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APPROXIMATION BY LINEAR MEANS OF FOURIER SERIES AND REALIZATION FUNCTIONALS IN WEIGHTED ORLICZ SPACES

Abstract. Using one-sided Steklov means, we introduce a new modulus of smoothness in weighted Orlicz spaces and state its equivalence with a special K-functional. We prove Stechkin-Nikol'skii-type inequality for trigonometric polynomials and direct estimates for the approximation by Riesz-Zygmund, Vallée-Poussin, and Euler means in weighted Orlicz spaces. By these results, several types of realization functionals equivalent to the above cited K-functional in points $1/n$, $n \in \mathbb{N}$, are constructed.

Key words: *weighted Orlicz spaces, K-functional, realization functional, Riesz-Zygmund means, Euler means*

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1. Introduction. Let f be a 2π-periodic continuous function ($f \in C_{2\pi}$), T_n be the space of trigonometric polynomials of degree at most n, $n \in \mathbb{Z}_+ = \{0, 1, \dots\}$, $\|f\|_\infty = \max\limits_{x\in[0,\, 2\pi]} |f(x)|$. For $r \in \mathbb{N} = \{1, 2, \dots\}$, let us consider the r-th difference with step h

$$\Delta_h^r f(x) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x + jh)$$

and the r-th modulus of continuity $\omega_r(f, \delta)_\infty = \sup\limits_{0\leqslant h\leqslant\delta} \|\Delta_h^r f\|_\infty$. By definition, the best approximation is given by $E_n(f)_\infty = \inf\{\|f - t_n\|_\infty : t_n \in T_n\}$, $n \in \mathbb{Z}_+$. Then the classical Jackson-Stechkin theorem states that

$$E_n(f)_\infty \leqslant C\omega_r(f, (n+1)^{-1})_\infty, \quad n \in \mathbb{Z}_+, \tag{1}$$

(see [6, Ch. 7, Theorem 2.3]). On the other hand, for $t_n \in T_n$, $n \in \mathbb{N}$, $r \in \mathbb{N}$, the Stechkin-Nikol'skii inequality

$$\|t_n^{(r)}\|_\infty \leqslant C_n^r \|\Delta^r_{\pi/2n} t_n\|_\infty \leqslant C n^r \omega_r(t_n, 2\pi/n)_\infty \tag{2}$$

holds (see [6, Ch. 4, § 12, (12.1)]).

If $\tau_n(f) \in T_n$ satisfies the equality $\|f - \tau_n(f)\|_\infty = E_n(f)_\infty$, then, by (1) and (2), it is easy to see that

$$\|\tau_n^{(r)}(f)\|_\infty \leqslant C n^r \omega_r(\tau_n(f), 2\pi/n)_\infty \leqslant C n^r \omega_r(f, 1/n)_\infty, \quad n \in \mathbb{N},$$

and the inequality

$$C_1 \omega_r(f, 1/n)_\infty \leqslant \|f - \tau_n(f)\|_\infty + n^{-r} \|\tau_n^{(r)}(f)\|_\infty \leqslant C_2 \omega_r(f, 1/n)_\infty \tag{3}$$

holds for all $n \in \mathbb{N}$. The inequality (3) shows that the modulus of smoothness $\omega_r(f, 1/n)_\infty$ and the K-functional

$$K(f, n^{-r}, C_{2\pi}, C^r_{2\pi}) = \inf\{\|f - g\|_\infty + n^{-r}\|g^{(r)}\|_\infty : g \in C^r_{2\pi}\}$$

are equivalent, where $C^r_{2\pi}$ is the space of 2π-periodic r times continuously differentiable functions on $\mathbb{R}$. Moreover, we can take $g = \tau_n(f)$ to realize this equivalence. The middle part of (3) is often called the realization functional for $K(f, n^{-r}, C_{2\pi}, C^r_{2\pi})$. The idea of such functionals belongs to Ditzian, Hristov, and Ivanov [7].

