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A BALL COMPARISON BETWEEN EXTENDED MODIFIED JARRATT METHODS UNDER THE SAME SET OF CONDITIONS FOR SOLVING EQUATIONS AND SYSTEMS OF EQUATIONS

Abstract. In this paper, we compare the radii of convergence of Jarratt-type methods under the same set of conditions for solving nonlinear equations and systems of equations. Our convergence analysis is based on the first Fréchet derivative that only appears on the method. Numerical examples where the theoretical results are tested complete the paper.

Key words: *Jarratt type method, Banach space, radious of convergence*

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1. Introduction. In this paper we compare the convergence radii of Jarratt-type methods under the same set of conditions for solving nonlinear equation

$$F(x) = 0,$$

where $F\colon \Omega \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is continuously Fréchet differentiable, $\mathcal{B}_1, \mathcal{B}_2$ are Banach spaces, and Ω is a nonempty convex set.

The methods under consideration in this paper are [14]:

$$
\begin{aligned}
y_n &= x_n - \frac{1}{2}F'(x_n)^{-1}F(x_n),\\
z_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n)),\\
x_{n+1} &= y_n + (F'(x_n) - 3F'(z_n))^{-1}(F(x_n) + 2F(z_n)),
\end{aligned}
\tag{1}
$$

and [13]

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n),
\end{aligned}
$$

$$
\begin{aligned}
z_n &= x_n - \frac{1}{2}(3F'(y_n) - F'(x_n))^{-1}(3F'(y_n) + \\
&\quad + F'(x_n))F'(x_n)^{-1}F(x_n), \\
x_{n+1} &= z_n - 2(3F'(y_n) - F'(x_n))^{-1}F(z_n).
\end{aligned} \tag{2}
$$

The sixth-order convergence of iterative methods was, in general, obtained in [13], [14], respectively, when $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^m$. Taylor expansions were used, and conditions up to the seven order derivative restrict applicability of the methods.

For example: Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define f on Ω by

$$
f(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases} \tag{3}
$$

Then, we have $t_* = 1$,

$$
f'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.
$$

Obviously $f'''(t)$ is not bounded on Ω. So, convergence of these methods is not guaranteed by the analysis in these papers. Our convergence analysis is based on the first Fréchet derivative that only appears on the method.

We also provide a computable radius of convergence, not given in [1–22]. This way, we locate a set of initial points for the convergence of the method. The numerical examples are chosen to show how the radii theoretically predicted are computed. In particular, the example given by (3) shows that earlier results cannot be used to show convergence of the method. Our results significantly extend the applicability of these methods and provide a new way of looking at iterative methods.

The article contains local convergence analysis in Section 2 and the numerical examples in Section 3.

2. Ball convergence. We develop real functions to be used in the ball-convergence analysis of method (1) and (2), respectively. Set $S = [0, \infty)$. Assume

(i) Equation

$$
\varphi_0(t) - 1 = 0
$$

has the least solution $\rho_0 \in S - \{0\}$ for some function $\varphi_0 \colon S \longrightarrow S$ continuous and nondecreasing. Let $S_0 = [0, \rho_0)$;

(ii) equation
$$\psi_1(t) - 1 = 0,$$

has the least solution $r_1 \in S_0 - \{0\}$, for some functions $\varphi \colon S_0 \longrightarrow S$ and $\varphi_1 \colon S_0 \longrightarrow S$ continuous and nondecreasing, where

$$\psi_1(t) = \frac{\int\limits_0^1 \varphi((1-\theta)t)d\theta + \frac{1}{2}\int\limits_0^1 \varphi_1(\theta t)d\theta}{1 - \varphi_0(t)};$$

(iii) equation
$$\psi_2(t) - 1 = 0$$

has the least solution $r_2 \in S_0 - \{0\}$, where

$$\psi_2(t) \quad = \quad \frac{\int\limits_0^1 \varphi((1-\theta)t)d\theta + \frac{1}{3}\int\limits_0^1 \varphi_1(\theta t)d\theta}{1 - \varphi_0(t)};$$

(iv) equation
$$p(t) - 1 = 0,$$

has the least positive solution ρ_p, where $p(t) = \frac{1}{3}(3\varphi_0(\psi_2(t)t) + \varphi_0(t))$. Let $\rho = \min\{\rho_0, \rho_p\}$ and $S_1 = [0, \rho)$. Assume equation

$$\psi_3(t) - 1 = 0$$

has the least solution $r_3 \in (0, \rho)$, where

$$\psi_3(t) = \frac{\int\limits_0^1 \varphi((1-\theta)t)d\theta}{1 - \varphi_0(t)} + \frac{3(\varphi_0(t) + \varphi_0(\psi_2(t)t))\int\limits_0^1 \varphi_1(\theta t)d\theta}{4(1 - \varphi_0(t))(1 - p(t))} +$$
$$+ \frac{\int\limits_0^1 \varphi_1(\theta\psi_2(t)t)d\theta\psi_2(t)}{1 - p(t)}.$$

