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MULTIPARAMETER FRACTIONAL DIFFERENTIATION WITH NON SINGULAR KERNEL

Abstract. We introduce here Caputo and Riemann-Liouville type non singular kernel very general multi parameter left and right side fractional derivatives and we prove their continuity. These have the advantage to describe accurately complex situations and phenomena and we can measure their fractional smoothness with memory and nonlocality. Then, we derive related left and right fractional integral inequalities of Hardy, Opial and Hilbert-Pachpatte types, also of Hardy type involving convexity.

Key words: *Non singular Kernel, fractional derivative, multi parameters, fractional integral inequalities*

2020 Mathematical Subject Classification: *26A33, 26D10, 26D15*

1. Background. Of great inspiration here are the articles [3], [4], [6]. But the most important for the author to write this article has been the work of R. K. Saxena, S. L. Kalla and Ravi Saxena, [7], about the multivariate analogue of generalized Mittag-Leffler function. Their extension allows at the same time to be treated both the univariate and multivariate level because of the many parameters involved. That gives us the advantage to record and treat lots of various data resulting from complicated natural phenomena and from diverse socioeconomical interactions, as well as mechanics, chemistry, etc. Also, their function serves well as a non singular kernel.

Here we use the multivariate analogue of generalized Mittag-Leffler function, see [7], defined for $\lambda, \gamma_j, \rho_j, z_j \in \mathbb{C}$, $\mathrm{Re}\,(\rho_j) > 0$ $(j = 1, \ldots, m)$ in terms of a multiple series of the form:

$$E_{(\rho_j),\lambda}^{(\gamma_j)}\,(z_1, \ldots, z_m) = E_{(\rho_1,\ldots,\rho_m),\lambda}^{(\gamma_1,\ldots,\gamma_m)}\,(z_1, \ldots, z_m) =$$

$$= \sum_{k_1,\ldots,k_m=0}^{\infty} \frac{(\gamma_1)_{k_1}\ldots(\gamma_m)_{k_m}}{\Gamma\left(\lambda+\sum_{j=1}^{m}k_j\rho_j\right)} \frac{z_1^{k_1}\ldots z_m^{k_m}}{k_1!\ldots k_m!}, \tag{1}$$

where $(\gamma_j)_{k_j}$ is the Pochhammer symbol, Γ is the gamma function. This is a special case of the generalized Lauricella series in several variables, see [8, p. 454] and [10].

By [9, p. 157], (1) converges for $\operatorname{Re}(\rho_j) > 0$, $j = 1, \ldots, m$.
In what follows we will use the particular case of $E_{(\rho,\ldots,\rho),\lambda}^{(\gamma_1,\ldots,\gamma_m t)}[\omega_1 t^\rho, \ldots, \omega_m t^\rho]$, denoted by $E_{(\rho),\lambda}^{(\gamma_j)}[\omega_1 t^\rho, \ldots, \omega_m t^\rho]$, where $0 < \rho < 1$, $t \geqslant 0$, $\lambda > 0$, $\gamma_j \in \mathbb{R}$ with $(\gamma_j)_{k_j} := \gamma_j(\gamma_j+1)\ldots(\gamma_j+k_j-1)$, $\omega_j \in \mathbb{R}-\{0\}$, for $j = 1, \ldots, m$. Let $f \in C^1([a,b])$, we define the following Caputo type generalized left fractional derivative with non singular kernel of order ρ, as

$$D_{a*}^{\rho} f(x) := {}_{(\gamma_j)(\omega_j)}^{CA} D_{a*}^{\rho,\lambda} f(x) :=$$

$$:= \frac{A(\rho)}{1-\rho} \int_a^x E_{(\rho),\lambda}^{(\gamma_j)} \left[\frac{-\omega_1 \rho}{1-\rho}(x-t)^\rho, \ldots, \frac{-\omega_m \rho}{1-\rho}(x-t)^\rho\right] f'(t)dt, \quad x \in [a,b], \tag{2}$$

where $A(\rho)$ is a normalizing constant.

Let now $f \in C^{n+1}([a,b])$, $n \in \mathbb{Z}_+$.

We define the Caputo type generalized left fractional derivative with non singular kernel of order $n+\rho$, as

$$D_{a*}^{n+\rho} f(x) := {}_{(\gamma_j)(\omega_j)}^{CA} D_{a*}^{n+\rho,\lambda} f(x) :=$$

$$:= \frac{A(\rho)}{1-\rho} \int_a^x E_{(\rho),\lambda}^{(\gamma_j)} \left[\frac{-\omega_1 \rho}{1-\rho}(x-t)^\rho, \ldots, \frac{-\omega_m \rho}{1-\rho}(x-t)^\rho\right] f^{(n+1)}(t)dt, \tag{3}$$

for $x \in [a,b]$.

Similarly, we define the Caputo type generalized right fractional derivative with non singular kernel of order ρ, as

$$D_{b-}^{\rho} f(x) := {}_{(\gamma_j)(\omega_j)}^{CA} D_{b-}^{\rho,\lambda} f(x) :=$$

$$:= \frac{-A(\rho)}{1-\rho} \int_x^b E_{(\rho),\lambda}^{(\gamma_j)} \left[\frac{-\omega_1 \rho}{1-\rho}(t-x)^\rho, \ldots, \frac{-\omega_m \rho}{1-\rho}(t-x)^\rho\right] f'(t)dt, \quad x \in [a,b]. \tag{4}$$

And, for $f \in C^{n+1}([a,b])$, $n \in \mathbb{Z}_+$, we define

$$D_{b-}^{n+\rho} f(x) := {}_{(\gamma_j)(\omega_j)}^{CA} D_{b-}^{n+\rho,\lambda} f(x) :=$$

$$:= (-1)^{n+1} \frac{A(\rho)}{1-\rho} \int_x^b E_{(\rho),\lambda}^{(\gamma_j)} \left[\frac{-\omega_1 \rho}{1-\rho}(t-x)^\rho, \ldots, \frac{-\omega_m \rho}{1-\rho}(t-x)^\rho \right] f^{(n+1)}(t) dt, \tag{5}$$

$x \in [a,b]$.

