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INTERPOLATION PROBLEMS FOR FUNCTIONS WITH ZERO BALL MEANS

Abstract. Let $n \geqslant 2$, $V_r(\mathbb{R}^n)$ be the set of functions with zero integrals over all balls in $\mathbb{R}^n$ of radius r. Various interpolation problems for the class $V_r(\mathbb{R}^n)$ are studied. In the case when the set of interpolation nodes is finite, we solve the interpolation problem under general conditions. For the problems with infinite set of nodes, some sufficient conditions of solvability are founded.

Note that an essential condition is that the definition of the class $V_r(\mathbb{R}^n)$ involves integration over balls. For instance, it can be shown that the analogues of our results in which the class of functions is defined using zero integrals over all shifts of a fixed parallelepiped in $\mathbb{R}^n$ do not hold true.

Key words: *interpolation problems, spherical means, mean periodicity*

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1. Introduction. Let $\mathbb{R}^n$ be real Euclidean space of dimension $n \geqslant 2$ with Euclidean norm $|\cdot|$. Assume that $f \in L_{\text{loc}}(\mathbb{R}^n)$ and the equality

$$\int_{|x|\leqslant r} f(x+y)dx = 0 \tag{1}$$

holds for some fixed $r > 0$ and all $y \in \mathbb{R}^n$. Is it true that $f = 0$? This question was addressed in 1929 by the well-known Romanian mathematician D. Pompeiu, who stated that the answer is positive for $n = 2$ (see, e. g., [15]). However, fifteen years later L. Chakalov [15] found an error in Pompeiu's proof. Moreover, he showed that the function $f(x_1, x_2) = \sin(\lambda x_1)$ has zero integrals over all unit disks in $\mathbb{R}^2$ if λ is a zero of the Bessel function J_1. Later, it was found that similar examples of nonzero functions with condition (1) can be constructed by a method proposed by J. Radon as early as 1917. This method is based on the mean value theorem for eigenfunctions of the Laplacian and can be extended to an arbitrary two-point homogeneous space X (see [10, Part 2, Sect. 2.4]). Additionally, the method allows one to construct nonzero functions on X with zero integrals over all spheres of fixed radius.

Let $V_r(\mathbb{R}^n)$ denote the set of functions $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying (1) for all $y \in \mathbb{R}^n$. Over the recent fifty years, this class of functions and its various analogues and generalizations have been intensively studied by F. John, J. Delsarte, J. D. Smith, L. Zalcman, C. A. Berenstein, and others (see the overviews in [1], [15], [16] and monographs [10–12], which provide extensive bibliographies). The basic directions in these studies can be listed as follows.

1. The study of zero sets and corresponding uniqueness theorems for the class $V_r(\mathbb{R}^n)$ [5], [8], [10–12]. This direction goes back to the uniqueness theorem of John [5, Chapter 6] for functions with zero spherical means.

2. The study of admissible constraints on the growth of nonzero functions of the class $V_r(\mathbb{R}^n)$ and its analogues on unbounded domains (theorem of the Liouville and Phragmén-Lindelöf types [4–6], [8–12]).

3. The study of functions with conditions of type (1), in which r belongs to a given two-element set [1], [7], [8], [10–12], [15], [16] (two-radius theorem). The first result in this direction is Delsarte's classical theorem on the characterization of harmonic functions by a mean-value equation satisfied by only two radii.

4. Description of functions of the class $V_r(\mathbb{R}^n)$ in the form of series in terms of spherical harmonics [10–13] (analogues of Taylor and Laurent expansions in the theory of analytic functions).

5. The problem of continuation [10–12] .

6. Theorems on removable singularities [10–14].

7. Integral geometry problems of reconstructing functions of specified classes from given spherical means [1], [2], [11], [12], [15], [16].

8. Approximation of functions with zero spherical means by linear combinations of special functions [10–12].

9. The study of analogues and generalizations of the class $V_r(\mathbb{R}^n)$ on various homogeneous spaces and groups (e. g., on Riemannian symmetric spaces) [1], [2], [7], [10–12], [14–16].

