0, B_1, B_2, \ldots, B_n\}$ be a system of mutually non-overlapping simply connected domains, such that $0 \in B_0 \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $|a_k| = 1$, $k = \overline{1,n}$, and $r^{\gamma}(B_0, 0) \prod\limits_{k=1}^{n} r(B_k, a_k) > I_n^0(\gamma)$. Then the following inequality holds:*

$$r\left(B_0, 0\right) \leqslant n^{-\frac{n}{2(n-\gamma)}}\, I_n^0(\gamma)^{-\frac{1}{n-\gamma}}.$$

Lemma 2. [4] *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in (0, n]$. Then for any system of different fixed points $A_n = \{a_k\}_{k=1}^{n} \subset \mathbb{C} \setminus \{0\}$, such that $\prod\limits_{k=1}^{n} |a_k| \leqslant 1$, and for any collection of mutually non-overlapping domains $\{B_0, B_1, B_2, \ldots, B_n\}$, such that $a_0 = 0$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, the following inequality holds:*

$$I_n(\gamma) \leqslant n^{-\frac{\gamma}{2}} \left(I_n(0)\right)^{1-\frac{\gamma}{n}}. \tag{3}$$

According to the condition of the problem $a_0 = 0$, $|a_k| = 1$, $k = \overline{1,n}$, further, we assume without loss of generality that $0 = \arg a_1 < \arg a_2 < \ldots < \arg a_n < 2\pi$. Since in [14] the problem was solved under conditions $0 < \alpha_k \leqslant 2/\sqrt{\gamma}$, $k = \overline{1,n}$, $n \geqslant 5$, then, for a given n, it is sufficient to consider only configurations of domains B_k and points a_k for which $\alpha_0 > \frac{2}{\sqrt{\gamma}}$. The results of this paper are addendum to the theorem of the work [14].

2. Main results. In this work, we prove the following propositions.

Theorem 4. *Let $n \in \mathbb{N}$, $n \geqslant 24$ and $1 < \gamma \leqslant n^{\frac{2}{3} - \frac{2}{3}\frac{\ln(\ln(n))}{\ln(n)}}$. Then the following inequality holds:*

$$r^{\gamma}(B_0, a_0) \prod_{k=1}^{n} r(B_k, a_k) \leqslant \left(\frac{4}{n}\right)^{n} \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1 - \frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \left(\frac{1 - \frac{\sqrt{\gamma}}{n}}{1 + \frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}},$$

where B_k, $k = \overline{0,n}$, are mutually non-overlapping simply connected domains in $\overline{\mathbb{C}}$, $a_0 = 0$, $|a_k| = 1$, $k = \overline{1,n}$, and the equality is attained, in particular, for points a_k and domains B_k that are, respectively, the poles and circular domains of the quadratic differential (1).

Proof. Using Lemma 1 and the result of the paper [14], consider the case $\alpha_0 > \frac{2}{\sqrt{\gamma}}$ and $r\left(B_0, 0\right) \leqslant n^{-\frac{n}{2(n-\gamma)}}\, I_n^0(\gamma)^{-\frac{1}{n-\gamma}}$. Prove that under these conditions

$$\frac{r^{\gamma}(B_0,a_0)\prod\limits_{k=1}^{n} r(B_k,a_k)}{\left(\frac{4}{n}\right)^{n}\frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}}\left(\frac{1-\frac{\sqrt{\gamma}}{n}}{1+\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}} < 1.$$

Further, from Lemma 1 $r^{\gamma}(B_0,a_0) \leqslant n^{-\frac{n\gamma}{2(n-\gamma)}}\, I_n^0(\gamma)^{-\frac{\gamma}{n-\gamma}}$. Then, using Theorem 5.2.3 [2], the following estimates hold:

$$\prod_{k=1}^{n} r(B_k,a_k) \leqslant 2^n \prod_{k=1}^{n}\alpha_k \leqslant 2^n\alpha_0\left(\frac{2-\alpha_0}{n-1}\right)^{n-1} < \\ < \frac{4^n}{(n-1)^{n-1}\sqrt{\gamma}}\left(1-\frac{1}{\sqrt{\gamma}}\right)^{n-1}.$$

