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A SOLUTION TO QI'S EIGHTH OPEN PROBLEM ON COMPLETE MONOTONICITY

Abstract. In this paper, the complete monotonicity of $\frac{1}{\arctan x}$ is proved. This problem was posted by F. Qi and R. P. Agarwal as the eighth open problem of collection of eight open problems.

Key words: *logarithmically completely monotonic functions, polygamma function, inequalities, Stieltjes function*

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1. Introduction. A function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n \left[\ln f(x)\right]^{(n)} \geqslant 0 \tag{1}$$

for $n \in \mathbb{N}$ on I.

A non-negative function f is called a Stieltjes function if there exist non-negative constants $a, b \geqslant 0$ and σ is a measure on $(0, \infty)$, such that

$$f(x) = \frac{a}{x} + b + \int_{(0,\infty)} \frac{1}{x+t}\sigma(dt) \tag{2}$$

and $\int_{(0,\infty)}(1+t)^{-1}\sigma(dt) < \infty$ (see [7]). The theorem of Widder (see Theorem 12.5 in [2] or Theorem 18b at p. 366 in [3]) states that a non-negative function f is a Stieltjes function if and only if it is smooth and such that

$$(-1)^{n-1}\left(x^n f(x)\right)^{(2n-1)} \geqslant 0, \quad n \geqslant 1. \tag{3}$$

In the paper [4], F. Qi and R. P. Agarwal posed eight open problems on complete monotonicity. The eighth open problem says:

Qi's Eighth Open Problem. The function $\frac{1}{\arctan x}$ is logarithmically completely monotonic on $(0,\infty)$ but not a Stieltjes transform.

In [1], [6], [8], there is *Faá di Bruno's formula:* if g and f are functions with a sufficient number of derivatives, then

$$\frac{d^n}{dt^n}g(f(t)) = \sum \frac{n!}{b_1!b_2!\cdots b_n!}g^{(s)}(f(t))\prod_{i=1}^{n}\left(\frac{f^{(i)}(t)}{i!}\right)^{b_i} \tag{4}$$

where the sum is over all different solutions in nonnegative integers $b_1, b_2, \dots, b_n$ of $b_1 + 2b_2 + \cdots + nb_n = n$, and $s := b_1 + \cdots + b_n$. And in [5], the *nth* derivative of $\arctan x$ is obtained:

$$\frac{d^n}{dx^n}(\arctan x) = \frac{(-1)^{n-1}(n-1)!}{(1+x^2)^{n/2}}\sin\left(n\arcsin\left(\frac{1}{\sqrt{1+x^2}}\right)\right). \tag{5}$$

Equations (4) and (5) are important tools in the solution.

The function $\psi(x) := \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, the logarithmic derivative of the gamma function, is called psi function or digamma function, and $\psi^{(m)}(x)$ for $m \in \mathbb{N}$ are called the polygamma functions. Properties and inequalities related to polygamma functions can be found in [9–12] and references therein. The only positive zero of digamma function is $c = 1.461632144\ldots$ and the series representation is [1]

$$\psi^{(m)}(x) = (-1)^{m+1}\sum_{n=0}^{\infty}\frac{m!}{(n+x)^{m+1}}, \quad x > 0, \quad m = 1, 2, \dots$$

It follows that [1]

$$(-1)^{(m+1)}\psi^{(m)}(x) > 0, \quad m = 1, 2, \dots \tag{6}$$

2. Main Results.

Theorem 1. *The function* $\frac{1}{\arctan x}$ *is logarithmically completely monotonic on* $(0,\infty)$ *but not a Stieltjes transform.*