In this paper, interpolation problems for the class $V_r(\mathbb{R}^n)$ are studied.

In the case when the set of interpolation nodes is finite, a theorem on the existence of a solution to the interpolation problem is obtained under general assumptions (see Theorem 1 below). Next, we provide some sufficient conditions for the solvability of multiple interpolation problems with an infinite number of nodes (see Theorem 2).

2. Formulations of the main results. As usual, the symbols $\mathbb{N}$, $\mathbb{Z}_+$, and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, and complex numbers, respectively.

First consider the interpolation problem for the class $V_r(\mathbb{R}^n)$ with a finite set of interpolation nodes.

Theorem 1. *Let $q \in \mathbb{N}$. Then, for any set of distinct points $a_1, \dots, a_q$ in $\mathbb{R}^n$ and for any collection of constants $b_k \in \mathbb{C}$, $k = 1, \dots, q$, there exists a real analytic function $f \in V_r(\mathbb{R}^n)$ satisfying the conditions*

$$f(a_k) = b_k, \quad k = 1, \dots, q. \tag{2}$$

Note that an essential condition in Theorem 1 is that the definition of the class $V_r(\mathbb{R}^n)$ involves integration over balls. It can be shown that the analogue of Theorem 1, in which the class of functions is defined using zero integrals over all shifts of a fixed parallelepiped in $\mathbb{R}^n$ does not hold true. Indeed, any such function of the class $C^\infty(\mathbb{R}^n)$ satisfies a linear difference equation relating the values of the function and its partial derivatives at the vertices of the given parallelepiped (see [11, Part 4]. Therefore, if the vertices of this parallelepiped are used as interpolation nodes, then the numbers b_k in condition (2) cannot be taken as arbitrary.

Theorem 1 has the following immediate consequence, which shows that the solution of interpolation problem (2) for the class $V_r(\mathbb{R}^n)$ is not unique.

Corollary 1. *Let $q \in \mathbb{N}$. Then, for any set of distinct points $a_1, \dots, a_q$ in $\mathbb{R}^n$, there exists a nonzero real analytic function $f \in V_r(\mathbb{R}^n)$ satisfying the conditions*

$$f(a_k) = 0, \quad k = 1, \dots, q.$$

In the general case, interpolation problems for the class $V_r(\mathbb{R}^n)$ with an infinite set of interpolation nodes is much more complicated than in the case of a finite set. Below, we present some sufficient conditions for the existence of a solution to the multiple interpolation problem.

We set

$$A = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geqslant 0, x_2 = \dots = x_n = 0\}.$$

Theorem 2. *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of distinct points in A such that*

$$\lim_{k\to\infty} a_k = \infty. \tag{3}$$

Then, for any sequence $\{m_k\}_{k=1}^{\infty}$ of nonnegative integers and for any collection of constants

$$b_{k,l} \in \mathbb{C} \quad (k \in \mathbb{N}, \quad l = 0, \dots, m_k),$$

there exists a real analytic function $f \in V_r(\mathbb{R}^n)$, such that

$$\left(\frac{\partial}{\partial x_1}\right)^l f(a_k) = b_{k,l} \tag{4}$$

for all $k \in \mathbb{N}$, $l = 0, \dots, m_k$.

This result yields the following analogue of Corollary 1.

Corollary 1. *Suppose that the sequence $\{a_k\}_{k=1}^{\infty}$ of distinct points in A satisfies condition (3). Then, for any sequence $\{m_k\}_{k=1}^{\infty}$ of nonnegative integers there exists a real analytic function $f \in V_r(\mathbb{R}^n)$, such that*

$$\left(\frac{\partial}{\partial x_1}\right)^l f(a_k) = 0, \quad k \in \mathbb{N}, \quad l = 0, \dots, m_k,$$

and $f|_A \neq 0$.