The above derivatives generalize the Atangana-Baleanu fractional derivative, see [3].

Similarly, we defie the Riemann-Liouville corresponding versions of above fractional derivatives for $f \in C([a,b])$, $n \in \mathbb{Z}_+$, $0 < \rho < 1$, the left one:

$$D_{a}^{n+\rho} f(x) := {}_{(\gamma_j)(\omega_j)}^{RLA} D_{a}^{n+\rho,\lambda} f(x) :=$$

$$:= \frac{A(\rho)}{1-\rho} \frac{d^{n+1}}{dx^{n+1}} \int_a^x E_{(\rho),\lambda}^{(\gamma_j)} \left[\frac{-\omega_1 \rho}{1-\rho}(x-t)^\rho, \ldots, \frac{-\omega_m \rho}{1-\rho}(x-t)^\rho \right] f(t) dt, \tag{6}$$

$x \in [a,b]$, and right one:

$$D_{b}^{n+\rho} f(x) := {}_{(\gamma_j)(\omega_j)}^{RLA} D_{b}^{n+\rho,\lambda} f(x) :=$$

$$:= \frac{(-1)^{n+1} A(\rho)}{1-\rho} \frac{d^{n+1}}{dx^{n+1}} \int_x^b E_{(\rho),\lambda}^{(\gamma_j)} \left[\frac{-\omega_1 \rho}{1-\rho}(t-x)^\rho, \ldots, \frac{-\omega_m \rho}{1-\rho}(t-x)^\rho \right] f(t) dt, \tag{7}$$

$x \in [a,b]$.

In this work we emphasize on the Caputo version. The advantage of our fractional derivatives here is, that they have non singular and very general kernels able to incorporate lots of various fixed data sets from complicated physical phenomena, carried by the different sets of their parameters.

We present the following basic Hardy type inequalities:

Theorem 1. *All as above with $\gamma_j > 0$, $j = 1, \ldots, m$; $\lambda = 1$. Then*

$$\left\{ \left\| D_{a*}^{n+\rho} f \right\|_\infty, \left\| D_{b-}^{n+\rho} f \right\|_\infty \right\} \leqslant \frac{(b-a)\,|A(\rho)|}{1-\rho} \times$$

$$\times E_{(\rho),2}^{(\gamma_j)} \left[\frac{|\omega_1|\,\rho}{1-\rho}(b-a)^\rho, \ldots, \frac{|\omega_m|\,\rho}{1-\rho}(b-a)^\rho \right] \left\| f^{(n+1)} \right\|_\infty < \infty, \tag{8}$$

where $n \in \mathbb{Z}_+$.

Proof. We prove the first inequality, for the second one as similar the proof is omitted. We have by (3) that

$$\left| D_{a*}^{n+\rho} f(x) \right| \leqslant$$

$$\leqslant \frac{|A(\rho)|}{1-\rho} \int_a^x \left(E_{(\rho),}^{(\gamma_j)} \left[\frac{|\omega_1| \rho}{1-\rho} (x-t)^{\rho}, \ldots, \frac{|\omega_m| \rho}{1-\rho} (x-t)^{\rho} \right] dt \right) \left\| f^{(n+1)} \right\|_\infty =$$

$$\overset{([?],\ \text{p. } 175)}{=} \frac{|A(\rho)|}{1-\rho} \Bigg[\sum_{k_1,\ldots,k_m=0}^{\infty} \frac{(\gamma_1)_{k_1} \cdots (\gamma_m)_{k_m}}{\Gamma\left(1+\rho\left(\sum_{i=1}^m k_i\right)\right) \prod_{i=1}^m k_i!} \left(\frac{\rho}{1-\rho}\right)^{\sum_{i=1}^m k_i} \times$$

$$\times \left(\prod_{i=1}^m |\omega_i|^{k_i} \right) \left(\int_a^x (x-t)^{\rho\left(\sum_{i=1}^m k_i\right)} dt \right) \Bigg] \left\| f^{(n+1)} \right\|_\infty =$$

$$= \frac{|A(\rho)|}{1-\rho} \Bigg[\sum_{k_1,\ldots,k_m=0}^{\infty} \frac{(\gamma_1)_{k_1} \cdots (\gamma_m)_{k_m}}{\Gamma\left(1+\rho\left(\sum_{i=1}^m k_i\right)\right) \prod_{i=1}^m k_i!} \left(\frac{\rho}{1-\rho}\right)^{\sum_{i=1}^m k_i} \times$$

$$\times \left(\prod_{i=1}^m |\omega_i|^{k_i} \right) \frac{(x-a)^{\rho\left(\sum_{i=1}^m k_i\right)+1}}{\left(\rho\left(\sum_{i=1}^m k_i\right)+1\right)} \Bigg] \left\| f^{(n+1)} \right\|_\infty \leqslant \tag{9}$$

$$\leqslant (b-a) \frac{|A(\rho)|}{1-\rho} \Bigg[\sum_{k_1,\ldots,k_m=0}^{\infty} \frac{(\gamma_1)_{k_1} \cdots (\gamma_m)_{k_m}}{\Gamma\left(2+\rho\left(\sum_{i=1}^m k_i\right)\right) \prod_{i=1}^m k_i!} \left(\frac{\rho}{1-\rho}\right)^{\sum_{i=1}^m k_i} \times$$

