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A. S. Krivosheev, O. A. Krivosheeva

INVARIANT SUBSPACES IN UNBOUNDED DOMAINS

Abstract. We study subspaces of functions analytic in an unbounded convex domain of the complex plane and invariant with respect to the differentiation operator. This paper is devoted to the study of the problem of representing all functions from an invariant subspace by series of exponential monomials. These exponential monomials are eigenfunctions and associated functions of the differentiation operator in the invariant subspace. A simple geometric criterion of the fundamental principle is obtained. It is formulated just in terms of the Krisvosheev condensation index for the sequence of exponents of the mentioned exponential monomials.

Key words: *invariant subspace, fundamental principle, exponential monomial, entire function, series of exponents*

2020 Mathematical Subject Classification: *30D10*

1. Introduction. Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ be a sequence of different complex numbers λ_k and their multiplicities n_k. We assume that $|\lambda_k|$ are non-decreasing and $|\lambda_k| \to \infty$, $k \to \infty$. By $\Xi(\Lambda)$ we denote the set of limits of converging sequences of the form $\left\{\dfrac{\bar{\lambda}_{k_j}}{|\lambda_{k_j}|}\right\}_{j=1}^{\infty}$ ($\bar{\lambda}$ is a complex conjugate of λ). The set $\Xi(\Lambda)$ is closed and it is contained in the unit circle $S(0,1)$. We introduce a family of exponential monomials

$$\mathcal{E}(\Lambda) = \{z^n e^{\lambda_k z}\}_{k=1,\, n=0}^{\infty,\, n_k-1}.$$

Let $D \subset \mathbb{C}$ be a convex domain and

$$H_D(\varphi) = \sup_{z \in D} \operatorname{Re}(z e^{-i\varphi}), \quad \varphi \in [0, 2\pi]$$

be its support function. Let

$$J(D) = \{e^{i\varphi} \in S(0,1) : H_D(\varphi) = +\infty\}.$$

If D is bounded, then $J(D) = \emptyset$. In the case of an unbounded domain, the following situation is possible:

1) $J(D) = S(0,1)$, i. e. $D = \mathbb{C}$;
2) D is the half-plane $\{z \in \mathbb{C}\colon \mathrm{Re}(ze^{-i\varphi}) < a\}$ and $J(D) = S(0,1) \backslash \{e^{i\varphi}\}$;
3) D is a strip $\{z \in \mathbb{C}\colon b < \mathrm{Re}(ze^{-i\varphi}) < a\}$ and $J(D) = S(0,1) \backslash \{e^{i\varphi}, e^{i(\varphi+\pi)}\}$;
4) in all other cases, $J(D)$ is an arc of the unit circle that subtends an angle of at least π.

By $\mathrm{int}\, J(D)$ and $\partial J(D)$ we denote the set of interior points and boundary points, respectively, (in the topology of the circle $S(0,1)$) of the set $J(D)$.

Let $H(D)$ be a space of analytic functions on the domain D with the topology of uniform convergence on compact sets $K \subset D$, and let $W \subset H(D)$ be a nontrivial ($W \neq \{0\}$, $H(D)$) closed subspace with respect to the differentiation operator. The spectrum of this operator in the subspace W is an at most countable set $\{\lambda_k\}$ [1, Chap. II, Sec. 7]. Let $\Lambda = \{\lambda_k, n_k\}$ be a multiple spectrum of the differentiation operator on W. Then $\mathcal{E}(\Lambda)$ is the family of its eigenfunctions and generalized eigenfunctions on W. We say that W admits spectral synthesis, if it coincides with the closure $W(\Lambda, D)$ (in $H(D)$) of the linear span of system $\mathcal{E}(\Lambda)$. We mention that the problem of spectral synthesis was completely solved in [2] and [3]. If D is an unbounded convex domain, then the identity $W = W(\Lambda, D)$ holds, that is, W admits the spectral synthesis [3, Theorem 8.2].

Particular cases of invariant subspaces are the spaces of solutions of linear homogeneous differential, difference, and differential-difference equations with constant coefficients both of finite and infinite orders, as well as of more general convolution equations and their systems.

The main problem in the theory of invariant subspaces is the problem of fundamental principle. The fundamental principle is said to hold in a subspace W with the spectrum Λ, if for each function $g \in W$ the following representation holds:

$$g(z) = \sum_{k=1,\, n=0}^{\infty,\, n_k-1} d_{k,n} z^n e^{\lambda_k z}, \quad z \in D, \tag{1}$$

and this series converges uniformly on compact sets in D. The name «fundamental principle» arises in relation with a particular case of the invariant subspace; namely, the space of solutions to a linear homogeneous differential equation with constant coefficients. It is known that each solution to such equation is a linear combination of elementary solutions, exponential monomials $z^n e^{\lambda_k z}$, with exponents being zeroes (probably, multiple) of a characteristic polynomial. The presence of this representation is called Euler fundamental principle.