3. Notation and some auxiliary statements.

For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$, we set

$$(z, \zeta) = \sum_{j=1}^{n} z_j \zeta_j.$$

Let $t = (t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$, $x \in \mathbb{R}^n$, $z \in \mathbb{C}$,

$$h(x, z, t) = e^{i(x_1 t_1 + \dots + x_{n-1} t_{n-1})} \cos\left(x_n \sqrt{z - t_1^2 - \dots - t_{n-1}^2}\right). \tag{5}$$

Also, let

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \ |x| = 1\},$$

$$U_{a,b} = \left\{t \in \mathbb{R}^{n-1} : \ a < t_1^2 + \dots + t_{n-1}^2 < b\right\}.$$

We denote the group of rotations of $\mathbb{R}^n$ by $SO(n)$. Let J_ν be the Bessel function of the first kind of order ν. For $z > 0$, we define

$$\mathbf{I}_\nu(z) = \frac{J_\nu(z)}{z^\nu}.$$

Note that the function $\mathbf{I}_\nu$ with $\nu > -1$ has an infinite number of positive zeros (see [3, Ch. 7, Sect. 7.9]). The following lemmas are needed in the proof of the main results.

Lemma 1. *Let $w_k = (w_{k,1}, \ldots, w_{k,n}) \in \mathbb{R}^n$, $k \in \{1, \ldots, q\}$ and assume that*

$$w_{i,1} \neq w_{j,1} \quad \text{for} \quad i, j \in \{1, \ldots, q\}, \quad i \neq j. \tag{6}$$

Let $0 < a < b$, $z > 0$. Suppose that there exist $c_k \in \mathbb{C}$, $k = 1, \ldots, q$, such that

$$\sum_{k=1}^{q} c_k h(w_k, z, t) = 0 \tag{7}$$

for all $t \in U_{a,b}$. Then $c_k = 0$ for all k.

Proof. First consider the case $n \geqslant 3$. We assume that $t_1 \in (\alpha, \beta)$ for some $0 < \alpha < \beta$, the numbers $t_2, \ldots, t_{n-1}$ are fixed and $t = (t_1, \ldots, t_{n-1}) \in U_{a,b}$.

Let

$$\lambda = \sum_{j=2}^{n-1} w_{k,j} t_j, \quad \mu = z - \sum_{j=2}^{n-1} t_j^2.$$

Consider the entire function

$$\varphi(\zeta) = \sum_{k=1}^{q} c_k e^{i(w_{k,1}\zeta + \lambda)} \cos\left(w_{k,n}\sqrt{\mu^2 - \zeta^2}\right), \quad \zeta \in \mathbb{C}. \tag{8}$$

By the definition of h and relation (7), we see that $\varphi(t_1) = 0$. Since $t_1 \in (\alpha, \beta)$, the function φ vanishes due to the uniqueness theorem for analytic functions. Suppose now that $\zeta^2 = \mu^2 - \eta$, where $\eta > 0$, $\eta \to +\infty$ and $\mathrm{Im}\zeta \to -\infty$. Then, in view of (6), the equality

$$\sum_{k=1}^{q} c_k e^{i(w_{k,1}\zeta + \lambda)} \cos\left(w_{k,n}\sqrt{\eta}\right) = 0$$

brings us to the conclusion that $c_p = 0$ with $w_{p,1} = \max_k w_{k,1}$. Similarly we obtain $c_k = 0$ for all k, as required.

For the case $n = 2$, it is sufficient to repeat the argument for the function φ in (8) with $\lambda = 0$, $\mu = z$. The proof of Lemma 1 is now complete. □

Lemma 2. *Let $t = (t_1, \dots, t_{n-1}) \in \mathbb{C}^{n-1}$, $z > 0$ and*

$$\mathbf{I}_{\frac{n}{2}}\left(r\sqrt{z}\right) = 0. \tag{9}$$

Then the function $h(x, z, t)$ is in the class $V_r(\mathbb{R}^n)$.