$$\times \left(\prod_{i=1}^m |\omega_i|^{k_i} \right) (b-a)^{\rho\left(\sum_{i=1}^m k_i\right)} \Bigg] \left\| f^{(n+1)} \right\|_\infty =$$

$$= \frac{(b-a)|A(\rho)|}{1-\rho} \left[E_{(\rho),2}^{(\gamma_j)} \left[\frac{|\omega_1| \rho}{1-\rho} (b-a)^{\rho}, \ldots, \frac{|\omega_m| \rho}{1-\rho} (b-a^{\rho} \right] \right] \left\| f^{(n+1)} \right\|_\infty < \infty,$$

proving the claim. □

We also give

Theorem 2. *All as above with $\gamma_j > 0$, $j = 1, \ldots, m$, and $0 < \rho < 1$, $\lambda > 0$, etc. Then*

$$D_{a*}^{n+\rho} f,\ D_{b-}^{n+\rho} f \in C\left([a,b]\right),\ \ n \in \mathbb{Z}_+.$$

Proof. We prove it only for the first one, the other one as similar is omitted. Let $x_N, x_0 \in [a,b]$ such that $x_N \to x_0$, as $N \to \infty$. Then

$$\chi_{[a,x_N]}\left(t\right) \to \chi_{[a,x_0]}\left(t\right),\ \ \text{a. e.}\ \left(t \in [a,b]\,\right),$$

where χ is the characteristic function.

Also it holds

$$E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\omega_1\rho}{1-\rho}\left(x_N-t\right)^{\rho},\ldots,\frac{-\omega_m\rho}{1-\rho}\left(x_N-t\right)^{\rho}\right] \to$$

$$\to E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\omega_1\rho}{1-\rho}\left(x_0-t\right)^{\rho},\ldots,\frac{-\omega_m\rho}{1-\rho}\left(x_0-t\right)^{\rho}\right]. \tag{10}$$

Therefore we get that

$$\chi_{[a,x_N]}\left(t\right) E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\omega_1\rho}{1-\rho}\left(x_N-t\right)^{\rho},\ldots,\frac{-\omega_m\rho}{1-\rho}\left(x_N-t\right)^{\rho}\right] f^{(n+1)}\left(t\right) \to$$

$$\to \chi_{[a,x_0]}\left(t\right) E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\omega_1\rho}{1-\rho}\left(x_0-t\right)^{\rho},\ldots,\frac{-\omega_m\rho}{1-\rho}\left(x_0-t\right)^{\rho}\right] f^{(n+1)}\left(t\right),$$

a. e. over $t \in [a,b]$, as $N \to \infty$.

Furthermore it holds

$$\chi_{[a,x_N]}(t)\left|E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\omega_1\rho}{1-\rho}\left(x_N-t\right)^{\rho},\ldots,\frac{-\omega_m\rho}{1-\rho}\left(x_N-t\right)^{\rho}\right]\right|\left|f^{(n+1)}\left(t\right)\right| \leqslant$$

$$\leqslant E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{\left|\omega_1\right|\rho}{1-\rho}\left(b-a\right)^{\rho},\ldots,\frac{\left|\omega_m\right|\rho}{1-\rho}\left(b-a\right)^{\rho}\right]\left\|f^{(n+1)}\right\|_{\infty} < \infty. \tag{11}$$

Thus, by the denominated convergence theorem, we derive

$$\frac{A(\rho)}{1-\rho}\int_a^b \chi_{[a,x_N]}(t) E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\omega_1\rho}{1-\rho}(x_N-t)^{\rho},\ldots,\frac{-\omega_m\rho}{1-\rho}(x_N-t)^{\rho}\right] f^{(n+1)}(t)dt \to \tag{12}$$

$$\to \frac{A(\rho)}{1-\rho}\int_a^b \chi_{[a,x_0]}(t)E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\omega_1\rho}{1-\rho}(x_0-t)^\rho,\ldots,\frac{-\omega_m\rho}{1-\rho}(x_0-t)^\rho\right]f^{(n+1)}(t)dt,$$

as $N\to\infty$, proving the continuity of $D_{a*}^{n+\rho}f$ over $[a,b]$, for $n\in\mathbb{Z}_+$. $\square$

In this work we derive left and right side fractional inequalities related to the introduced fractional derivatives (2)–(7) of non singular kernel and multiple parameters. These are of Hardy, Opial and Hilbert-Pachpatte types, also of Hardy type involving convexity.

2. Main results. From now on we denote by

$$\frac{-\vec{\omega}\rho}{1-\rho}(x-t)^\rho := \left(\frac{-\omega_1\rho}{1-\rho}(x-t)^\rho,\ldots,\frac{-\omega_m\rho}{1-\rho}(x-t)^\rho\right), \tag{13}$$

and

$$\frac{-\vec{\omega}\rho}{1-\rho}(t-x)^\rho := \left(\frac{-\omega_1\rho}{1-\rho}(t-x)^\rho,\ldots,\frac{-\omega_m\rho}{1-\rho}(t-x)^\rho\right). \tag{14}$$

So, we rewrite

$$D_{a*}^{n+\rho}f(x) = \frac{A(\rho)}{1-\rho}\int_a^x E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\vec{\omega}\rho}{1-\rho}(x-t)^\rho\right]f^{(n+1)}(t)dt, \tag{15}$$

and

$$D_{b-}^{n+\rho}f(x) = \frac{(-1)^{n+1}A(\rho)}{1-\rho}\int_x^b E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\vec{\omega}\rho}{1-\rho}(t-x)^\rho\right]f^{(n+1)}(t)\,dt, \tag{16}$$

$\forall\, x\in[a,b]$, $f\in C^{n+1}([a,b])$, $n\in\mathbb{Z}_+$.