By means of the Laplace transform, the fundamental principle problem is reduced to a dual problem of multiple Interpolation in the space of entire functions of exponential type. The study of both problems has a rich history. Initially, they were studied independently. The main milestones of this study are reflected in works [4] and [5]. In the case of a bounded convex domain, the fundamental principle problem was completely solved in [5]– [8]. There a simple geometric criterion of fundamental principle was obtained [8, Theorem 3.2] for invariant subspaces admitting the spectral synthesis. It was formulated only in terms of the Krivosheev's condensation index S_Λ (it will be introduced below), of the maximal angular density of the sequence Λ, and of the length of an arc of the boundary of D.

The situation with unbounded convex domains is much worse. In work [5], a criterion of fundamental principle for arbitrary convex domains was obtained. It, however has, two disadvantages. It involves some restriction for the multiplicity n_k of the points λ_k. Moreover, it involves the following condition, which is equivalent to the validity of the fundamental principle. Namely, it requires the existence of a family of entire functions vanishing at the points λ_k with multiplicities at least n_k, the growth of which is close to a regular one and is related with D. There remained an open question: under which conditions for Λ and D such family exists. The problem on constructing such family is rather complicated.

A complete solution of fundamental principle problem for nontrivial invariant subspaces of entire functions was obtained in work [9]. It was proved that the validity of the fundamental principle in each such subspace is equivalent to the finiteness of the condensation index S_Λ. Invariant subspaces in a half-plane were studied in the case of a simple positive spectrum (i. e., $n_k = 1$, $k \geqslant 1$), having a finite upper density. In work [10], this problem was solved completely for an arbitrary convex domain D. The solution was given in terms of simple geometric characteristics of the sequence Λ and the domain D. This involves a principally new aspect. It turned out that the measurability of the sequence Λ is not necessary for the validity of the fundamental principle in the case of a vertical half-plane; even the finiteness of its maximal density is not necessary, despite the support function of the half-plane is bounded in the positive direction. A necessary and sufficient condition in this situation is the vanishing of the characteristics S_Λ. In work [11], this result was extended for the case of invariant subspaces with almost real spectrum Λ (i. e., $\Xi(\Lambda) = \{1\}$). Significant difficulties related to multiplicities n_k of the points λ_k were overcome. We note that the result of the work [11] it easily extended for the case of invariant subspaces with the spectrum Λ, for which $\Xi(\Lambda)$ is a one-point set. In the work [12], by decomposing an invariant subspace into the sum of two invariant subspaces and basing on the results in [9] and [11], a criterion of fundamental principle for invariant subspaces in a half-plane with an arbitrary spectrum was obtained. It is formulated only in terms of condensation index S_Λ. The same applies to the result from the work [13]. It provides a fundamental principle criterion for invariant subspaces in an arbitrary convex domain under the condition $\Xi(\Lambda) \subseteq J(D)$.

In this work, the result from [13] extends for the case when $\Xi(\Lambda)$ lies in the closure $\overline{J(D)}$ of the set $J(D)$. We note that this case is fundamentally different from the case $\Xi(\Lambda) \subseteq J(D)$. A simple geometric criterion of the fundamental principle is obtained; it is based only on the concept of the condensation index of the sequence that makes up the spectrum of an invariant subspace.

2. Preliminaries. We begin with recalling some notions and facts related with interpolating Leont'ev functions.

Let $\Lambda = \{\lambda_k, n_k\}$ and f be an entire function of exponential type, i. e.,

$$\ln |f(\lambda)| \leqslant A + B|\lambda|, \quad \lambda \in \mathbb{C}, \quad A, B \geqslant 0.$$

We write $f(\Lambda) = 0$, if f vanishes at the points λ_k with multiplicity at least n_k. By definition, an indicator of function f is the function

$$h_f(\varphi) = \varlimsup_{t\to\infty} \frac{\ln |f(te^{i\varphi})|}{t}, \quad \varphi \in [0, 2\pi].$$

It coincides with the support function of some convex compact set $T \subset \mathbb{C}$ called an indicator diagram of f. By $\gamma(t, f)$ we denote the function associated with f in the Borel sense [1, Chap. I, Sec. 5]. An adjoint diagram K of the function f is a convex hull of the set of singular points $\gamma(t, f)$. Thus, $\gamma(t, f)$ is analytic outside some compact set K. By the Polya theorem [1, Chap. I, Sec. 5, Theorem 5.4],

$$h_f(\varphi) = H_T(\varphi) = H_K(-\varphi), \quad \varphi \in [0, 2\pi]. \tag{2}$$

Therefore, K is a compact set, complex conjugate with the compact set T.