Proof. We set $\zeta_j = t_j$ for $j \in \{1, \dots, n-1\}$ and

$$\zeta_n = \sqrt{z - \sum_{j=1}^{n-1} t_j^2}.$$

Then $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$ and $(\zeta, \zeta) = z$. Consider the function

$$u_\zeta(x) = e^{i(x,\zeta)}, \quad x \in \mathbb{R}^n.$$

For each $y \in \mathbb{R}^n$, one has

$$\int_{|x|\leqslant r} u_\zeta(x+y)dx = e^{i(y,\zeta)} \int_{|x|\leqslant r} e^{i(x,\zeta)}dx = (2\pi)^{\frac{n}{2}} r^n \mathbf{I}_{\frac{n}{2}}(r\sqrt{z}) e^{i(y,\zeta)}$$

(see [3, Ch. 7, Sect. 7.12 (7)]). Together with (9), this shows that $u_\zeta \in V_r(\mathbb{R}^n)$. Taking into account that

$$h(x, z, t) = \frac{1}{2}\left(u_\zeta(x_1, \dots, x_n) + u_\zeta(x_1, \dots, -x_n)\right)$$

we obtain the required assertion. □

4. Proof of Theorem 1. Bearing in mind that the points $a_1, \dots, a_q$ are pairwise different, we infer that there exists $\xi \in \mathbb{S}^{n-1}$ such that

$$(\xi, a_i - a_j) \neq 0 \quad \text{for all} \quad i, j \in \{1, \dots, q\}, \quad i \neq j. \tag{10}$$

Since the group $SO(n)$ acts transitively on $\mathbb{S}^{n-1}$, one has

$$\tau\xi = (1, 0, \dots, 0) \quad \text{for some} \quad \tau \in SO(n). \tag{11}$$

We set

$$w_k = \tau a_k = (w_{k,1}, \dots, w_{k,n}) \in \mathbb{R}^n, \quad k \in \{1, \dots, q\}. \tag{12}$$

Condition (10) shows that

$$(\tau\xi, w_i - w_j) \neq 0, \quad i, j \in \{1, \dots, q\}, \quad i \neq j.$$

Together with (11), this yields

$$w_{i,1} \neq w_{j,1} \quad \text{for all} \quad i, j \in \{1, \dots, q\}, \quad i \neq j.$$

Assume now that $0 < a < b$, $\zeta > 0$, and $J_{\frac{n}{2}}(r\zeta) = 0$. Consider the functions

$$g_k(t) = h(w_k, \zeta^2, t), \quad t \in U_{a,b},$$

where $k = 1, \dots, q$ and h are defined by (5). Due to Lemma 1, the functions g_k are linearly independent on $U_{a,b}$. For each $m \in \{1, \dots, q\}$, let L_m denote the linear subspace of $L^2(U_{a,b})$ generated by the functions g_k, such that $k \neq m$. Since $g_m \notin L_m$, we obtain, by the Hahn-Banach theorem, that there exists a continuous linear functional Φ_m on $L^2(U_{a,b})$, such that

$$\Phi_m|_{L_m} = 0 \quad \text{and} \quad \Phi_m(g_m) = 1. \tag{13}$$

Using the Riesz theorem, we see that Φ_m has the form

$$\Phi_m(u) = \int_{U_{a,b}} u(t)\varphi_m(t)dt, \quad u \in L^2(U_{a,b})$$

for some function $\varphi_m \in L^2(U_{a,b})$. We now define the function G by the formula

$$G(x) = \sum_{m=1}^{q} b_m \int_{U_{a,b}} h(x, \zeta^2, t)\varphi_m(t)dt, \quad x \in \mathbb{R}^n. \tag{14}$$

Formula (14) guarantees that

$$G(w_k) = \sum_{m=1}^{q} b_m \int_{U_{a,b}} h(w_k, \zeta^2, t)\varphi_m(t)dt =$$

$$= \sum_{m=1}^{q} b_m \Phi_m(g_k) = b_k \tag{15}$$

for all $k \in \{1, \dots, q\}$. Next, for each $y \in \mathbb{R}^n$ we have

$$\int_{|x|\leqslant r} G(x+y)dx = \sum_{m=1}^{q} b_m \int_{U_{a,b}} \int_{|x|\leqslant r} h(x+y, \zeta^2, t)dx\varphi_m(t)dt = 0 \tag{16}$$

because of Lemma 2. Setting $F(x) = G(\tau x)$, we see, from (15) and (12), that

$$F(a_k) = G(\tau a_k) = G(w_k) = b_k \quad \text{for all} \quad k.$$

In addition, (16) shows that $F \in V_r(\mathbb{R}^n)$. Since the function F is real analytic, this completes the proof of Theorem 1.