In this paper, we study another forms of realization functionals, using Riesz-Zygmund, generalized Vallée-Poussin, and Euler means of Fourier series in weighted Orlicz spaces. The main tool is a Stechkin-Nikol'skii-type inequality (Theorem 4) and direct estimates of approximation by the cited means. In this paper, We use a new type of a modulus of smoothness in the weighted Orlicz space similar to that introduced in [19] for variable exponent spaces and in [20] for weighted Lorentz spaces.

2. Definitions. Let $M(u)$ be a continuous convex strictly increasing on $[0, +\infty)$ function, such that $M(0) = 0$, $\lim\limits_{u\to 0+0} M(u)/u = 0$, $\lim\limits_{u\to+\infty} M(u)/u = +\infty$. Then $M(u)$ is called a Young function or an N-function.

We denote by $N(v)$ the complementary function $N(v) = \max\{uv - M(u) : u \geqslant 0\}$, $v \in [0, +\infty)$. It is known that $N(v)$ is also a Young function.

For a Young function $M(u)$, we denote by $L_{M,2\pi}$ the linear space of 2π-periodic measurable functions $f:[0,2\pi]\to\mathbb{R}$, such that the integral $\int\limits_0^{2\pi} M(\lambda|f(x)|)\,dx$ is finite for some $\lambda>0$. It is well known that $L_{M,2\pi}$ equipped with the Orlicz norm

$$\|f\|_M := \sup\left\{\int\limits_0^{2\pi} |f(x)g(x)|\,dx : \int\limits_0^{2\pi} N(|g(x)|)\,dx \leqslant 1\right\}$$

is a Banach space called the Orlicz space generated by M. The Orlicz spaces may be considered as a generalization of the Lebesgue spaces $L^p_{2\pi}$, $1<p<\infty$, which are obtained in the case $M_p(u)=u^p/p$. If $L^p_{2\pi}$, $1\leqslant p<\infty$, we consider the norm $\|f\|_p=\left(\int\limits_0^{2\pi}|f(t)|^p\,dt\right)^{1/p}$. More about Orlicz spaces can be found in [13] and [17]. Matuszewska and Orlicz [15] introduced a pair of indices associated with a given Orlicz space $L_{M,2\pi}$.

Let $M^{-1}:[0,+\infty)\to[0,+\infty)$ be the inverse function of a Young function M and $h(t)=\limsup\limits_{x\to\infty} M^{-1}(x)/M^{-1}(tx)$, $t>0$. Then

$$\alpha_M=\lim_{t\to+\infty} -\frac{\log h(t)}{\log t}, \quad \beta_M=\lim_{t\to 0+0} -\frac{\log h(t)}{\log t} \tag{4}$$

are called the lower and upper Boyd indices of the function M and of the Orlicz space $L_{M,2\pi}$. Boyd [5] proved that the indices defined above coincide with the general indices for rearrangement invariant spaces named after him (see [3, Ch. 3, Definition 5.12]). Matuszewska and Orlicz [15] considered reciprocals to α_M and β_M. It is known that $0\leqslant\alpha_M\leqslant\beta_M\leqslant 1$, $\alpha_N+\beta_M=1$, $\alpha_M+\beta_N=1$. If $0<\alpha_M\leqslant\beta_M<1$, then we say that M and $L_{M,2\pi}$ has nontrivial Boyd indices.

A nonnegative 2π-periodic measurable function w is called a weight function if the set $w^{-1}(\{0,+\infty\})$ has zero Lebesgue measure. For a Young function M and a weight function w, we denote by $L_{M,w}$ the space of all 2π-periodic measurable functions f, such that $fw\in L_{2\pi}$ with the norm $\|f\|_{M,w}=\|fw\|_M$. We will often use the spaces L^p_w, $1\leqslant p<\infty$, with the norm $\|f\|_{p,w}=\|fw\|_p$.