Let D be a convex domain, $g \in H(D)$, $0 \in K$, and $\sigma \in \mathbb{C}$ be such that the shift $K + \sigma$ of the conjugate diagram K of the function f lies in the domain D. The function

$$\omega_f(\lambda, \sigma, g) = e^{-\sigma\lambda} \frac{1}{2\pi i} \int\limits_{\Omega} \gamma(t, f) \biggl(\int\limits_0^t g(t + \sigma - \eta) e^{\lambda\eta} d\eta \biggr) dt, \qquad (3)$$

is called the interpolation function for the function g [14, Chap. I, Sec. 2], where Ω is a contour: a simple closed continuous rectifiable curve, enveloping the compact set K and located in the domain $D - \sigma$.

We are going to omit the restriction $0 \in K$. We choose an arbitrary point $w \in K$. Conjugate diagram of the function $f_w(z) = f(z)e^{-wz}$ coincides with the compact set $K_w = K - w$, which contains the origin. Then, by formula (3), we define the function $\omega_{f_w}(\lambda, \sigma, g)$ for all $\sigma \in \mathbb{C}$, such that the compact set $K_w + \sigma$ lies in the domain D.

Let us mention some properties of the function $\omega_{f_w}(\lambda, \sigma, g)$. It follows from (3), that this function is entire and linear in the third independent variable. Let $K(\varepsilon) = K + B(0, \varepsilon)$ be an ε-expansion of the compact set K, $\Omega(\varepsilon) = \partial(K(\varepsilon)) - w$ and $\Omega_\sigma(\varepsilon) = \Omega(\varepsilon) + \sigma \subset G$. By (3), we have:

$$|\omega_{f_w}(\lambda, \sigma, g)| \leqslant \frac{1}{2\pi} |e^{-\sigma\lambda}| \max_{z \in \Omega(\varepsilon)} |e^{\lambda z}| \max_{z \in \Omega_\sigma(\varepsilon)} |g(z)| \int\limits_{\Omega(\varepsilon)} |\gamma(t, f_w)||t||dt| \leqslant$$

$$\leqslant \tau_\varepsilon \exp\bigl(r H_{\Omega(\varepsilon)}(-\varphi) - \operatorname{Re}(\sigma\lambda)\bigr) \max_{z \in \Omega_\sigma(\varepsilon)} |g(z)| \int\limits_{\partial K(\varepsilon)} |\gamma(t, f)||dt| =$$

$$= A(f, \varepsilon) \exp\left(r H_K(-\varphi) + \varepsilon r - \operatorname{Re}(w\lambda) - \operatorname{Re}(\sigma\lambda)\right) \sup_{z \in \Omega_\sigma(\varepsilon)} |g(z)|, \lambda = re^{i\varphi},$$

where $A(f, \varepsilon) = (2\pi)^{-1} \tau_\varepsilon(f) d_\varepsilon$, d_ε is the diameter of the domain $K(\varepsilon)$, and $\tau_\varepsilon(f)$ is the latter integral. In view of identity (2) for all $\lambda \in \mathbb{C}$, we have:

$$|\omega_{f_w}(\lambda, \sigma, g)| \leqslant A(f, \varepsilon) \exp((h_f(\varphi) + \varepsilon)r - \operatorname{Re}((w + \sigma)\lambda)) \max_{z \in \Omega_\sigma(\varepsilon)} |g(z)|. \quad (4)$$

We now remind the main property of the interpolation function. Let $\Lambda = \{\lambda_k, n_k\}$ be a multiple zero set of the function f and

$$P(z) = \sum_{k=1}^{p} \sum_{n=0}^{n_k - 1} a_{k,n} z^n e^{\lambda_k z}.$$

Then the following identities hold [14, Chap. I, Sec. 2, Theorem 1.2.4]:

$$\frac{1}{2\pi i}\int\limits_{\partial B(\lambda_k,b_k)}\frac{\omega_{f_w}(\lambda,\sigma,P)}{f_w(\lambda)}e^{\lambda z}d\lambda=\sum_{n=0}^{n_k-1}a_{k,n}z^n e^{\lambda_k z}, \tag{5}$$

where $\sigma\in\mathbb{C}$, $k=\overline{1,p}$ and $\partial B(\lambda_k,b_k)$ is a circumference containing no points λ_s, $s\neq k$.

The following statements are particular cases of, respectively, Theorems 2.1.1 and 2.1.2 from the book [14].