And, thus, we obtain the inequality

$$\frac{I_n(\gamma)}{I_n^0(\gamma)} \leqslant \frac{\frac{4^n}{(n-1)^{n-1}\sqrt{\gamma}}\left(1-\frac{1}{\sqrt{\gamma}}\right)^{n-1}}{n^{\frac{n\gamma}{2(n-\gamma)}}\, I_n^0(\gamma)^{\frac{n}{n-\gamma}}} := G_n(\gamma).$$

Combining the previous inequality and inequality (2), we get

$$G_n(\gamma) = n^{\frac{n\gamma+2n+2\gamma}{2(n-\gamma)}}\left(\frac{n}{n-1}\right)^{n-1}\left(1-\frac{1}{\sqrt{\gamma}}\right)^{n-1}\left(1-\frac{\gamma}{n^2}\right)^{\frac{n^2+\gamma}{n-\gamma}}\times \\ \times\left(\frac{1}{4\sqrt{\gamma}}\right)^{\frac{n+\gamma}{n-\gamma}}\left(\frac{1+\frac{\sqrt{\gamma}}{n}}{1-\frac{\sqrt{\gamma}}{n}}\right)^{\frac{2n\sqrt{\gamma}}{n-\gamma}}.$$

Note that in order to prove Theorem 4, we need to show that $G_n(\gamma) < 1$ for given n and γ.

Evaluate the expression $G_n(\gamma)$ under the conditions of the theorem. It is not difficult to show that $\left(\frac{n}{n-1}\right)^{n-1} < e$ and $\left(1-\frac{\gamma}{n^2}\right)^{\frac{n^2+\gamma}{n-\gamma}} < 1$. Also, the assessment $\left(\frac{1+\frac{\sqrt{\gamma}}{n}}{1-\frac{\sqrt{\gamma}}{n}}\right)^{\frac{2n\sqrt{\gamma}}{n-\gamma}}\left(\frac{1}{4\sqrt{\gamma}}\right)^{\frac{n+\gamma}{n-\gamma}} < 0{,}06 < \frac{1}{e}$ is correct. Accordingly,

$$G_n(\gamma) < n^{\frac{n\gamma+2n+2\gamma}{2(n-\gamma)}}\left(1-\frac{1}{\sqrt{\gamma}}\right)^{n-1}. \tag{4}$$

Evaluate the expression $n^{\frac{n\gamma+2n+2\gamma}{2(n-\gamma)}}\left(1-\frac{1}{\sqrt{\gamma}}\right)^{n-1}$. We get the following transformations:

$$n^{\frac{n\gamma+2n+2\gamma}{2(n-\gamma)}}\left(1-\frac{1}{\sqrt{\gamma}}\right)^{n-1}=n^{\frac{n\gamma+2n+2\gamma}{2(n-\gamma)}}\left(\left(1-\frac{1}{\sqrt{\gamma}}\right)^{\sqrt{\gamma}}\right)^{\frac{n-1}{\sqrt{\gamma}}}<$$

$$<\left(n\left(\frac{1}{e}\right)^{\frac{2n^2-2n-2n\gamma+2\gamma}{n\gamma\sqrt{\gamma}+2n\sqrt{\gamma}+2\gamma\sqrt{\gamma}}}\right)^{\frac{n\gamma+2n+2\gamma}{2(n-\gamma)}}=\left(n\left(\frac{1}{e}\right)^{\frac{n}{\gamma^{\frac{3}{2}}}\frac{2-\frac{2}{n}-\frac{2\gamma}{n}+\frac{2\gamma}{n^2}}{1+\frac{2}{\gamma}+\frac{2}{n}}}\right)^{\frac{n\gamma+2n+2\gamma}{2(n-\gamma)}}.$$

Since in the paper [18] the problem is solved for $n \geqslant 12$ and $1 < \gamma \leqslant n^{0,45}$, it is enough to consider only $\gamma > n^{0,45}$.