A weight w belongs to the Muckenhoupt class $A_p(\mathbb{T})$, $1<p<\infty$, if

$$|w|_{A_p}=\sup\left(|I|^{-1}\int\limits_I w^p(x)\,dx\right)^{1/p}\left(|I|^{-1}\int\limits_I w^{-p'}(x)\,dx\right)^{1/p'}<\infty,$$

where $p' = p/(p-1)$ and the supremum is taken with respect to all intervals $I \subset \mathbb{R}$ whose lengths $|I|$ do not exceed 2π (see [16], where w^p was substituted by ω).

The class of trigonometric polynomials was shown to be dense in $L_{M,w}$ for M with nontrivial Boyd indices and $w \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$ in [10, Lemma 4]. Thus, we may define the best approximations $E_n(f)_{M,w} = \inf_{t_n \in T_n} \|f - t_n\|_{M,w}$, $n \in \mathbb{Z}_+$, and the sequence $\{E_n(f)_{M,w}\}_{n=0}^{\infty}$ decreases to zero. On the other hand, for a 2π-periodic locally integrable function f, we can consider two variants of Steklov operators

$$s_h(f)(x) = h^{-1} \int_x^{x+h} f(u)\,du, \quad s_h^{(2)}(f)(x) = (2h)^{-1} \int_{x-h}^{x+h} f(u)\,du.$$

In the present paper, we use the following modulus of smoothness:

$$\Omega_r(f,\delta)_{M,w} = \sup_{0 \leqslant h_i \leqslant \delta} \left\| \prod_{i=1}^{r} (I - s_{h_i})(f) \right\|_{M,w},$$

where I is the identity operator. In [10], [8], [11] a similar construction $\Omega_r^*(f,\delta)_{M,w}$ with $s_{h_i}^{(2)}$ instead of s_{h_i} was used. The Fourier series of $s_h^{(2)}(f)$ is more appropriate for the treatment than one of $s_h(f)$, while $\Omega_r(f,\delta)_{M,w}$ are more general than $\Omega_r^*(f,\delta)_{M,w}$, since the last modulus of smoothness is equivalent to $\Omega_{2r}(f,\delta)_{M,w}$ for $r \in \mathbb{N}$ under some conditions on M and w (see Corollary 1). These concepts are well-defined: this follows from the boundedness of s_h and $s_h^{(2)}$ in $L_{M,w}$ under the conditions at the beginning of this paragraph (see Lemma 2 and Lemma 1 in [10]). Unfortunately, the definition of a fractional modulus of smoothness in such a way is connected with the problem of uniform boundedness of iterates of the Steklov operators. We note also the papers [18], [9] that study the moduli of smoothness defined with help of $s_h(f)$.

For an $r \in \mathbb{N}$, a Young function M, and a weight w, we denote by $W^r L_{M,w}$ the collection of all absolutely continuous on each period functions f ($f \in AC_{2\pi}$), such that $f', \ldots, f^{(r-1)} \in AC_{2\pi}$ and $f^{(r)} \in L_{M,w}$. If $f \in L_{M,w}$ and $r \in \mathbb{N}$, then we define the Peetre's K-functional by

$$K(f,t,L_{M,w},W^r L_{M,w}) = \inf\{\|f-g\|_{M,w} + t\|g^{(r)}\|_{M,w} : g \in W^r L_{M,w}\}.$$

It is well known that $L_w^p \subset L_{2\pi}^1$ for $1 < p < \infty$ and $w \in A_p(\mathbb{T})$. By the proof of Lemma 2 below, under conditions $0 < \alpha_M \leqslant \beta_M < 1$ and

$w \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$, there exists q, p, such that $1 < q < 1/\beta_M \leqslant \leqslant 1/\alpha_M < p < \infty$ and $w \in A_p(\mathbb{T}) \cap A_q(\mathbb{T})$. Moreover, from the results of Boyd [5] we see that $fw \in L^p_{2\pi} + L^q_{2\pi} = L^q_{2\pi}$, i. e., $fw = g_1 + g_2$, $g_1 \in L^p_{2\pi}$, $g_2 \in L^q_{2\pi}$, or $f \in L^q_w \subset L^1_{2\pi}$. Thus, if $0 < \alpha_M \leqslant \beta_M < 1$ and $w \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$, then $f \in L_{M,w}$ has the Fourier series

$$a_0(f)/2 + \sum_{k=1}^{\infty}(a_k(f)\cos kx + b_k(f)\sin kx) =: \sum_{k=0}^{\infty} A_k(f)(x).$$