Lemma 1. *Let $\Lambda=\{\lambda_k,n_k\}$, D be a convex domain and the system $\mathcal{E}(\Lambda)$ be incomplete in the space $H(D)$. Assume that*

$$g(z)=\lim_{\mu\to\infty}P_\mu(z),\quad P_\mu(z)=\sum_{k=1}^{\mu}\sum_{n=0}^{n_k-1}a_{k,n,\mu}z^n e^{\lambda_k z}, \tag{6}$$

where the convergence is uniform on compact sets in the domain D. Then there exist limits

$$a_{k,n}=\lim_{\mu\to\infty}a_{k,n,\mu},\quad n=\overline{0,n_k-1},\quad k\geqslant 1.$$

Lemma 2. *Let $\Lambda=\{\lambda_k,n_k\}$, D be a convex domain and the system $\mathcal{E}(\Lambda)$ be incomplete in the space $H(D)$. Assume that (6) holds and*

$$g(z)=\lim_{\mu\to\infty}Q_\mu(z),\quad Q_\mu(z)=\sum_{k=1}^{\mu}\sum_{n=0}^{n_k-1}b_{k,n,\mu}z^n e^{\lambda_k z},$$

where the convergence is uniform on compact sets in the domain D. Then we have

$$\lim_{\mu\to\infty}a_{k,n,\mu}=\lim_{\mu\to\infty}b_{k,n,\mu},\quad n=\overline{0,n_k-1},\quad k\geqslant 1.$$

3. Fundamental principle. Let $\Lambda=\{\lambda_k,n_k\}$, $n(r,\Lambda)$ be the number of the points λ_k, counted with their multiplicities n_k, in the open disk $B(0,r)$, and

$$\overline{n}(\Lambda)=\lim_{r\to+\infty}\frac{n(r,\Lambda)}{r}$$

be the upper density of the sequence Λ. According to Lindelöf's well-known theorem [15, Chap. I, Sec. 11, Theorem 15] $\overline{n}(\Lambda)<+\infty$ if and only if there exists an entire function f of exponential type, such that $f(\Lambda) = 0$.

Let D be an unbounded convex domain and W be a nontrivial closed invariant with respect to the differentiation operator subspace in $H(D)$. As noted above, in this case W admits spectral synthesis, i. e., the identity $W = W(\Lambda, D)$ holds, where Λ is the multiple spectrum of the differentiation operator in the subspace W.

The subspace $W(\Lambda, D)$ is nontrivial if and only if the system $\mathcal{E}(\Lambda)$ is not complete in the space $H(D)$. Therefore, in studying invariant subspaces $W \subset H(D)$ for unbounded convex domain, it suffices to consider the case where $W = W(\Lambda, D)$ and $\mathcal{E}(\Lambda)$ is not complete in $H(D)$.

The system $\mathcal{E}(\Lambda)$ is not complete in the space $H(D)$ if and only if [1, Chap. I, Sec. 7, Theorem 7.2 and Sec. 5, Theorem 5.2] there exist an entire function f of exponential type such that $f(\Lambda) = 0$, and some translate $K + \sigma$ of its conjugate diagram K lies in D.

The completeness problem of the system $\mathcal{E}(\Lambda)$ in the space $H(D)$ has a simple solution in the case where the domain D is not contained in any strip. Such domains are called large convex domains. They contain translates of any convex compact set. Therefore, in view of Lindelöf's theorem, in the case of a large domain, the system $\mathcal{E}(\Lambda)$ is not complete in $H(D)$ if and only if $\overline{n}(\Lambda) < +\infty$. In the case when D lies in a strip, a simple criterion of completeness of the system $\mathcal{E}(\Lambda)$ was obtained in [16, 17]. It was formulated in terms of the logarithmic block-density of the sequence Λ.

According to [5], we introduce the index of condensation

$$S_\Lambda = \lim_{\delta\to 0} \varliminf_{k\to\infty} \frac{\ln |q^k_\Lambda(\lambda_k, \delta)|}{|\lambda_k|}, q^k_\Lambda(z, \delta) = \prod_{\lambda_m \in B(\lambda_k,\, \delta|\lambda_k|,\, \lambda_m \neq \lambda_k)} \left(\frac{z - \lambda_m}{3\delta|\lambda_m|}\right)^{n_m}.$$

Let $\Lambda = \{\lambda_k, n_k\}$, $\Lambda_1 = \{\xi_p, m_p\}$ and $\Lambda_2 = \{\varsigma_j, l_j\}$. We write $\Lambda_1 \subseteq \Lambda$, if $\xi_p = \lambda_{k_p}$ and $m_p \leqslant n_{k_p}$, $p \geqslant 1$. We write $\Lambda = \Lambda_1 \cup \Lambda_2$, if for any $k \geqslant 1$ there exists either $p \geqslant 1$, such that $\lambda_k = \xi_p$, or $j \geqslant 1$, such that $\lambda_k = \varsigma_j$, and the following conditions hold:

1. if there exists $p \geqslant 1$ such that $\lambda_k = \xi_p$ and $\lambda_k \neq \varsigma_j$ for any $j \geqslant 1$, then $n_k = m_p$;
2. if there exists $j \geqslant 1$ such that $\lambda_k = \varsigma_j$ and $\lambda_k \neq \xi_p$ for any $p \geqslant 1$, then $n_k = l_j$;

3. if there exists $p \geqslant 1$ and $j \geqslant 1$ such that $\lambda_k = \xi_p = \varsigma_j$, then $n_k = m_p + l_j$.