The parameter r defined by

$$r = \min\{r_k\}, \ k = 1, 2, 3, \tag{4}$$

shall be shown to be a convergence radius for method (1). We have, by (4), that for all $t \in [0, r)$

$$0 \leqslant \varphi_0(t) < 1, \tag{5}$$

$$0 \leqslant p(t) < 1, \tag{6}$$

and

$$0 \leqslant \psi_k(t) < 1. \tag{7}$$

The notation $C(x, a), \bar{C}(x, a)$ is used for the open and closed balls, respectively, in $\mathcal{B}_1$ with center $x \in \mathcal{B}_1$ and of radius $a > 0$.

The conditions (A) are needed with functions φ_0, φ, and φ_1 as defined previously. Further, assume:

(a1) $F \colon \Omega \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is Fréchet-continuously differentiable; there exists a simple $x_* \in \Omega$ such that $F(x_*) = 0$.

(a2) For each $x \in \Omega$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leqslant \varphi_0(\|x - x_*\|).$$

Set $\Omega_0 = \Omega \cap C(x_*, \rho_0)$.

(a3) For each $x, y \in \Omega_0$,

$$\|F'(x_*)^{-1}(F'(y) - F'(x))\| \leqslant \varphi(\|y - x\|),$$

$$\|F'(x_*)^{-1}F'(x)\| \leqslant \varphi_1(\|x - x_*\|).$$

(a4) $\bar{C}(x_*, \gamma) \subset \Omega$ for some $\gamma > 0$ to be determined.

(a5) There exists $b \geqslant r$ such that $\int_0^1 \varphi_0(\theta b)d\theta < 1$. Set $\Omega_1 = \Omega \cap \bar{T}(x_*, b)$.

The ball convergence of method (1) follows, based on conditions (A).

Theorem 1. *Assume conditions (A) with $\gamma = r$ hold and choose $x_0 \in C(x_*, r) - \{x_*\}$. Then, iteration $\{x_n\}$ generated by method (1) is well-defined in $C(x_*, r)$, remains in $C(x_*, r)$ for all $n = 0, 1, 2, \ldots$, with $\lim_{n \longrightarrow \infty} x_n = x_*$,*

$$\|y_n - x_*\| \leqslant \psi_1(\|x_n - x_*\|)\|x_n - x_*\| \leqslant \|x_n - x_*\| < r, \tag{8}$$

$$\|z_n - x_*\| \leqslant \psi_2(\|x_n - x_*\|)\|x_n - x_*\| \leqslant \|x_n - x_*\|, \tag{9}$$

and

$$\|x_{n+1} - x_*\| \leqslant \psi_3(\|x_n - x_*\|)\|x_n - x_*\| \leqslant \|x_n - x_*\|, \tag{10}$$

where the functions ψ_k, $k = 1, 2, 3$ are given previously and r is given by (6). Moreover, x_* the only solution of equation $F(x) = 0$ in the set Ω_1 given in (a5).

Proof. Estimates (8)–(10) are proved by induction on i. Using (4), (5), (a1) and (a2), we get, in turn, for $u \in C(x_*, r) - \{x_*\}$:

$$\|F'(x_*)^{-1}(F'(u) - F'(x_*))\| \leqslant \varphi_0(\|u - x_*\|) \leqslant \varphi_0(r) < 1. \quad (11)$$

Then, the Banach lemma on invertible operators [8] and (11) gives $F'(u)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$, with

$$\|F'(u)^{-1}F'(x_*)\| \leqslant \frac{1}{1 - \varphi_0(\|u - x_*\|)}. \quad (12)$$