Further we will consider the Fourier partial sums $S_n(f)(x) = \sum\limits_{k=0}^{n} A_k(f)(x)$, the Riesz-Zygmund means of order $r \in \mathbb{N}$

$$Z^r_n(f)(x) = \sum_{k=0}^{n}(1 - k^r/(n+1)^r)A_k(f)(x), \quad n \in \mathbb{N},$$

the generalized Vallée-Poussin means

$$v^r_n(f)(x) = \frac{(2n+1)^r Z^r_{2n}(f)(x)}{(2n+1)^r - (n+1)^r} - \frac{(n+1)^r Z^r_n(f)(x)}{(2n+1)^r - (n+1)^r}, \quad n \in \mathbb{N},$$

and the Euler means

$$e^t_n(f)(x) = (1+t)^{-n}\sum_{k=0}^{n}\binom{n}{k}t^{n-k}S_k(f)(x), \quad t > 0, \quad n \in \mathbb{N}.$$

More about these means can be found in the monograph by Hardy [14]. It is well known that for $f \in L^1_{2\pi}$ the following limit

$$\widetilde{f}(x) = (2\pi)^{-1}\lim_{t\to 0+0}\int_t^{\pi}(f(x-u) - f(x+u))\,\mathrm{ctg}(u/2)\,du$$

exists a. e. on $\mathbb{R}$ (see [1, Ch. VIII, § 7]). The function $\widetilde{f}(x)$ is called the conjugate function to f. If $\widetilde{f} \in L^1_{2\pi}$, then its Fourier series has the form $\sum\limits_{k=1}^{\infty}(a_k(f)\sin kx - b_k(f)\cos kx)$.

3. Auxiliary propositions. Lemma 1 is proved in [3, p. 153].

Lemma 1. *Let $1 < q < p < \infty$. If a linear operator is bounded in the Lebesgue spaces $L^p_{2\pi}$ and $L^q_{2\pi}$, then it is bounded in every Orlicz space $L_{M,2\pi}$ provided that $1/p < \alpha_M \leqslant \beta_M < 1/q$.*

Lemma 2 is proved for $s_h^{(2)}$ in [10, Lemma 1].

Lemma 2. *Let M be a Young function with nontrivial indices and $w \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$. Then the operators s_h are uniformly bounded in $L_{M,w}$ with respect to $h > 0$.*

Proof. Since $0 < \alpha_M \leqslant \beta_M < 1$, by [4, Theorem 2.31] we find q and p such that $1 < q < 1/\beta_M \leqslant 1/\alpha_M < p < \infty$ and $w \in A_p(\mathbb{T}) \cap A_q(\mathbb{T})$. It is known that the maximal operator is bounded in all L_w^r, $1 < r < \infty$, $w \in A_r(\mathbb{T})$ (see [16]). We use the multiplication operator $F_w(f) = wf$ to define $A_h(f) = F_w(s_h(F_{w^{-1}}(f)))$. Then A_h is uniformly in $h > 0$ bounded in $L_{2\pi}^p$ and $L_{2\pi}^q$. Indeed, let $f = f_1 w$, $f_1 \in L_w^p$. Then $s_h(f_1) = s_h(T_{w^{-1}} f) \in L_w^p$ and

$$\|A_h(f)\|_p = \|w s_h(f_1)\|_{p,w} \leqslant C_1 \|f_1\|_{p,w} = C_1 \|f\|_p.$$

By Lemma 1, the operator A_h is uniformly in $h > 0$ bounded in $L_{M,2\pi}$, i. e., for $f = f_1 w^{-1}$, $f \in L_{M,w}$, one has

$$\|s_h(f)\|_{M,w} = \|w s_h(f)\|_M = \|A_h(f_1)\|_M \leqslant C_2 \|f_1\|_M = C_2 \|f\|_{M,w}.$$

The proof is completed. □

Lemma 3 is stated in [10, (15) and (16)]. The statement concerning conjugation operator can be proved by the same method as in Lemma 2, while the inequalities (5) can be proved as in [1, Ch. VIII, § 20].