From now on we take $0<\rho<1$, $\lambda>0$, $\gamma_j>0$, $\omega_j\in\mathbb{R}-\{0\}$, for $j=1,\ldots,m$.

We present the following L_p Hardy type inequalities.

Theorem 3. *Let $f_\nu\in C^{n+1}([a,b])$, $n\in Z_+$, $\nu=1,\ldots,N$; $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\left\|\prod_{\nu=1}^N D_{a*}^{n+\rho}f_\nu\right\|_p \leqslant \left(\frac{A(\rho)}{1-\rho}\right)^N \times$$

$$\times\left(\int_a^b\left(\int_a^x\left|E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\vec{\omega}\rho}{1-\rho}(x-t)^\rho\right]\right|^p dt\right)^N dx\right)^{\frac{1}{p}}\left(\prod_{\nu=1}^N\left\|f_\nu^{(n+1)}\right\|_q\right), \tag{17}$$

and

$$\left\| \prod_{\nu=1}^{N} D_{b-}^{n+\rho} f_\nu \right\|_p \leqslant \left(\frac{A(\rho)}{1-\rho} \right)^N \times$$

$$\times \left(\int_a^b \left(\int_x^b \left| E_{(\rho),\lambda}^{(\gamma_j)} \left[\frac{-\vec{\omega}\rho}{1-\rho} (t-x)^\rho \right] \right|^p dt \right)^N dx \right)^{\frac{1}{p}} \left(\prod_{\nu=1}^{N} \left\| f_\nu^{(n+1)} \right\|_q \right). \quad (18)$$

Proof. We prove only (17).

We have that

$$D_{a*}^{n+\rho} f_\nu(x) = \frac{A(\rho)}{1-\rho} \int_a^x E_{(\rho),\lambda}^{(\gamma_j)} \left[\frac{-\vec{\omega}\rho}{1-\rho} (x-t)^\rho \right] f_\nu^{(n+1)}(t)\, dt, \quad (19)$$

$\nu = 1, \ldots, N$, $\forall\ x \in [a,b]$.

By Hölder's inequality and (19) we obtain

$$\left| D_{a*}^{n+\rho} f_\nu(x) \right| \leqslant \frac{|A(\rho)|}{1-\rho} \int_a^x \left| E_{(\rho),\lambda}^{(\gamma_j)} \left[\frac{-\vec{\omega}\rho}{1-\rho} (x-t)^\rho \right] \right| \left| f_\nu^{(n+1)}(t) \right| dt \leqslant$$

$$\leqslant \frac{|A(\rho)|}{1-\rho} \left(\int_a^x \left| E_{(\rho),\lambda}^{(\gamma_j)} \left[\frac{-\vec{\omega}\rho}{1-\rho} (x-t)^\rho \right] \right|^p dt \right)^{\frac{1}{p}} \left\| f_\nu^{(n+1)} \right\|_q, \quad (20)$$

and

$$\prod_{\nu=1}^{N} \left| D_{a*}^{n+\rho} f_\nu(x) \right| \leqslant$$

$$\leqslant \left(\frac{|A(\rho)|}{1-\rho} \right)^N \left(\int_a^x \left| E_{(\rho),\lambda}^{(\gamma_j)} \left[\frac{-\vec{\omega}\rho}{1-\rho} (x-t)^\rho \right] \right|^p dt \right)^{\frac{N}{p}} \prod_{\nu=1}^{N} \left\| f_\nu^{(n+1)} \right\|_q,$$

$\forall\ x \in [a,b]$.

Hence it holds

$$\left(\prod_{\nu=1}^{N} \left| D_{a*}^{n+\rho} f_\nu(x) \right| \right)^p \leqslant$$

$$\leqslant \left(\frac{|A(\rho)|}{1-\rho}\right)^{Np} \left(\int_a^x \left|E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\vec{\omega}\rho}{1-\rho}(x-t)^{\rho}\right]\right|^p dt\right)^N \left(\prod_{\nu=1}^{N} \left\|f_{\nu}^{(n+1)}\right\|_q\right)^p, \tag{21}$$

and

$$\int_a^b \left(\prod_{\nu=1}^{N} \left|D_{a*}^{n+\rho} f_{\nu}(x)\right|\right)^p dx \leqslant \left(\frac{|A(\rho)|}{1-\rho}\right)^{Np} \times$$

$$\times \left(\int_a^b \left(\int_a^x \left|E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\vec{\omega}\rho}{1-\rho}(x-t)^{\rho}\right]\right|^p dt\right)^N dx\right) \left(\prod_{\nu=1}^{N} \left\|f_{\nu}^{(n+1)}\right\|_q\right)^p, \tag{22}$$

proving the claim. □

We continue with fractional Opial type inequalities:

Theorem 4. *Let $f \in C^{n+1}([a,b])$, $n \in \mathbb{Z}_+$; $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$. Then*

i) the left side one:

$$\int_a^x \left|D_{a*}^{n+\rho} f(w)\right| \left|f^{(n+1)}(w)\right| dw \leqslant 2^{-\frac{1}{q}} \frac{|A(\rho)|}{1-\rho} \times$$

$$\times \left(\int_a^x \left(\int_a^w \left|E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\vec{\omega}\rho}{1-\rho}(w-t)^{\rho}\right]\right|^p dt\right) dw\right)^{\frac{1}{p}} \left(\int_a^x \left|f^{(n+1)}(w)\right|^q dw\right)^{\frac{2}{q}}, \tag{23}$$

ii) the right side one:

$$\int_x^b \left|D_{b-}^{n+\rho} f(w)\right| \left|f^{(n+1)}(w)\right| dw \leqslant 2^{-\frac{1}{q}} \frac{|A(\rho)|}{1-\rho} \times$$

$$\times \left(\int_x^b \left(\int_w^b \left|E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\vec{\omega}\rho}{1-\rho}(t-w)^{\rho}\right]\right|^p dt\right) dw\right)^{\frac{1}{p}} \left(\int_x^b \left|f^{(n+1)}(w)\right|^q dw\right)^{\frac{2}{q}}, \tag{24}$$

$\forall\ x \in [a,b]$.