We put

$$f(z,\Lambda) = z^{n_1} \prod_{k=2}^{\infty} \left(1 - \frac{z^2}{\lambda_k^2}\right)^{n_k}.$$

If $\lambda_k \neq 0$, then the multiplier z^{n_1} is missing, and the product starts with $k = 1$.

Let $\varphi \in \mathbb{R}$ and $a \leqslant +\infty$. Let

$$\Pi(a,\varphi) = \{z \in \mathbb{C} : \mathrm{Re}(ze^{-i\varphi}) < a\}.$$

The set $\Pi(a,\varphi)$ is a half-plane, when $a \in \mathbb{R}$. If $a = +\infty$, then $\Pi(a,\varphi) = \mathbb{C}$.

Let D be an unbounded convex domain and $D \neq \Pi(a,\varphi)$, $\varphi \in \mathbb{R}$, $a \leqslant +\infty$. Then $\partial J(D) = \{e^{i\varphi_1}, e^{i\varphi_2}\}$ is a two-element set.

By symbol $J_0(D)$, we denote a subset of the set $J(D)$ that consists of all points $e^{i\varphi}$, such that

$$\{e^{i\alpha} : \alpha \in (\varphi - \pi/2, \varphi + \pi/2)\} \subset J(D).$$

Lemma 3. *Let $\Lambda = \{\lambda_k, n_k\}$, D be an unbounded convex domain, $D \neq \Pi(a,\varphi)$, $\varphi \in \mathbb{R}$, $a \leqslant +\infty$, and the system $\mathcal{E}(\Lambda)$ be not complete in $H(D)$. Suppose that $\Xi(\Lambda) = \Lambda_1 \cup \Lambda_2$, $\Xi(\Lambda_j) \subseteq \{e^{i\varphi_j}\}$, $j = 1,2$, where $\{e^{i\varphi_1}, e^{i\varphi_2}\} = \partial J(D)$. Then, for each function $g \in W(\Lambda, D)$, the representation $g = g_1 + g_2$ is true, where $g_j \in W(\Lambda_j, \Pi(H_D(\varphi_j), \varphi_j))$, $j = 1,2$.*

Proof. Let $g \in W(\Lambda, D)$. Then we have

$$g(z) = \lim_{\mu\to\infty} P_\mu(z), \quad P_\mu(z) = \sum_{k=1}^{\mu} \sum_{n=0}^{n_k-1} a_{k,n,\mu} z^n e^{\lambda_k z},$$

and the convergence is uniform on the compacts set in the domain D. Let

$$P_{\mu,j}(z) = \sum_{k=1,\lambda_k,n_k\in\Lambda_j}^{\mu} \sum_{n=0}^{n_k-1} a_{k,n,\mu} z^n e^{\lambda_k z}, \quad j = 1,2.$$

We show that a certain subsequence of the sequence $\{P_{\mu,j}\}$ converges uniformly on the compact sets in the domain $\Pi(H_D(\varphi_j), \varphi_j)$, $j = 1,2$. Then we have $g \in W(\Lambda, \mathbb{C})$. To do this, it is sufficient to estimate the modules of exponential polynomials $P_{\mu,j}$ uniformly on compact sets in the domain $\Pi(H_D(\varphi_j),\varphi_j)$. To get these estimates, we use a method based on the use of an interpolating function. First of all, we define an entire function f_0 of exponential type with suitable growth estimates and such that $f_0(\Lambda)=0$, which will be used to construct the interpolating function.

Consider two cases:

Case 1. $e^{i\varphi_1}\neq -e^{i\varphi_2}$. In this case, we can assume that $J(D)$ is an arc of the circle $S(0,1)$, $\pi<\varphi_2-\varphi_1<2\pi$ and $J(D)\subseteq\{e^{i\varphi}:\varphi\in[\varphi_1,\varphi_2]\}$.

Let $j=1,2$. Since $\mathcal{E}(\Lambda)$ is not complete in $H(D)$, then $\overline{n}(\Lambda_j)\leqslant \leqslant\overline{n}(\Lambda)<+\infty$. And, according to Theorem 2.3 in [12], for any $\varepsilon>0$ and $\delta\in(0,1)$ there exist $\gamma_j>0$, the sequence Λ_j^0, and a strictly increasing unbounded sequence of positive integers $\{\tau_{s,j}\}_{s=1}^{\infty}$, such that

$$\tau_{s+1,j}\leqslant(1+\delta)\tau_{s,j},\quad s\geqslant 1,\quad \Lambda_j\subseteq\Lambda_j^0. \tag{7}$$