It follows that iterates y_0 and z_0 are well-defined by method (1) for $n = 0$. We can also write

$$y_0 - x_* = (x_0 - x_* - F'(x_0)^{-1}F(x_0)) + \frac{1}{2}F'(x_0)^{-1}F'(x_0) \quad (13)$$

and

$$z_0 - x^* = x_0 - x^* - \frac{2}{3}F'(x_0)^{-1}F(x_0). \quad (14)$$

In view of (4), (7) (for $k = 1, 2$), (a3), (12) (for $u = x_0$) and (13), we have, in turn:

$$\|y_0 - x_*\| \leqslant \|F'(x_0)^{-1}F(x_*)\| \times$$

$$\times \| \int_0^1 F'(x_*)^{-1}(F'(x_* + \theta(x_0 - x_*)) - F'(x_0))d\theta(x_0 - x_*) +$$

$$+ \frac{1}{2}\|F'(x_0)^{-1}F(x_*)\|\|F'(x_0)^{-1}F(x_0)\| \leqslant$$

$$\leqslant \frac{(\int_0^1 \varphi((1-\theta)\|x_0 - x_*\|)d\theta + \frac{1}{2}\int_0^1 \varphi_1(\theta\|x_0 - x_*\|)d\theta}{1 - \varphi_0(\|x_0 - x_*\|)}\|x_0 - x_*\| =$$

$$= \psi_1(\|x_0 - x_*\|)\|x_0 - x_*\| \leqslant \|x_0 - x_*\| < r, \quad (15)$$

and

$$\|z_0 - x_*\| \leqslant$$

$$\leqslant \left[\frac{\int_0^1 \varphi((1-\theta)\|x_0 - x_*\|)d\theta + \frac{1}{3}\int_0^1 \varphi_1(\theta\|x_0 - x_*\|)d\theta)}{1 - \varphi_0(\|x_0 - x_*\|)} \right]\|x_0 - x_*\| \leqslant$$

$$\leqslant \psi_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leqslant \|x_0 - x_*\|; \quad (16)$$

this shows that $y_0, z_0 \in C(x_*, r)$ and (8), (9) hold for $n = 0$, respectively.

Next, we show that $(F'(x_0) - 3F'(z_0))$ is invertible; this will imply the existence of iterate x_1. Indeed, using (4), (6) and (16), we obtain, in turn, that

$$\|(-2F'(x_*))^{-1}(F'(x_0) - 3F'(z_0) + 2F'(x_*))\| \leqslant$$
$$\leqslant \frac{1}{2}(3\|F'(x_*)^{-1}(F'(z_0) - F'(x_*))\| + \|F'(x_*)^{-1}(F'(x_0) - F(x_*))\|) \leqslant$$
$$\leqslant \frac{1}{3}(3\varphi_0(\|z_0 - x_*\|) + \varphi_0(\|x_0 - x_*\|)) \leqslant$$
$$\leqslant p(\|x_0 - x_*\| \leqslant p(r) < 1, \quad (17)$$

so

$$\|(F'(x_0) - 3F'(z_0))^{-1}F'(x_*)\| \leqslant \frac{1}{2(1 - p(\|x_0 - x_*\|))}. \quad (18)$$

It follows from (4), (7) (for $k = 3$), (15), (16) and (18), and

$$\begin{aligned}
x_1 - x_* &= x_0 - x_* - F'(x_0)^{-1}F(x_0) + \\
&\quad + \frac{1}{2}F'(x_0)^{-1}F(x_0) + \\
&\quad + (F'(x_0) - 3F(z_0))^{-1}F'(x_0) + 2(F'(x_0) - 3F'(z_0))^{-1}F(z_0) = \\
&= x_0 - x_* - F'(x_0)^{-1}F(x_0) + \\
&\quad + \frac{3}{2}F'(x_0)^{-1}(F'(z_0) - F'(x_0))(F'(x_0) - 3F'(z_0))^{-1}F(x_0) + \\
&\quad + 2(F'(x_0) - 3F'(z_0))^{-1}F(z_0). \quad (19)
\end{aligned}$$

We have, in turn,

$$\|x_1 - x_*\| \leqslant$$
$$\leqslant \frac{\int_0^1 \varphi((1-\theta)\|x_0 - x_*\|)d\theta\|x_0 - x_*\|}{1 - \varphi_0(\|x_0 - x_*\|)} +$$
$$+ \frac{3(\varphi_0(\|x_0 - x_*\|) + \varphi_0(\|z_0 - x_*\|))\int_0^1 \varphi_1(\theta\|x_0 - x_*\|)d\theta\|x_0 - x_*\|}{4(1 - \varphi_0(\|x_0 - x_*\|))(1 - p(\|x_0 - x_*\|))} +$$
$$+ \frac{(\varphi_1(\theta\|z_0 - x_*\|)d\theta\|z_0 - x_*\|}{1 - p(\|x_0 - x_*\|)} \leqslant$$

$$\leqslant \psi_3(\|x_0 - x_*\|)\|x_0 - x_*\| \leqslant \|x_0 - x_*\|, \quad (20)$$