Lemma 3. *Let M be a Young function with nontrivial indices and $w \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$. Then the conjugation operator is bounded in $L_{M,w}$ and for $f \in L_{M,w}$ the inequalities*

$$\|S_n(f)\|_{M,w} \leqslant C_1 \|f\|_{M,w}, \quad \|f - S_n(f)\|_{M,w} \leqslant (C_1 + 1) E_n(f)_{M,w}, \quad (5)$$

hold for $n \in \mathbb{Z}_+$.

Lemma 4 is a variant of the Bernstein inequality proved in [10, Lemma 3]. It will be used in the proof of its extension, Theorem 4.

Lemma 4. *Let M and w be as in Lemma 3, $t_n \in T_n$, $n \in \mathbb{N}$, $r \in \mathbb{N}$. Then*

$$\|t_n^{(r)}\|_{M,w} \leqslant C n^r \|t_n\|_{M,w}. \quad (6)$$

4. Direct and inverse approximation theorems. The relation $A(t) \asymp B(t)$, $t \in T$, means that there exist $C_1, C_2 > 0$, such that $C_1 A(t) \leqslant \leqslant B(t) \leqslant C_2 A(t)$, $t \in T$. Since the proofs of Theorems 1–3 are similar to that of the corresponding results in [20, Theorems 1-3] and [19], we omit them. Theorems 2 and 3 are de facto proved in [10, Theorems 2 and 4] for even r (see Corollary 1).

Theorem 1. *Let M be a Young function with nontrivial indices and $w \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$, $r \in \mathbb{N}$. Then, for $f \in L_{M,w}$, we have:*

$$\Omega_r(f,t)_{M,w} \asymp K(f,t^r,L_{M,w},W^r L_{M,w}), \quad t \in [0,2\pi].$$

From the comparison of Theorem 1 and [10, Theorem 8], we obtain:

Corollary 1. *Under conditions of Theorem 1, one has:*

$$\Omega_{2r}(f,\delta)_{M,w} \asymp \Omega_r^*(f,\delta)_{M,w}, \quad \delta \in [0,2\pi].$$

Theorem 2. *Let M be a Young function with nontrivial indices and $w \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$, $r \in \mathbb{N}$. Then, for $f \in L_{M,w}$, the direct approximation theorem*

$$E_n(f)_{M,w} \leqslant C\Omega_r(f,(n+1)^{-1})_{M,w}, \quad n \in \mathbb{Z}_+,$$

holds.

Theorem 3. *Let M, w and r be as in Theorem 2. Then*

$$\Omega_r(f,n^{-1})_{M,w} \leqslant Cn^{-r}\sum_{k=1}^{n} k^{r-1}E_{k-1}(f)_{M,w}, \quad n \in \mathbb{N}.$$

If ω is increasing and continuous on $[0;2\pi]$, $\omega(0)=0$, then $\omega \in \Phi$. A function $\omega \in \Phi$ belongs to the Bary-Stechkin class B_α, $\alpha > 0$, if $\sum_{k=1}^{n} k^{\alpha-1}\omega(k^{-1}) = O(n^\alpha \omega(n^{-1}))$, $n \in \mathbb{N}$ (see [2]). Corollary 1 easily follows from Theorems 2 and 3.

Corollary 1. *Let M, w, and r be as in Theorem 2, $\omega \in B_r$. Then the conditions $E_n(f)_{M,w} = O(\omega((n+1)^{-1}))$, $n \in \mathbb{Z}_+$, and $\Omega_r(f,\delta)_{M,w} = O(\omega(\delta))$, $\delta \in [0,2\pi]$, are equivalent.*

5. Approximation by linear means of Fourier series and realization functionals. The following analogue of the Stechkin-Nikol'skii inequality (2) plays an important role in this paper. It also generalizes Lemma 4.