For $n \geqslant 24$ and $n^{0,45} < \gamma < n^{\frac{2}{3}-\frac{2}{3}\frac{\ln(\ln(n))}{\ln(n)}}$, the inequality

$$\frac{2-\frac{2}{n}-\frac{2\gamma}{n}+\frac{2\gamma}{n^2}}{1+\frac{2}{\gamma}+\frac{2}{n}}>1$$

is satisfied. So, we have

$$n\left(\frac{1}{e}\right)^{\frac{n}{\gamma^{\frac{3}{2}}}\frac{2-\frac{2}{n}-\frac{2\gamma}{n}+\frac{2\gamma}{n^2}}{1+\frac{2}{\gamma}+\frac{2}{n}}}<n\left(\frac{1}{e}\right)^{\frac{n}{\gamma^{\frac{3}{2}}}}<n\left(\frac{1}{e}\right)^{n^{\frac{\ln(\ln(n))}{\ln(n)}}}=1.$$

Thus, from inequality (4) we obtain that for n and γ given in Theorem 4 $G_n(\gamma) < 1$, which means that

$$r^{\gamma}(B_0,a_0)\prod_{k=1}^{n} r(B_k,a_k) \leqslant I_n^0(\gamma).$$

Theorem 4 is proved. □

Theorem 5. *Let $n \in \mathbb{N}$, $n \geqslant 12$ and $1 < \gamma \leqslant n^{\frac{2}{3}-\frac{2}{3}\frac{\ln(2\ln(n))}{\ln(n)}}$. Then the result of the Theorem 4 remains valid without the condition of simply connected domains B_k, $k=\overline{0,n}$.*

Proof. Taking into account the paper [14], consider the case $\alpha_0 > \frac{2}{\sqrt{\gamma}}$. Prove that under this condition

$$\frac{r^{\gamma}(B_0,a_0)\prod\limits_{k=1}^{n} r(B_k,a_k)}{I_n^0(\gamma)}<1.$$

Then, using Theorem 5.2.3 [2], we obtain

$$\prod_{k=1}^{n} r(B_k, a_k) < \frac{4^n}{(n-1)^{n-1}\sqrt{\gamma}} \left(1 - \frac{1}{\sqrt{\gamma}}\right)^{n-1}.$$

Thus, using inequality (3) and the previous inequality,

$$r^{\gamma}(B_0, a_0) \prod_{k=1}^{n} r(B_k, a_k) <$$
$$< n^{-\frac{\gamma}{2}} 4^{n-\gamma} \left(\frac{1}{n-1}\right)^{\frac{n^2-n-n\gamma+\gamma}{n}} \left(\frac{1}{\gamma}\right)^{\frac{n-\gamma}{2n}} \left(1 - \frac{1}{\sqrt{\gamma}}\right)^{\frac{n^2-n-n\gamma+\gamma}{n}}.$$

So, we have the inequality

$$\frac{I_n(\gamma)}{I_n^0(\gamma)} \leqslant 4^{-\gamma-\frac{\gamma}{n}} n^{n-\frac{\gamma}{2}+\frac{2\gamma}{n}} \left(\frac{1}{\gamma}\right)^{\frac{n+\gamma}{2n}} \left(\frac{1}{n-1}\right)^{\frac{n^2-n-n\gamma+\gamma}{n}} \times$$
$$\times \left(1 - \frac{1}{\sqrt{\gamma}}\right)^{\frac{n^2-n-n\gamma+\gamma}{n}} \left(1 - \frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}} \left(\frac{1+\frac{\sqrt{\gamma}}{n}}{1-\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}} := P_n(\gamma).$$

It is easy to see that

$$n^{n-\frac{\gamma}{2}+\frac{2\gamma}{n}} \left(\frac{1}{n-1}\right)^{\frac{n^2-n-n\gamma+\gamma}{n}} = n^{\gamma+1+\frac{\gamma}{n}} \left(\frac{n}{n-1}\right)^{n-1-\gamma+\frac{\gamma}{n}}.$$

Taking into account the previous equality, we get the following expression:

$$P_n(\gamma) = 4^{-\gamma-\frac{\gamma}{n}} \left(\frac{1}{\gamma}\right)^{\frac{n+\gamma}{2n}} n^{\gamma+1+\frac{\gamma}{n}} \left(1 - \frac{1}{\sqrt{\gamma}}\right)^{\frac{n^2-n-n\gamma+\gamma}{n}} \times$$
$$\times \left(\frac{n}{n-1}\right)^{n-1-\gamma+\frac{\gamma}{n}} \left(1 - \frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}} \left(\frac{1+\frac{\sqrt{\gamma}}{n}}{1-\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}.$$