showing (10) for $n = 0$ and $x_1 \in C(x_*, r_*)$. Then, repalce x_0, y_0, z_0, x_1 by x_i, y_i, z_i, x_{i+1} in the preceding calculations to complete the induction for items (8)–(10). In view of the estimation

$$\|x_{i+1} - x_*\| \leqslant \beta\|x_i - x_*\| < r, \quad (21)$$

where $\beta = \psi_3(\|x_0 - x_*\|) \in [0, 1)$, we conclude that $\lim_{i \to \infty} x_i = x_*$ and $x_{i+1} \in C(x_*, r_*)$. The uniqueness part is left to complete the proof. Consider $v \in \Omega_1$ with $F(v) = 0$. Let $G = \int_0^1 F'(x_* + \theta(v - x_*))d\theta$. Then, using (a2) and (a5), we obtain

$$\|F'(x_*)^{-1}(G - F'(x_*))\| \leqslant \int_0^1 \varphi_0(\theta\|x_* - v\|)d\theta \leqslant \int_0^1 \varphi_0(\theta b)d\theta < 1,$$

so $G^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$. Consequently, from $0 = F(v) - F(x_*) = G(v - x_*)$, we obtain $v = x_*$. □

Remark.

1. In view of (a2) and the estimate

$$\|F'(x^*)^{-1}F'(x)\| = \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \leqslant$$
$$\leqslant 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leqslant 1 + \varphi_0(\|x - x^*\|)$$

   the second condition in (h3) can be dropped and φ_1 can be replaced by

$$\varphi_1(t) = 1 + \varphi_0(t)$$

   or

$$\varphi_1(t) = 1 + \varphi_0(\rho_0),$$

   since $t \in [0, \rho_0)$.

2. The results obtained here can be used for operators F satisfying autonomous differential equations [2] of the form

$$F'(x) = P(F(x))$$

   where P is a continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then we can choose $P(x) = x + 1$.

3. Let $\varphi_0(t) = L_0 t$, and $\varphi(t) = Lt$. In [2], [3], we showed that $r_A = \frac{2}{2L_0 + L}$ is the convergence radius of Newton's method:

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \dots \qquad (22)$$

under the conditions (a1)–(a3). It follows from the definition that the convergence radius r of the method (1) cannot be larger than the convergence radius r_A of the second-order Newton's method (22). As already noted in [2], [3], r_A is at least as large as the convergence radius given by Rheinboldt [18]

$$r_R = \frac{2}{3L_1},$$

where L_1 is the Lipschitz constant on D. The same value for r_R was given by Traub [19]. In particular, for $L_0 < L_1$ we have

$$r_R < r_A$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \text{ as } \frac{L_0}{L_1} \to 0.$$

That is, the radius of convergence r_A is at most three times larger than Rheinboldt's.

4. We can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) \Big/ \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) \Big/ \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

The ball convergence of method (2) follows in a similar way. Let $\bar{\psi}_1 = \psi_2$ and define functions

$$\bar{\psi}_2(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta}{1 - \varphi_0(t)} + \frac{3}{4}\frac{(\varphi_0(t) + \varphi_0(\bar{\psi}_1(t)t))}{(1 - \varphi_0(t))(1 - q(t))},$$

$$q(t) = \frac{1}{2}(\varphi_0(t) + 3\varphi_0(\bar{\psi}_1)t)t)),$$

and

$$\bar{\psi}_3(t) = \left[\frac{\int\limits_0^1 \varphi((1-\theta)\bar{\psi}_2(t)t)d\theta}{1 - \varphi_0(\bar{\psi}_2(t)t)} + \right.$$

$$\left. + \frac{(3\varphi_0(\bar{\psi}_1(t)t) + 2\varphi_0(\bar{\psi}_2(t)t) + \varphi_0(t))\int\limits_0^1 \varphi_1(\theta\bar{\psi}_2(t)t)d\theta}{2(1 - \varphi_0(\bar{\psi}_2(t)t))(1 - q(t))}\right]\bar{\psi}_2(t).$$