Proof. We prove only (23). We have that (by Hölder's inequality)

$$\left|D_{a*}^{n+\rho}f\left(x\right)\right|\leqslant\frac{\left|A\left(\rho\right)\right|}{1-\rho}\int\limits_{a}^{x}\left|E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\vec{\omega}\rho}{1-\rho}\left(x-t\right)^{\rho}\right]\right|\left|f^{(n+1)}(t)\right|dt\leqslant$$

$$\leqslant\frac{\left|A\left(\rho\right)\right|}{1-\rho}\left(\int\limits_{a}^{x}\left|E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\vec{\omega}\rho}{1-\rho}(x-t)^{\rho}\right]\right|^{p}dt\right)^{\frac{1}{p}}\left(\int\limits_{a}^{x}\left|f^{(n+1)}\left(t\right)\right|^{q}dt\right)^{\frac{1}{q}}. \tag{25}$$

Call

$$\Phi\left(x\right):=\int\limits_{a}^{x}\left|f^{(n+1)}\left(t\right)\right|^{q}dt,\ \ \Phi\left(a\right)=0. \tag{26}$$

Then

$$\Phi'\left(x\right)=\left|f^{(n+1)}\left(x\right)\right|^{q}\geqslant 0, \tag{27}$$

and

$$\left(\Phi'\left(x\right)\right)^{\frac{1}{q}}=\left|f^{(n+1)}\left(x\right)\right|\geqslant 0,\ \ \forall\ x\in\left[a,b\right].$$

Consequently, we get

$$\left|D_{a*}^{n+\rho}f\left(w\right)\right|\left|f^{(n+1)}\left(w\right)\right|\leqslant$$

$$\leqslant\frac{\left|A\left(\rho\right)\right|}{1-\rho}\left(\int\limits_{a}^{w}\left|E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\vec{\omega}\rho}{1-\rho}\left(w-t\right)^{\rho}\right]\right|^{p}dt\right)^{\frac{1}{p}}\left(\Phi\left(w\right)\Phi'\left(w\right)\right)^{\frac{1}{q}}, \tag{28}$$

$\forall\ w\in\left[a,b\right].$

Thus, by applying again Hölder's inequality:

$$\int\limits_{a}^{x}\left|D_{a*}^{n+\rho}f\left(w\right)\right|\left|f^{(n+1)}\left(w\right)\right|dw\leqslant$$

$$\leqslant\frac{|A(\rho)|}{1-\rho}\int\limits_{a}^{x}\left(\int\limits_{a}^{w}\left|E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\vec{\omega}\rho}{1-\rho}(w-t)^{\rho}\right]\right|^{p}dt\right)^{\frac{1}{p}}(\Phi(w)\Phi'(w))^{\frac{1}{q}}dw\leqslant$$

$$\leqslant\frac{\left|A\left(\rho\right)\right|}{1-\rho}\left(\int\limits_{a}^{x}\left(\int\limits_{a}^{w}\left|E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\vec{\omega}\rho}{1-\rho}(w-t)^{\rho}\right]\right|^{p}dt\right)dw\right)^{\frac{1}{p}}\left(\int\limits_{a}^{x}\Phi(w)\Phi'(w)dw\right)^{\frac{1}{q}} \tag{29}$$

$$= \frac{|A(\rho)|}{1-\rho}\left(\int_a^x\left(\int_a^w\left|E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\vec{\omega}\rho}{1-\rho}(w-t)^{\rho}\right]\right|^p dt\right)dw\right)^{\frac{1}{p}}\left(\frac{\Phi^2(x)}{2}\right)^{\frac{1}{q}} =$$

$$2^{-\frac{1}{q}}\frac{|A(\rho)|}{1-\rho}\left(\int_a^x\left(\int_a^w\left|E_{(\rho),\lambda}^{(\gamma_j)}\left[\frac{-\vec{\omega}\rho}{1-\rho}(w-t)^{\rho}\right]\right|^p dt\right)dw\right)^{\frac{1}{p}}\left(\int_a^x\left|f^{(n+1)}(w)\right|^q dw\right)^{\frac{2}{q}},$$

proving the claim. □

We continue with fractional Hilbert-Pachpatte type inequalities. First comes the left side one:

Theorem 5. *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\mu=1,2$. Let $[a_\mu,b_\mu]\subset\mathbb{R}$, $f_\mu\in C^{n_\mu+1}([a_\mu,b_\mu])$, $n_\mu\in\mathbb{Z}_+$; $0<\rho_\mu<1$, $\lambda_\mu>0$, $\gamma_{j\mu}>0$, $\omega_{j\mu}\in\mathbb{R}-\{0\}$, $j=1,\dots,m$. Also we denote*

$$\frac{-\vec{\omega}_\mu\rho_\mu}{1-\rho_\mu}(x_\mu-t_\mu)^{\rho_\mu} := \left(\frac{-\omega_{1\mu}\rho_\mu}{1-\rho_\mu}(x_\mu-t_\mu)^{\rho_\mu},\dots,\frac{-\omega_{m\mu}\rho_\mu}{1-\rho_\mu}(x_\mu-t_\mu)^{\rho_\mu}\right), \tag{30}$$