$$\big|\ln|f_j(re^{i\varphi})|-\pi\gamma_j r|\sin(\varphi+\varphi_j)|\big|\leqslant\varepsilon r,$$

$$re^{i\varphi}\in(\mathbb{C}\setminus(\Gamma(\delta,\varphi_j)\cup B(0,\tau_{1,j})))\cup\left(\bigcup_{j=1}^{\infty}S(0,\tau_{s,j})\right), \tag{8}$$

$$h_{f_j}(\varphi)<\pi\gamma_j|\sin(\varphi+\varphi_j)|+\varepsilon,\quad \varphi\in[0,2\pi], \tag{9}$$

where $f_j(z)=f(z,\Lambda_j^0)$ and $\Gamma(\delta,\varphi_j)=\{t\lambda:\lambda\in B(e^{-i\varphi_j},\delta),t\in\mathbb{R}\}$.

Let

$$f_0(z)=f(z,\Lambda_1^0)f(z,\Lambda_2^0)e^{a_1+a_2}z,\quad a_1=\gamma_1e^{i(\varphi_1+\pi/2)},\quad a_2=\gamma_2e^{i(\varphi_2-\pi/2)}.$$

Find the estimates for the function $|f_0|$. Let T be a parallelogram with vertices $0,2\pi\bar a_1,2\pi\bar a_2$ and $2\pi(\bar a_1+\bar a_2)$. The convex compact set T is the sum of the segments $I_1=[0,2\pi\bar a_1]$ and $I_2=[0,2\pi\bar a_2]$ (if, for example, $\gamma_1=0$, then $T=I_2$; similarly in the other case). Therefore, for the support function of this compact set, the following equalities are true:

$$H_T(\varphi)=H_{I_1}(\varphi)+H_{I_2}(\varphi)=\pi\gamma_1(|\sin(\varphi+\varphi_1)|-\sin(\varphi+\varphi_1))+\\ +\pi\gamma_2(|\sin(\varphi+\varphi_2)|+\sin(\varphi+\varphi_2)). \tag{10}$$

Choose a number $\delta_0>0$, such that $\Gamma(\delta_0,\varphi_1)\cap\Gamma(\delta_0,\varphi_2)=\{0\}$, and let $\varepsilon>0$, $\delta\in(0,\delta_0)$. By (8)–(10), we have

$$\ln|f_0(re^{i\varphi})|\geqslant(H_T(\varphi)-\varepsilon)r,$$

$$re^{i\varphi}\in(\partial\Gamma(\delta,\varphi_j)\setminus B(0,\tau_{1,j}))\cup\left(\Gamma(\delta,\varphi_j)\cap\bigcup_{j=1}^{\infty}S(0,\tau_{s,j})\right), \tag{11}$$

where $j = 1, 2$.

$$h_{f_0}(\varphi) < H_T(\varphi) + \varepsilon, \quad \varphi \in [0, 2\pi]. \tag{12}$$

By (11), the inequality follows:

$$h_{f_0}(\varphi) \geqslant H_T(\varphi) - \varepsilon, \quad \varphi \in [0{,}2\pi]. \tag{13}$$

Let

$$\Gamma_{s,j} = \Gamma(\delta, \varphi_j) \cap \{z : \tau_{s,j} < |z| < \tau_{s+1,j}\}, \quad s \geqslant 1, \quad j = 1, 2.$$

Let K be the conjugate diagram of the function f_0 and w be the point of the compact set K closest to the origin. By (12) and (13), there exists a constant $c > 0$ (it depends only on the number $2\pi - (\varphi_2 - \varphi_1)$), such that $|w| \leqslant c\varepsilon$.

The interpolating function $\omega_{f_w}(\lambda, \sigma, P_\mu)$ is determined by formula (3) for any $\mu \geqslant 1$ and $\sigma \in \mathbb{C}$. Now, using this function, we find the upper estimates on $|P_{\mu,j}|$.

Let $a_{k,n,\mu} = 0$, $k > \mu$. By (5) and by the residue theorem:

$$\frac{1}{2\pi i} \int\limits_{\partial\Gamma_{s,j}} \frac{\omega_{f_w}(\lambda, \sigma, P_\mu)}{f_w(\lambda)} e^{\lambda z} d\lambda = \sum_{\lambda_k \in \Gamma_{s,j}} a_{k,n,\mu} z^n e^{\lambda_k z}, \tag{14}$$

where $\sigma \in \mathbb{C}$, $k \geqslant 1$ and $f_w(\lambda) = f_0(\lambda) e^{-w\lambda}$.

Let K_0 be the compact set complex conjugate to the compact set T. By (2) and (12), it follows that $K \subset K_0 + B(0, \varepsilon)$.