Proof. Let $h(x) = \ln\left(\frac{1}{\arctan x}\right) = -\ln(\arctan x)$, $g(x) = \ln(x)$ and $f(x) = \arctan x$, then $h(x) = -g(f(x))$. Now,

$$(-1)^n h^{(n)}(x) = (-1)^{n+1}\left(\ln(\arctan x)\right)^{(n)}.$$

By (4) and (5), we obtain

$$(-1)^n h^{(n)}(x) = \sum \frac{(-1)^{n+s} n!}{b_1! b_2! \cdots b_n!} \frac{(s-1)!}{(\arctan x)^s} \times$$

$$\times \prod_{r=1}^{n} \left(\frac{(-1)^{r-1}(r-1)!}{(1+x^2)^{r/2}} \sin \left(r \arcsin \left(\frac{1}{\sqrt{1+x^2}} \right) \right) \right)^{b_r} =$$

using $(-1)^{n+s} = (-1)^{2b_1+3b_2+4b_3+\cdots+(n+1)b_n}$ we obtain

$$= \sum \frac{n!}{b_1! b_2! \ldots b_n!} \frac{(s-1)!}{(\arctan x)^s} \times$$

$$\times \prod_{r=1}^{n} \left(\frac{(r-1)!}{(1+x^2)^{r/2}} \sin \left(r \arcsin \left(\frac{1}{\sqrt{1+x^2}} \right) \right) \right)^{b_r} > 0 \quad \text{for all } x \in (0, \infty).$$

Suppose that $f(x) = \frac{1}{\arctan x}$ is a Steiltjes transform; then the equation (3) must hold for all $n \in \mathbb{N}$. However, we claim that for $n = 2$ the equation (3) does not hold for $f(x) = \frac{1}{\arctan x}$. Now, for $n = 2$

$$(-1)^1 \left(x^2 f(x)\right)^{(3)} = -\left(\frac{x^2}{\arctan x} \right)^{(3)} =$$

$$= \frac{6\arctan(x)^2 - 2\arctan(x)^2 x^2 - 12x\arctan(x) + 6x^2}{\arctan(x)^4 (1+x^2)^3}.$$

We have

$$\lim_{x \to 0+} \left(\frac{6\arctan(x)^2 - 2\arctan(x)^2 x^2 - 12x\arctan(x) + 6x^2}{\arctan(x)^4 (1+x^2)^3} \right) = -2$$

and

$$\lim_{x \to \infty} \left(\frac{6\arctan(x)^2 - 2\arctan(x)^2 x^2 - 12x\arctan(x) + 6x^2}{\arctan(x)^4 (1+x^2)^3} \right) = 0$$

This implies

$$-2 < (-1)^1 \left(\frac{x^2}{\arctan x} \right)^{(3)} \leqslant 0, \quad \forall x \in (0, \infty),$$

which contradicts (3). Thus, the function $\frac{1}{\arctan x}$ is logarithmically completely monotonic but not a Steiltjes transform. The proof of Theorem 1 is complete. □

Theorem 2. *The function* $g(x) = \frac{1}{\psi(x)}$ *is strictly logarithmically completely monotonic on* (c, ∞)*, where* c *is the only positive zero of* $\psi(x)$*.*

Proof. Consider

$$(-1)^n\Big(\ln\Big(\frac{1}{\psi(x)}\Big)\Big)^{(n)} = (-1)^{n+1}\left(\ln(\psi(x))\right)^{(n)} =$$

using (4) we have

$$= \sum \frac{n!}{b_1!b_2!\dots b_n!}\frac{(-1)^{n+s}(s-1)!}{\psi^s(x)}\Big(\frac{\psi'(x)}{1!}\Big)^{b_1}\Big(\frac{\psi''(x)}{2!}\Big)^{b_2}\dots\Big(\frac{\psi^{(n)}(x)}{n!}\Big)^{b_n} =$$

since $n+s = 2b_1+3b_2+4b_3+\dots+(n+1)b_n$ and using (6), we have

$$= \sum \frac{n!}{b_1!b_2!\dots b_n!}\frac{(s-1)!}{\psi^s(x)}\Big(\frac{\psi'(x)}{1!}\Big)^{b_1}\Big(\frac{-\psi''(x)}{2!}\Big)^{b_2}\dots\Big(\frac{(-1)^{n+1}\psi^{(n)}(x)}{n!}\Big)^{b_n}.$$

So, $(-1)^n\Big(\ln\big(\frac{1}{\psi(x)}\big)\Big)^{(n)} > 0$, for all $x \in (c,\infty)$. The proof of Theorem 2 is complete. □

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