$\mu=1,2.$

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\left|D_{a_1*}^{n_1+\rho_1}f_1(x_1)\right|\left|D_{a_2*}^{n_2+\rho_2}f_2(x_2)\right|dx_1dx_2}{\left\{\frac{\int_{a_1}^{x_1}\left|E_{(\rho_1),\lambda_1}^{(\gamma_j1)}\left[\frac{-\vec{\omega}_1\rho_1}{1-\rho_1}(x_1-t_1)^{\rho_1}\right]\right|^p dt_1}{p}+\frac{\int_{a_2}^{x_2}\left|E_{(\rho_2),\lambda_2}^{(\gamma_j2)}\left[\frac{-\vec{\omega}_2\rho_2}{1-\rho_2}(x_2-t_2)^{\rho_2}\right]\right|^q dt_2}{q}\right\}}\leqslant$$

$$\leqslant(b_1-a_1)(b_2-a_2)\frac{|A_1(\rho_1)||A_2(\rho_2)|}{(1-\rho_1)(1-\rho_2)}\left\|f_1^{(n_1+1)}\right\|_q\left\|f_2^{(n_2+1)}\right\|_p. \tag{31}$$

Proof. Here we have ($\mu=1,2$)

$$D_{a_\mu*}^{n_\mu+\rho_\mu}f_\mu(x_\mu)\overset{(15)}{=}\frac{A_\mu(\rho_\mu)}{1-\rho_\mu}\int_{a_\mu}^{x_\mu}E_{(\rho_\mu),\lambda_\mu}^{(\gamma_{j\mu})}\left[\frac{-\vec{\omega}_\mu\rho_\mu}{1-\rho_\mu}(x_\mu-t_\mu)^{\rho_\mu}\right]f_\mu^{(n_\mu+1)}(t_\mu)\,dt_\mu, \tag{32}$$

$\forall\ x_\mu\in[a_\mu,b_\mu]$, $n_\mu\in\mathbb{Z}_+$.

Then

$$\left|D_{a_\mu*}^{n_\mu+\rho_\mu}f_\mu(x_\mu)\right|\leqslant$$

$$\leqslant\frac{|A_\mu(\rho_\mu)|}{1-\rho_\mu}\int_{a_\mu}^{x_\mu}\left|E_{(\rho_\mu),\lambda_\mu}^{(\gamma_{j\mu})}\left[\frac{-\vec{\omega}_\mu\rho_\mu}{1-\rho_\mu}(x_\mu-t_\mu)^{\rho_\mu}\right]\right|\left|f_\mu^{(n_\mu+1)}(t_\mu)\right|dt_\mu, \tag{33}$$

$\mu = 1, 2.$

By applying Hölder's inequality twice we get:

$$\left| D_{a_1*}^{n_1+\rho_1} f_1(x_1) \right| \leqslant \frac{|A_1(\rho_1)|}{1-\rho_1} \times$$

$$\times \left(\int_{a_1}^{x_1} \left| E_{(\rho_1),\lambda_1}^{(\gamma_{j1})} \left[\frac{-\vec{\omega}_1 \rho_1}{1-\rho_1} (x_1 - t_1)^{\rho_1} \right] \right|^p dt_1 \right)^{\frac{1}{p}} \left(\int_{a_1}^{x_1} \left| f_1^{(n_1+1)}(t_1) \right|^q dt_1 \right)^{\frac{1}{q}}, \tag{34}$$

$\forall\ x_1 \in [a_1, b_1]$, and

$$\left| D_{a_2*}^{n_2+\rho_2} f_2(x_2) \right| \leqslant \frac{|A_2(\rho_2)|}{1-\rho_2} \times$$

$$\times \left(\int_{a_2}^{x_2} \left| E_{(\rho_2),\lambda_2}^{(\gamma_{j2})} \left[\frac{-\vec{\omega}_2 \rho_2}{1-\rho_2} (x_2 - t_2)^{\rho_2} \right] \right|^q dt_2 \right)^{\frac{1}{q}} \left(\int_{a_2}^{x_2} \left| f_2^{(n_2+1)}(t_2) \right|^p dt_2 \right)^{\frac{1}{p}}, \tag{35}$$

$\forall\ x_2 \in [a_2, b_2]$.

Hence we have (by (34), (35))

$$\left| D_{a_1*}^{n_1+\rho_1} f_1(x_1) \right| \left| D_{a_2*}^{n_2+\rho_2} f_2(x_2) \right| \leqslant \frac{|A_1(\rho_1)| |A_2(\rho_2)|}{(1-\rho_1)(1-\rho_2)} \times$$

$$\times \left(\int_{a_1}^{x_1} \left| E_{(\rho_1),\lambda_1}^{(\gamma_{j1})} \left[\frac{-\vec{\omega}_1 \rho_1}{1-\rho_1} (x_1 - t_1)^{\rho_1} \right] \right|^p dt_1 \right)^{\frac{1}{p}} \times \tag{36}$$

$$\times \left(\int_{a_2}^{x_2} \left| E_{(\rho_2),\lambda_2}^{(\gamma_{j2})} \left[\frac{-\vec{\omega}_2 \rho_2}{1-\rho_2} (x_2 - t_2)^{\rho_2} \right] \right|^q dt_2 \right)^{\frac{1}{q}} \left\| f_1^{(n_1+1)} \right\|_q \left\| f_2^{(n_2+1)} \right\|_p \leqslant$$

$\left(\text{using Young's inequality for } a, b \geqslant 0,\ a^{\frac{1}{p}} b^{\frac{1}{q}} \leqslant \frac{a}{p} + \frac{b}{q}\right)$

$$\leqslant \frac{|A_1(\rho_1)| |A_2(\rho_2)|}{(1-\rho_1)(1-\rho_2)} \left\{ \frac{\int_{a_1}^{x_1} \left| E_{(\rho_1),\lambda_1}^{(\gamma_j 1)} \left[\frac{-\vec{\omega}_1 \rho_1}{1-\rho_1} (x_1 - t_1)^{\rho_1} \right] \right|^p dt_1}{p} + \right.$$

$$+\frac{\int_{a_2}^{x_2}\left|E_{(\rho_2),\lambda_2}^{(\gamma_j 2)}\left[\frac{-\vec{\omega}_2\rho_2}{1-\rho_2}(x_2-t_2)^{\rho_2}\right]\right|^q dt_2}{q}\Bigg\}\left\|f_1^{(n_1+1)}\right\|_q\left\|f_2^{(n_2+1)}\right\|_p, \tag{37}$$

$\forall\ x_\mu \in [a_\mu, b_\mu]$, $\mu = 1, 2$.