Since $e^{i(\varphi_1 + \pi/2)}$, $e^{i(\varphi_2 - \pi/2)} \in J_0(D)$, then, by Lemma 3.1 in [12], for each $\alpha \in D$ the rays $\alpha + \{te^{i(\varphi_1+\pi/2)}, t \geqslant 0\}$ and $\alpha + \{te^{i(\varphi_2 - \pi/2)}, t \geqslant 0\}$ lie in the domain D. Then, according to its convexity, for each $\alpha \in D$ the embedding $\alpha + K_0 \subset D$ is also true.

Choose $\varepsilon(\alpha) > 0$ such that $\alpha + K_0 + B(0, 3\varepsilon(\alpha)) \subset D$ and let $\varepsilon \in (0, \varepsilon(\alpha))$. By (4), (11), (12) and the definition of the function f_w, we have:

$$\left| \int\limits_{\partial\Gamma_{s,j}} \frac{\omega_{f_w}(re^{i\varphi}, \sigma, P_\mu)}{f_w(re^{i\varphi})} e^{\lambda z} d\lambda \right| \leqslant$$

$$\leqslant Q \max_{re^{i\varphi} \in \partial\Gamma_{s,j}} \left(\frac{\exp(r(H_T(\varphi) + 2\varepsilon - \operatorname{Re}((w + \sigma - z)e^{i\varphi})))}{\exp((H_T(\varphi) - \varepsilon - \operatorname{Re}(we^{i\varphi}))r)} \right) =$$

$$= Q \max_{re^{i\varphi} \in \partial\Gamma_{s,j}} \exp(r(3\varepsilon - \operatorname{Re}((\sigma - z)e^{i\varphi}))), \quad s, \mu \geqslant 1, \tag{15}$$

where $Q = A(f_0, \varepsilon) b_{s,j} \max_{x \in \Omega_\sigma(\varepsilon)} |P_\mu(x)|$, $b_{s,j}$ is the length of the boundary $\partial\Gamma_{s,j}$, $\sigma = w + \alpha$, $\alpha \in D$.

Let $j = 1, 2$. By (7) there exist $c_1, c_0 > 0$, such that

$$b_{s,j} \leqslant c_1 \delta \tau_{s,j} \leqslant c_0 e^{\varepsilon \tau_{s,j}}, \quad s \geqslant 1. \tag{16}$$

Let L be an arbitrary compact set in the domain $\Pi(H_D(\varphi_j), \varphi_j)$. Then, by (7), there exists $c_2 > 0$, such that

$$\max_{re^{i\varphi} \in \partial\Gamma_{s,j}} \exp(r(3\varepsilon + \mathrm{Re}(ze^{i\varphi}))) \leqslant \exp(\tau_{s,j}(H_L(\varphi_j) + c_2(\varepsilon + \delta))) \tag{17}$$

for each $z \in L$ and $s \geqslant 1$. Let $\sigma = w + \alpha$, $\alpha \in D$. Since $|w| \leqslant c\varepsilon$, we have:

$$\max_{re^{i\varphi} \in \partial\Gamma_{s,j}} \exp(r(-\mathrm{Re}(we^{i\varphi}))) \leqslant \exp(c_3 \varepsilon \tau_{s,j}), \quad s \geqslant 1, \tag{18}$$

where $c_3 > 0$. By definition of the function $H_D(\varphi_j)$, we choose a point $\alpha \in D$ and a number $A > 0$ such that

$$\mathrm{Re}(\alpha e^{-i\varphi_j}) \geqslant H_L(\varphi_j) + A.$$

Then, by (7), we have for some $c_4 > 0$:

$$\max_{re^{i\varphi} \in \partial\Gamma_{s,j}} \exp(r(-\mathrm{Re}(\alpha e^{i\varphi}))) \leqslant \exp(\tau_{s,j}(-H_L(\varphi_j) - A + c_4 \delta)). \tag{19}$$

Given that the sequence P_μ converges uniformly on the compact set $\Omega_\sigma(\varepsilon)$, and by inequalities (15)–(19), we find $C(f_0,\varepsilon) > 0$ such that

$$\left| \int_{\partial\Gamma_{s,j}} \frac{\omega_{f_w}(re^{i\varphi}, \sigma, P_\mu)}{f_w(\lambda)} e^{\lambda z} d\lambda \right| \leqslant C(f_0, \varepsilon) \exp((\varepsilon + c_2(\varepsilon + \delta) + c_3 \varepsilon + c_4 \delta - A)\tau_{s,j}),$$

where $z \in L$. Choose $\varepsilon \in (0, \varepsilon(\alpha))$ and $\delta \in (0, \delta_0)$ such that $\delta \leqslant \varepsilon$ and

$$\varepsilon + c_2(\varepsilon + \delta) + c_3 \varepsilon + c_4 \delta - A \leqslant -\varepsilon.$$