Theorem 4. *Let M, w and r be as in Theorem 2. Then, for $t_n \in T_n$, $n \in \mathbb{N}$, the following inequality holds:*

$$\|t_n^{(r)}\|_{M,w} \leqslant Cn^r \Omega_r(t_n, 1/n)_{M,w}.$$

Proof. By the definition of K-functional, we find $C_1 > 1$ and $g \in W^r L_{M,w}$, such that

$$\|t_n - g\|_{M,w} + n^{-r}\|g^{(r)}\|_{M,w} \leqslant C_1 K(f, n^{-r}, L_{M,w}, W^r L_{M,w}).$$

Note two famous properties $S_n(t_n) = t_n$ for $t_n \in T_n$ and $S_n^{(r)}(g) = S_n(g^{(r)})$ for $g \in W^r L_{M,w}$. Using Lemma 3, (6) from Lemma 4, and Theorem 1, we obtain:

$$\begin{aligned} n^{-r}\|t_n^{(r)}\|_{M,w} &\leqslant n^{-r}(\|t_n^{(r)} - S_n^{(r)}(g)\|_{M,w} + \|S_n^{(r)}(g)\|_{M,w}) = \\ &= n^{-r}(\|S_n^{(r)}(t_n - g)\|_{M,w} + \|S_n(g^{(r)})\|_{M,w}) \leqslant \\ &\leqslant n^{-r}(C_2 n^r \|t_n - g\|_{M,w} + C_3\|g^{(r)}\|_{M,w}) \leqslant \\ &\leqslant \max(C_2, C_3)(\|t_n - g\|_{M,w} + \|g^{(r)}\|_{M,w}) \leqslant \\ &\leqslant C_4 K(t_n, n^{-r}, L_{M,w}, W^r L_{M,w}) \leqslant C_5 \Omega_r(t_n, 1/n)_{M,w}. \end{aligned}$$

The proof is completed. □

Now we can prove a direct approximation result for the Riesz-Zygmund means.

Theorem 5. *Let M be a Young function with nontrivial indices and $w \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$, $r \in \mathbb{N}$. Then, for $f \in L_{M,w}$, we have*

$$\|f - Z_n^r(f)\|_{M,w} \leqslant C\Omega_r(f, 1/n)_{M,w}, \quad n \in \mathbb{N}.$$

Proof. Note that

$$|S_n(f)(x) - Z_n^r(f)(x)| = (n+1)^{-r}\left|\sum_{k=1}^{n} k^r A_k(f)(x)\right| = \frac{|S_n^{(r)}(f)(x)|}{(n+1)^r}$$

for even r and

$$|S_n(f)(x) - Z_n^r(f)(x)| = (n+1)^{-r}|\widetilde{S_n^{(r)}}(f)(x)|$$

for odd r. By Lemma 3 and Theorem 4, in both cases we obtain

$$\|S_n(f) - Z_n^r(f)\|_{M,w} \leqslant C_1(n+1)^{-r}\|S_n^{(r)}(f)\|_{M,w} \leqslant$$

$$\leqslant C_2(n+1)^{-r}n^r\Omega_r(S_n(f),1/n)_{M,w}. \tag{7}$$

By Lemma 3, Theorem 2, and the inequality $\Omega_r(g,\delta)_{M,w} \leqslant C_3\|g\|_{M,w}$, which follows from Lemma 2, we find that

$$\Omega_r(S_n(f),1/n)_{M,w} \leqslant \Omega_r(S_n(f)-f,1/n)_{M,w} + \Omega_r(f,1/n)_{M,w} \leqslant$$

$$\leqslant C_3\|S_n(f)-f\|_{M,w} + \Omega_r(f,1/n)_{M,w} \leqslant C_4\Omega_r(f,1/n)_{M,w}. \tag{8}$$