So far we have that

$$\frac{\left|D_{a_1*}^{n_1+\rho_1} f_1(x_1)\right|\left|D_{a_2*}^{n_2+\rho_2} f_2(x_2)\right|}{\frac{\int_{a_1}^{x_1}\left|E_{(\rho_1),\lambda_1}^{(\gamma_j 1)}\left[\frac{-\vec{\omega}_1\rho_1}{1-\rho_1}(x_1-t_1)^{\rho_1}\right]\right|^p dt_1}{p}+\frac{\int_{a_2}^{x_2}\left|E_{(\rho_2),\lambda_2}^{(\gamma_j 2)}\left[\frac{-\vec{\omega}_2\rho_2}{1-\rho_2}(x_2-t_2)^{\rho_2}\right]\right|^q dt_2}{q}} \leqslant$$

$$\leqslant \frac{|A_1(\rho_1)|\,|A_2(\rho_2)|}{(1-\rho_1)(1-\rho_2)}\left\|f_1^{(n_1+1)}\right\|_q\left\|f_2^{(n_2+1)}\right\|_p, \tag{38}$$

$\forall\ x_\mu \in [a_\mu, b_\mu]$, $\mu = 1, 2$.

The denominator in (38) can be zero only when $x_1 = a_1$ and $x_2 = a_2$. Therefore we obtain (31) by integrating (38) over $[a_1, b_1] \times [a_2, b_2]$. □

The counterpart of Theorem 5 follows (the right side inequality)

Theorem 6. *All as in Theorem 5, but now we denote*

$$\frac{-\vec{\omega}_\mu\rho_\mu}{1-\rho_\mu}(t_\mu-x_\mu)^{\rho_\mu} := \left(\frac{-\omega_{1\mu}\rho_\mu}{1-\rho_\mu}(t_\mu-x_\mu)^{\rho_\mu}, \dots, \frac{-\omega_{m\mu}\rho_\mu}{1-\rho_\mu}(t_\mu-x_\mu)^{\rho_\mu}\right), \tag{39}$$

$\mu = 1, 2$.

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\left|D_{b_1-}^{n_1+\rho_1} f_1(x_1)\right|\left|D_{b_2-}^{n_2+\rho_2} f_2(x_2)\right| dx_1 dx_2}{\frac{\int_{x_1}^{b_1}\left|E_{(\rho_1),\lambda_1}^{(\gamma_j 1)}\left[\frac{-\vec{\omega}_1\rho_1}{1-\rho_1}(t_1-x_1)^{\rho_1}\right]\right|^p dt_1}{p}+\frac{\int_{x_2}^{b_2}\left|E_{(\rho_2),\lambda_2}^{(\gamma_j 2)}\left[\frac{-\vec{\omega}_2\rho_2}{1-\rho_2}(t_2-x_2)^{\rho_2}\right]\right|^q dt_2}{q}} \leqslant$$

$$\leqslant (b_1-a_1)(b_2-a_2)\frac{|A_1(\rho_1)|\,|A_2(\rho_2)|}{(1-\rho_1)(1-\rho_2)}\left\|f_1^{(n_1+1)}\right\|_q\left\|f_2^{(n_2+1)}\right\|_p. \tag{40}$$

Proof. As similar to the proof of Theorem 5 is omitted. □

We need

Notation 7. *Let* $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$ *be measure spaces with positive* σ*-finite measures, and let* $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *be nonnegative measurable functions,* $k(x, \cdot)$ *measurable on* Ω_2*, and*

$$K(x) = \int_{\Omega_2} k(x, y)\,d\mu_2(y), \quad \textit{for any } x \in \Omega_1. \tag{41}$$

We suppose that $K(x) > 0$ *a. e. on* Ω_1 *and by a weight function* u *(shortly: a weight), we mean a nonnegative measurable function on the actual set. Let the measurable functions* $g_j : \Omega_1 \to \mathbb{R}$, $j = 1, \ldots, r$, *with the representation*

$$g_j(x) = \int_{\Omega_2} k(x,y) f_j(y) d\mu_2(y), \tag{42}$$

where $f_j : \Omega_2 \to \mathbb{R}$ *are measurable functions,* $j = 1, \ldots, r$.

Denote by $\vec{x} = x := (x_1, \ldots, x_r) \in \mathbb{R}^r$, $\vec{g} := (g_1, \ldots, g_r)$ and $\vec{f} := (f_1, \ldots, f_r)$.