Assume the equations

$$\bar{\psi}_2(t) - 1 = 0 \text{ and } \bar{\psi}_3(t) - 1 = 0$$

have the minimal positive solutions $\bar{r}_2$ and $\bar{r}_3$ in A_0, respectively. Let

$$\bar{r} = \min\{r_2, \bar{r}_2, \bar{r}_3\}.$$

We need the estimates under conditions (A) for $\gamma = \bar{r}$:

$$\begin{aligned}
\|y_n - x_*\| &\leqslant \psi_2(\|x_n - x_*\|)\|x_n - x_*\| = \\
&= \bar{\psi}_1(\|x_n - x_*\|)\|x_n - x_*\| \leqslant \|x_n - x_*\| < \bar{r},
\end{aligned}$$

$$z_n - x_* = x_n - x_* - F'(x_n)^{-1}F(x_n) +$$
$$+ [I - \frac{1}{2}(3(F'(y_n) - F'(x_n))^{-1}(3F'(y_n) + F'(x_n))]F'(x_0)^{-1}F(x_n) =$$
$$= x_n - x_* - F'(x_n)^{-1}F(x_n) +$$
$$+ \frac{3}{2}(3F'(y_n) - F'(x_n))^{-1}(F'(y_n) - F'(x_n))F'(x_n)^{-1}F(x_n),$$

so

$$\|z_n - x_*\| \leqslant \left[\frac{\int\limits_0^1 \varphi((1-\theta)\|x_n - x_*\|)d\theta}{1 - \varphi_0(\|x_n - x_*\|)} + \right.$$

$$\left. + \frac{3}{4}\frac{(\varphi_1(\|y_n - x_*\|) + \varphi_0(\|x_n - x_*\|))\int\limits_0^1 \varphi_1(\theta\|x_n - x_*\|)d\theta}{(1 - \varphi_0(\|x_n - x_*\|))(1 - q(\|x_n - x_*\|))}\right]\|x_n - x_*\| \leqslant$$

$$\leqslant \bar{\psi}_2(\|x_n - x_*\|)\|x_n - x_*\| \leqslant \|x_n - x_*\| < \bar{r},$$

$$
\begin{aligned}
x_{n+1} - x_* &= z_n - x_* - F'(z_n)^{-1}F(z_n) + \\
&\quad + (F'(z_n)^{-1} - 2(3F'(y_n) - F'(x_n)))^{-1}F(z_n) = \\
&= z_n - x_* - F'(z_n)^{-1}F(z_n) + F'(z_n)^{-1}(3F'(y_n) - F'(x_n) - 2F'(z_n)) \times \\
&\quad \times (3F'(y_n) - F'(x_n))^{-1}F(z_n),
\end{aligned}
$$

so

$$
\begin{aligned}
\|x_{n+1} - x_*\| &\leqslant \left[\frac{\int_0^1 \varphi((1-\theta)\|z_n - x_*\|)d\theta}{1 - \varphi_0(\|z_n - x_*\|)} + \right. \\
&\quad + \frac{(3\varphi_0(\|y_n - x_*\|) + 2\varphi_0(\|z_n - x_*\|) + \varphi_0(\|x_n - x_*\|))}{(1 - \varphi_0(\|z_n - x_*\|))(1 - q(\|x_n - x_*\|))} \times \\
&\quad \left. \times \int_0^1 \varphi_1(\theta\|z_n - x_*\|)d\theta \right] \|z_n - x_*\| \leqslant \\
&\leqslant \bar{\psi}_3(\|x_n - x_*\|)\|x_n - x_*\| \leqslant \|x_n - x_*\|,
\end{aligned}
$$

where we also used as in (18) (for $y_n = z_n$)

$$\|(3F'(y_n) - F'(x_n))^{-1}F'(x_*)\| \leqslant \frac{1}{2(1 - q(\|x_n - x_*\|))}.$$

Hence, we arrive at:

Theorem 2. *Assume conditions (A) with $\gamma = \bar{r}$ hold and choose $x_0 \in C(x_*, \bar{r}_*) - \{x_*\}$. Then, the conclusions of Theorem 1 hold for method (2) with $\bar{\psi}_2, \bar{\psi}_3, \bar{r}$ replacing ψ_2, ψ_3 and r, respectively.*

3. Numerical Example. Consider the kinematic system

$$F_1'(x) = e^x,\ F_2'(y) = (e - 1)y + 1,\ F_3'(z) = 1$$

with $F_1(0) = F_2(0) = F_3(0) = 0$. Let $F = (F_1, F_2, F_3)$. Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3$, $\Omega = \bar{U}(0,1)$, $x_* = (0,0,0)^T$. Define the function F on Ω for $w = (x, y, z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then we get

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so $\varphi_0(t) = (e-1)t$, $\varphi(t) = e^{\frac{1}{e-1}}t$, $\varphi_1(t) = e^{\frac{1}{e-1}}$. Then the radii are

$$r_1 = 0.0402645, r_2 = 0.154407, r_3 = 0.688759, \bar{r}_2 = 0.138442, \bar{r}_3 = 0.112838;$$

so, we conclude that $r = r_1$ and $\bar{r} = \bar{r}_3$.

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