Finally, from (7), (8), Lemma 3, and Theorem 2, we deduce that

$$\|f-Z_n^r(f)\|_{M,w} \leqslant \|f-S_n(f)\|_{M,w} + \|S_n(f)-Z_n^r(f)\|_{M,w} \leqslant$$

$$\leqslant C_5(E_n(f)_{M,w} + \Omega_r(f,1/n)_{M,w}) \leqslant C_6\Omega_r(f,1/n)_{M,w}, \quad n \in \mathbb{N}.$$

□

Remark 1. *In [11, Corollary 2.1], the estimate*

$$\|f-Z_n^r(f)\|_{M,w} \leqslant C\Omega_r^*(f,1/n)_{M,w}, \quad n \in \mathbb{N}, \tag{9}$$

is stated. This corollary follows from Theorem 2.1 in [11], where the fulfilment of condition

$$n^{2r}|1-\lambda_\nu^{(n)}| \leqslant C\nu^{2r}, \quad \nu = 1,2,\ldots,n, \quad n \in \mathbb{N}, \tag{10}$$

is necessary. However, for $\lambda_\nu^{(n)} = 1-\nu^r/(n+1)^r$ *the condition (10) is not valid. The more correct variant of (9) is* $\|f-Z_n^{2r}(f)\|_{M,w} \leqslant \leqslant C\Omega_r^*(f,1/n)_{M,w}$, $n \in \mathbb{N}$.

Corollary 1. *Under conditions of Theorem 5, for* $k \in \mathbb{N}$, *the inequality*

$$\|f-v_n^r(f)\|_{M,w} \leqslant CE_n(f)_{M,w} \leqslant C\Omega_k(f,1/n)_{M,w}, \quad n \in \mathbb{N}, \tag{11}$$

holds.

Proof. Using summation by parts, we obtain:

$$v_n^r(f) = \frac{1}{(2n+1)^r-(n+1)^r}\sum_{k=n+1}^{2n}((k+1)^r-k^r)S_k(f)$$

and $v_n^r(t_n) = t_n$ for $t_n \in T_n$, $n \in \mathbb{N}$. From Theorem 5, we deduce that Z_n^r and v_n^r are uniformly in $n \in \mathbb{N}$ bounded in $L_{M,w}$. By a standard procedure, we deduce the first inequality in (11), while the second one follows from it and Theorem 2. □

Theorem 6 is proved similarly to Theorem 4 in [20].

Theorem 6. *Let M be a Young function with nontrivial indices and $w \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$, $r \in \mathbb{N}$. Then, for $f \in L_{M,w}$, we have:*

$$\|f - e_n^t(f)\|_{M,w} \leqslant C\Omega_r(f, 1/n)_{M,w}, \quad n \in \mathbb{N}.$$

Let $r, k \in \mathbb{N}$, $t > 0$, and $\tau_n(f) \in T_n$ be the polynomial of the best approximation for $f \in L_{M,w}$, i. e., $\|f - \tau_n(f)\|_{M,w} = E_n(f)_{M,w}$. Now we define five types of realization functionals:

$$R_r^\tau(f, n^{-r}, L_{M,w}) := \|f - \tau_n(f)\|_{M,w} + n^{-r}\|\tau_n^{(r)}(f)\|_{M,w},$$

$$R_r^z(f, n^{-r}, L_{M,w}) := \|f - Z_n^r(f)\|_{M,w} + n^{-r}\|(Z_n^r(f))^{(r)}\|_{M,w},$$

$$R_r^S(f, n^{-r}, L_{M,w}) := \|f - S_n(f)\|_{M,w} + n^{-r}\|S_n^{(r)}(f)\|_{M,w},$$

$$R_r^{v,k}(f, n^{-r}, L_{M,w}) := \|f - v_n^k(f)\|_{M,w} + n^{-r}\|(v_n^k(f))^{(r)}\|_{M,w},$$

$$R_r^{e,t}(f, n^{-r}, L_{M,w}) := \|f - e_n^t(f)\|_{M,w} + n^{-r}\|(e_n^t(f))^{(r)}\|_{M,w}.$$

The partial cases of Theorem 7 for $R_r^S(f, n^{-r}, L_{M,w})$ in the case of even r and for $R_r^{v,k}(f, n^{-r}, L_{M,w})$ in the case of even r and $k = 1$ were proved by Jafarov [12, Theorem 1.7].