We consider here $\Phi : \mathbb{R}_+^r \to \mathbb{R}$ a convex function, which is increasing per coordinate, i. e. if $x_j \leqslant y_j$, $j = 1, \ldots, r$, then

$$\Phi(x_1, \ldots, x_r) \leqslant \Phi(y_1, \ldots, y_r).$$

In [2, p. 588], we proved that

Theorem 8. *Let* u *be a weight function on* Ω_1, *and* k, K, g_j, f_j, $j = 1, \ldots, r \in \mathbb{N}$, *and* Φ *defined as above. Assume that the function* $x \to u(x) \frac{k(x,y)}{K(x)}$ *is integrable on* Ω_1 *for each fixed* $y \in \Omega_2$. *Define* v *on* Ω_2 *by*

$$v(y) := \int_{\Omega_1} u(x) \frac{k(x,y)}{K(x)} d\mu_1(x) < \infty. \tag{43}$$

Then

$$\int_{\Omega_1} u(x) \Phi\left(\frac{|g_1(x)|}{K(x)}, \ldots, \frac{|g_r(x)|}{K(x)}\right) d\mu_1(x) \leqslant$$

$$\leqslant \int_{\Omega_2} v(y) \Phi(|f_1(y)|, \ldots, |f_r(y)|) d\mu_2(y), \tag{44}$$

under the assumptions:

(i) f_j, $\Phi(|f_1|, \ldots, |f_r|)$, *are* $k(x,y) d\mu_2(y)$*-integrable,* μ_1*-a. e. in* $x \in \Omega_1$, *for all* $j = 1, \ldots, r$;

(ii) $v(y) \Phi(|f_1(y)|, \ldots, |f_r(y)|)$ *is* μ_2*-integrable.*

We give under convexity a left fractional generalized Hardy type inequality.

Theorem 9. *Let u_1 be a weight function on $[a, b]$,*

$$k_{a*}(x,y) := \frac{A(\rho)}{1-\rho} \chi_{[a,x]}(y) E_{(\rho),\lambda}^{\gamma_j} \left[\frac{-\vec{\omega}\rho}{1-\rho} (x-y)^{\rho} \right],$$

$$K_{a*}(x) := \frac{A(\rho)}{1-\rho} \int_a^b \chi_{[a,x]}(y) E_{(\rho),\lambda}^{\gamma_j} \left[\frac{-\vec{\omega}\rho}{1-\rho} (x-y)^{\rho} \right] dy, \tag{45}$$

for any $x \in [a, b]$. Also consider $D_{a*}^{n+\rho} f_{\mu}$ and $f_{\mu}^{(n+1)} \in C([a, b])$, $\mu = 1, \ldots, r \in \mathbb{N}$, $n \in \mathbb{Z}_+$, and $\Phi : \mathbb{R}_+^r \to \mathbb{R}$ a convex function, which is increasing per coordinate. Assume that the function $x \to u_1(x) \frac{k_{a*}(x,y)}{K_{a*}(x)}$ is integrable on $[a, b]$ for each $y \in [a, b]$. Define ν_1 on $[a, b]$ by

$$\nu_1(y) := \int_a^b u_1(x) \frac{k_{a*}(x,y)}{K_{a*}(x)} dx < \infty, \tag{46}$$

and ν_1 is assumed to be integrable on $[a, b]$.

Then

$$\int_a^b u_1(x) \Phi \left(\frac{\left| D_{a*}^{n+\rho} f_1(x) \right|}{K_{a*}(x)}, \ldots, \frac{\left| D_{a*}^{n+\rho} f_r(x) \right|}{K_{a*}(x)} \right) dx \leqslant$$

$$\leqslant \int_a^b \nu_1(y) \Phi \left(\left| f_1^{(n+1)}(y) \right|, \ldots, \left| f_r^{(n+1)}(y) \right| \right) dy. \tag{47}$$

Proof. See Notation 7 and apply Theorem 8. □

Also we give the right side corresponding to (47) Hardy type inequality.

Theorem 10. *Let u_2 be a weight function on $[a,b]$,*

$$k_{b-}(x,y) := \frac{(-1)^{n+1} A(\rho)}{1-\rho} \chi_{[x,b]}(y) E_{(\rho),\lambda}^{\gamma_j} \left[\frac{-\vec{\omega}\rho}{1-\rho} (y-x)^{\rho} \right],$$

$$K_{b-}(x) := \frac{(-1)^{n+1} A(\rho)}{1-\rho} \int_a^b \chi_{[x,b]}(y) E_{(\rho),\lambda}^{\gamma_j} \left[\frac{-\vec{\omega}\rho}{1-\rho} (y-x)^{\rho} \right] dy, \tag{48}$$

for any $x \in [a,b]$. *Also consider* $D_{b-}^{n+\rho} f_\mu$ *and* $f_\mu^{(n+1)} \in C([a,b])$, $\mu = 1,\ldots,r \in \mathbb{N}$, $n \in \mathbb{Z}_+$, *and* $\Phi : \mathbb{R}_+^r \to \mathbb{R}$ *a convex function, which is increasing per coordinate. Assume that the function* $x \to u_2(x) \frac{k_{b-}(x,y)}{K_{b-}(x)}$ *is integrable on* $[a,b]$ *for each* $y \in [a,b]$. *Define* ν_2 *on* $[a,b]$ *by*

$$\nu_2(y) := \int_a^b u_2(x) \frac{k_{b-}(x,y)}{K_{b-}(x)} dx < \infty, \tag{49}$$

and ν_2 *is assumed to be integrable on* $[a,b]$.

Then

$$\int_a^b u_2(x) \Phi\left(\frac{\left| D_{b-}^{n+\rho} f_1(x) \right|}{K_{b-}(x)}, \ldots, \frac{\left| D_{b-}^{n+\rho} f_r(x) \right|}{K_{b-}(x)} \right) dx \leqslant$$

$$\leqslant \int_a^b \nu_2(y) \Phi\left(\left| f_1^{(n+1)}(y) \right|, \ldots, \left| f_r^{(n+1)}(y) \right| \right) dy. \tag{50}$$

Proof. See Notation 7 and apply Theorem 8. □

Note 11. *One can create a vast number of similar very interesting theorems using* (3) *and* (5) *and based on the author's monographs* [1], [2]. *To stay short we choose to skip this task.*

Conclusion 12. *The highlight of this work is the introduction of the fractional derivatives* (2)–(7) *with non singular kernel and many parameters, able to incorporate lots of data, so they can describe complex settings and situations in complete ways.*

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