Since in the paper [18] the problem is solved for $n \geqslant 12$ and $1 < \gamma \leqslant n^{0{,}45}$, it is enough to consider only $\gamma > n^{0{,}45}$. Evaluate the expression $P_n(\gamma)$ under the conditions of the theorem. Using estimates

$$\left(\frac{n}{n-1}\right)^{n-1-\gamma+\frac{\gamma}{n}} < e, \qquad \left(1 - \frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}} < 1,$$

$$\left(\frac{1+\frac{\sqrt{\gamma}}{n}}{1-\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}} 4^{-\gamma-\frac{\gamma}{n}} \left(\frac{1}{\gamma}\right)^{\frac{n+\gamma}{2n}} < 0{,}06 < \frac{1}{e},$$

we get

$$P_n(\gamma) < n^{\gamma+1+\frac{\gamma}{n}} \left(1-\frac{1}{\sqrt{\gamma}}\right)^{\frac{n^2-n-n\gamma+\gamma}{n}}. \tag{5}$$

Evaluate the expression $n^{\gamma+1+\frac{\gamma}{n}} \left(1-\frac{1}{\sqrt{\gamma}}\right)^{\frac{n^2-n-n\gamma+\gamma}{n}}$ under the conditions of the theorem. The following transformations are also correct:

$$n^{\gamma+1+\frac{\gamma}{n}} \left(1-\frac{1}{\sqrt{\gamma}}\right)^{\frac{n^2-n-n\gamma+\gamma}{n}} = n^{\gamma+1+\frac{\gamma}{n}} \left(\left(1-\frac{1}{\sqrt{\gamma}}\right)^{\sqrt{\gamma}}\right)^{\frac{n^2-n-n\gamma+\gamma}{\sqrt{n\gamma}}} <$$

$$< \left(n\left(\frac{1}{e}\right)^{\frac{n^2-n-n\gamma+\gamma}{n\gamma\sqrt{\gamma}+n\sqrt{\gamma}+\gamma\sqrt{\gamma}}}\right)^{\gamma+1+\frac{\gamma}{n}} = \left(n\left(\frac{1}{e}\right)^{\frac{n}{\gamma^{\frac{3}{2}}}\frac{1-\frac{1}{n}-\frac{\gamma}{n}+\frac{\gamma}{n^2}}{1+\frac{1}{\gamma}+\frac{1}{n}}}\right)^{\gamma+1+\frac{\gamma}{n}}.$$

For $n \geqslant 12$ and $n^{0{,}45} < \gamma < n^{\frac{2}{3}-\frac{2}{3}\frac{\ln(2\ln(n))}{\ln(n)}}$, the inequality

$$\frac{1-\frac{1}{n}-\frac{\gamma}{n}+\frac{\gamma}{n^2}}{1+\frac{1}{\gamma}+\frac{1}{n}} > 0{,}5$$

holds. Consequently,

$$n\left(\frac{1}{e}\right)^{\frac{n}{\gamma^{\frac{3}{2}}}\frac{1-\frac{1}{n}-\frac{\gamma}{n}+\frac{\gamma}{n^2}}{1+\frac{1}{\gamma}+\frac{1}{n}}} < n\left(\frac{1}{e}\right)^{\frac{n}{2\gamma^{\frac{3}{2}}}} <$$

$$< n\left(\frac{1}{e}\right)^{\frac{n}{2n^{\frac{3}{2}\left(\frac{2}{3}-\frac{2}{3}\frac{\ln(2\ln(n))}{\ln(n)}\right)}}} = n\left(\frac{1}{e}\right)^{\frac{1}{2}n^{\frac{\ln(2\ln(n))}{\ln(n)}}} = 1.$$

Thus, from inequality (5) we obtain that for n and γ given in Theorem 5 $P_n(\gamma) < 1$, that is

$$r^{\gamma}(B_0, a_0)\prod_{k=1}^{n} r(B_k, a_k) \leqslant I_n^0(\gamma).$$

Theorem 5 is proved. □

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