Theorem 7. *Let M be a Young function with nontrivial indices and $w \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$, k,$r \in \mathbb{N}$. Then for $n \in \mathbb{N}$ and $f \in L_{M,w}$, we have:*

$$R_r^\tau(f, n^{-r}, L_{M,w}) \asymp K(f, n^{-r}, L_{M,w}, W^r L_{M,w}) \asymp R_r^z(f, n^{-r}, L_{M,w}) \asymp$$

$$\asymp R_r^S(f, n^{-r}, L_{M,w}) \asymp R_r^{v,k}(f, n^{-r}, L_{M,w}) \asymp R_r^{e,t}(f, n^{-r}, L_{M,w}).$$

Proof. It is clear that $K(f, n^{-r}, L_{M,w}, W^r L_{M,w})$ is majorized by all the realization functionals. Also, by Theorem 2, Theorem 5, Lemma 3, Corollary 1, and Theorem 6, we obtain that the first term of the realization functionals (in the same order as they are introduced above) is $O(\Omega_r(f, 1/n)_{M,w})$, $n \in \mathbb{N}$. By Theorem 1, we conclude that the first term of the realization functionals is majorized by $K(f, n^{-r}, L_{M,w}, W^r L_{M,w})$.

Finally, we consider, e. g., the case of $R_r^z(f, n^{-r}, L_{M,w})$. By Theorems 4, 5 and 1 we have

$$n^{-r}\|(Z_n^r(f))^{(r)}\|_{M,w} \leqslant C_1\Omega_r(Z_n^r(f), 1/n)_{M,w} \leqslant$$

$$\leqslant C_1(\Omega_r(Z_n^r(f)-f,1/n)_{M,w}+\Omega_r(f,1/n)_{M,w})\leqslant$$
$$\leqslant C_2(\|Z_n^r(f)-f\|_{M,w}+\Omega_r(f,1/n)_{M,w})\leqslant C_3\Omega_r(f,1/n)_{M,w}\leqslant$$
$$\leqslant C_4K(f,n^{-r},L_{M,w},W^rL_{M,w}),\quad n\in\mathbb{N}. \tag{12}$$

Thus, both terms of $R_r^z(f,n^{-r},L_{M,w})$ are majorized by $K(f,n^{-r},L_{M,w},W^rL_{M,w})$. To finish the proof for other realization functionals, we apply again Theorems 4 and 1, and also Theorem 2 for $R_r^\tau(f,n^{-r},L_{M,w})$, Lemma 3 for $R_r^S(f,n^{-r},L_{M,w})$, Corollary 1 for $R_r^{v,k}(f,n^{-r},L_{M,w})$, and Theorem 6 for $R_r^{e,t}(f,n^{-r},L_{M,w})$. $\square$

Note the property established in the proof of Theorem 7. For Z_n^r, see (12); for S_n, see (8); in other cases, the proof is similar.

Corollary 1. *Let M be a Young function with nontrivial indices and $w\in A_{1/\alpha_M}(\mathbb{T})\cap A_{1/\beta_M}(\mathbb{T})$, $k,r\in\mathbb{N}$. Then, for $n\in\mathbb{N}$ and $f\in L_{M,w}$, we have*

$$\Omega_r(\tau_n(f),1/n)_{M,w}\leqslant C\Omega_r(f,1/n)_{M,w},$$
$$\Omega_r(Z_n^r(f),1/n)_{M,w}\leqslant C\Omega_r(f,1/n)_{M,w},$$
$$\Omega_r(S_n(f),1/n)_{M,w}\leqslant C\Omega_r(f,1/n)_{M,w},$$
$$\Omega_r(v_n^k(f),1/n)_{M,w}\leqslant C\Omega_r(f,1/n)_{M,w},$$
$$\Omega_r(e_n^t(f),1/n)_{M,w}\leqslant C\Omega_r(f,1/n)_{M,w}.$$

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