5. Proof of Theorem 2. We set

$$\alpha_k = \exp|a_k|, \quad k \in \mathbb{N}.$$

By the hypotheses on $\{a_k\}$, the numbers α_k are pairwise different positive numbers such that

$$\lim_{k\to\infty} \alpha_k = +\infty.$$

Owing to the classical Hadamard theorem, there exists an entire function $H : \mathbb{C} \to \mathbb{C}$ satisfying the conditions

$$\begin{cases} H^{(j)}(\alpha_k) = 0 & \text{for } j \in \{0, \dots, m_k\}, \quad k \in \mathbb{N}, \\ H^{(j)}(\alpha_k) \neq 0 & \text{if } \ j = m_k + 1, \quad k \in \mathbb{N}. \end{cases} \tag{17}$$

Then the functions

$$H_{k,\nu}(z) = \frac{H(z)}{(z-\alpha_k)^\nu}, \quad k \in \mathbb{N}, \quad \nu \in \{1, \dots, m_k + 1\} \tag{18}$$

are entire and

$$\begin{cases} H_{k,\nu}^{(j)}(\alpha_k) = 0 & \text{if } 0 \leqslant j \leqslant m_k - \nu, \\ H_{k,\nu}^{(j)}(\alpha_k) \neq 0 & \text{for } \ j = m_k + 1 - \nu. \end{cases} \tag{19}$$

Let $w : \mathbb{C} \to \mathbb{C}$ be an arbitrary entire function. It is easy to see that for each $l \in \mathbb{Z}_+$ there exist algebraic polynomials $p_{l,j}$, $j \in \{0, \dots, l\}$ such that

$$\left(\frac{d}{dt}\right)^l (w(e^t)) = \sum_{j=0}^{l} w^{(j)}(e^t) p_{l,j}(e^t), \quad t \in \mathbb{C}. \tag{20}$$

Moreover, the polynomials $p_{l,j}$ are independent of w and $p_{l,l}(z) = z^l$. This shows that for each $k \in \mathbb{N}$ there exist the constants $\beta_{k,j} \in \mathbb{C}$, $j \in \{0, \dots, m_k\}$, such that

$$\sum_{j=0}^{l} \beta_{k,j} p_{l,j}(\alpha_k) = b_{k,l} \quad \text{for all} \quad l \in \{0, \dots, m_k\}. \tag{21}$$

Using now (19), we see that there exist the constants $\gamma_{k,\nu} \in \mathbb{C}$ ($k \in \mathbb{N}$, $\nu \in \{1, \dots, m_k + 1\}$), such that the functions

$$H_k(z) = \sum_{\nu=1}^{m_k+1} \gamma_{k,\nu} H_{k,\nu}(z) \tag{22}$$

satisfy the conditions

$$H_k^{(j)}(\alpha_k) = \beta_{k,j}, \quad j \in \{0, \dots, m_k\}. \tag{23}$$

In addition, it follows from (17), (18) and (22) that

$$H_k^{(j)}(\alpha_p) = 0 \quad \text{if} \quad p \in \mathbb{N}, \quad p \neq k, \quad j \in \{0, \dots, m_p\}. \tag{24}$$

Assume that

$$M_k = \max_{|z| \leqslant \alpha_k/4} |H_k(z)|, \quad \lambda_k \in \mathbb{N}$$

and

$$\lambda_k > 2m_k + k + M_k \quad \text{for all} \quad k \in \mathbb{N}. \tag{25}$$

We now define the function g_k by the formula

$$g_k(z) = \eta_k \int_0^z (1 - \zeta)^{m_k} \zeta^{\lambda_k} d\zeta, \quad z \in \mathbb{C}, \tag{26}$$

where

$$\eta_k = \frac{\Gamma(m_k + \lambda_k + 2)}{\Gamma(m_k + 1)\Gamma(\lambda_k + 1)}, \quad k \in \mathbb{N}. \tag{27}$$

Relation (27) yields

$$\eta_k \leqslant (m_k + \lambda_k + 1) \sum_{j=0}^{m_k+\lambda_k} \binom{m_k + \lambda_k}{j} \leqslant (m_k + \lambda_k + 1) 2^{m_k+\lambda_k}.$$

Hence,

$$|g_k(z)| \leqslant (m_k + \lambda_k + 1) 2^{m_k+\lambda_k} (1 + |z|)^{m_k} |z|^{\lambda_k+1}, \quad z \in \mathbb{C}. \tag{28}$$

In addition, it follows from (26) and (27) that

$$\begin{cases} g_k(1) = 1, \\ g_k^{(s)}(1) = 0 \quad \text{if } 1 \leqslant s \leqslant m_k. \end{cases} \tag{29}$$

Consider the function

$$w(z)=\sum_{k=1}^{\infty} g_k\left(\frac{z}{\alpha_k}\right)H_k(z), \quad z\in\mathbb{C}. \tag{30}$$

We claim that the series in (30) converges locally uniformly in $\mathbb{C}$. Let $R>0$, $\alpha_k>4R$ and assume that $|z|\leqslant R$. Then

$$|H_k(z)|\leqslant \max_{|z|\leqslant R}|H_k(z)|\leqslant \max_{|z|\leqslant \alpha_k/4}|H_k(z)|=M_k.$$

This together with (28) implies that

$$\left|g_k\Big(\frac{z}{\alpha_k}\Big)H_k(z)\right|\leqslant (m_k+\lambda_k+1)2^{m_k+\lambda_k}\Big(\frac{R}{\alpha_k}\Big)^{\lambda_k+1}\Big(1+\frac{R}{\alpha_k}\Big)^{m_k}M_k.$$

Bearing in mind that

$$m_k<\frac{\lambda_k}{2}, \quad \frac{R}{\alpha_k}\leqslant\frac{1}{4} \quad \text{and} \quad M_k<\lambda_k,$$

we obtain

$$\left|g_k\Big(\frac{z}{\alpha_k}\Big)H_k(z)\right|\leqslant \Big(\frac{3\lambda_k}{2}+1\Big)\lambda_k\Big(\frac{5}{8}\Big)^{\lambda_k/2}.$$

Since $\lambda_k>k$ (see (25)), this shows that the series in (30) converges uniformly in $|z|\leqslant R$. Consequently, the series converges locally uniformly in $\mathbb{C}$ and the function w is entire.

Next, relations (23) and (29) show that

$$w^{(j)}(\alpha_k)=\beta_{k,j}, \quad j\in\{0,\dots,m_k\}, \quad k\in\mathbb{N}. \tag{31}$$

Assume that Taylor's expansion of w has the form

$$w(z)=\sum_{p=0}^{\infty}c_p z^p, \quad z\in\mathbb{C}. \tag{32}$$

Then

$$\sum_{p=0}^{\infty}|c_p|R^p<+\infty \tag{33}$$

for each $R>0$.

Let $\nu > 0$, $\mathbf{I}_{\frac{n}{2}}(r\nu) = 0$ and $p \in \mathbb{Z}_+$. Take $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ such that $\zeta_1 = -ip$, $\zeta_2^2 + \cdots + \zeta_n^2 = p^2 + \nu^2$ and define

$$f_p(x) = e^{i(x,\zeta)}, \quad x \in \mathbb{R}^n.$$

The proof of Lemma 2 shows that $f_p \in V_r(\mathbb{R}^n)$. Consider the function

$$f(x) = \sum_{p=0}^{\infty} c_p f_p(x), \quad x \in \mathbb{R}^n.$$

Condition (33) ensures us that the function f is real analytic and $f \in V_r(\mathbb{R}^n)$. It follows by the definition of f_p and (32) that

$$f(x_1, 0, \ldots, 0) = w(e^{x_1}), \quad x_1 \in \mathbb{R}.$$

Using now relations (31), (20) and (21) we see that f satisfies (4). Thus the proof of Theorem 2 is complete.

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