Then

$$\left| \int_{\partial\Gamma_{s,j}} \frac{\omega_{f_w}(re^{i\varphi}, \sigma, P_\mu)}{f_w(\lambda)} e^{\lambda z} d\lambda \right| \leqslant C(f_0, \varepsilon) e^{-\varepsilon \tau_{s,j}}, \quad z \in L, \quad s, \mu \geqslant 1. \tag{20}$$

Since $\Xi(\Lambda_j) \subseteq \{e^{i\varphi_j}\}$, there exists a number s_0, such that for each $s \geqslant s_0$ and $\lambda_k \in \Gamma_{s,j}$ the pair λ_k, n_k is an element of the sequence Λ_j. Outside of the union of the sets

$$\bigcup_{j=1}^{2} \bigcup_{s=s_0}^{\infty} \Gamma_{s,j}$$

there exists only a finite number of points λ_k. We can assume that the points $\lambda_1, \ldots, \lambda_{k_0}$ exist. By (4) for each $\mu \geqslant 1$, we have:

$$\sum_{k=1}^{k_0} \left| \frac{1}{2\pi i} \int\limits_{\partial B(\lambda_k, b_k)} \frac{\omega_{f_w}(re^{i\varphi}, \sigma, P_\mu)}{f_w(\lambda)} e^{\lambda z} d\lambda \right| \leqslant \sum_{k=1}^{k_0} A(f_0, \varepsilon) 2\pi b_k \times$$

$$\times \max_{re^{i\varphi} \in \partial B(\lambda_k, b_k)} \left(\frac{\exp((h_f(\varphi) + \varepsilon - \operatorname{Re}((w + \sigma - z) re^{i\varphi}))r)}{f_w(re^{i\varphi})} \right) \max_{x \in \Omega_\sigma(\varepsilon)} |P_\mu(x)|$$

Since the sequence P_μ converges uniformly on the compact set $\Omega_\sigma(\varepsilon)$,

$$\sum_{k=1}^{k_0} \left| \frac{1}{2\pi i} \int\limits_{\partial B(\lambda_k, b_k)} \frac{\omega_{f_w}(re^{i\varphi}, \sigma, P_\mu)}{f_w(\lambda)} e^{\lambda z} d\lambda \right| \leqslant B, \quad z \in L, \quad \mu \geqslant 1. \tag{21}$$

By (5), (14), and the definition of polynomials $P_{\mu,j}$, we have:

$$P_{\mu,j}(z) = \sum_{k=1}^{k_0} \frac{1}{2\pi i} \int\limits_{\partial B(\lambda_k, b_k)} \frac{\omega_{f_w}(re^{i\varphi}, \sigma, P_\mu)}{f_w(\lambda)} e^{\lambda z} d\lambda +$$

$$+ \sum_{s=s_0}^{\infty} \int\limits_{\partial \Gamma_{s,j}} \frac{\omega_{f_w}(re^{i\varphi}, \sigma, P_\mu)}{f_w(re^{i\varphi})} e^{\lambda z} d\lambda.$$

Note that in the right=hand side of this formula there exists only a finite number of nonzero terms. Let $s(l)$, $l \geqslant 1$, all be numbers $s \geqslant s_0$, such that the domain $\Gamma_{s,j}$ contains at least one point λ_k. Then, by (20) and (21),

$$|P_{\mu,j}(z)| \leqslant B + C(f_0, \varepsilon) \sum_{l=1}^{\infty} e^{-\varepsilon \tau_{s(l),j}}, \quad z \in L, \quad \mu \geqslant 1. \tag{22}$$

Since Λ has finite upper density, the sequence $\{\tau_{s(l),j}\}$ also has finite upper density, by the choice of the numbers $s(l)$. It follows that the last series converges. Thus, by (22), the sequence $\{|P_{\mu,1}|\}$ is uniformly bounded on any compact set in the domain $\Pi(H_D(\varphi_1),\varphi_1)$. Applying Montel's theorem, we find the subsequence $\{P_{\mu(m),1}\}_{m=1}^{\infty}$ that converges uniformly on each compact set in $\Pi(H_D(\varphi_1),\varphi_1)$ to some function $g_1 \in W(\Lambda_1, \Pi(H_D(\varphi_1),\varphi_1))$.

By (22), the sequence $\{P_{\mu(m),2}\}$ is uniformly bounded on any compact set in the domain $\Pi(H_D(\varphi_2),\varphi_2)$. According to Montel's theorem, it is possible to choose its subsequence that converges uniformly on each compact set in $\Pi(H_D(\varphi_2),\varphi_2)$ to some function $g_2 \in W(\Lambda_1, \Pi(H_D(\varphi_2),\varphi_2))$.

It remains to note that $P_{\mu,1}+P_{\mu,2}=P_\mu$ converges uniformly on any compact set in the domain D to the function g. Therefore, we have $g = = g_1 + g_2$.

Case 2. $e^{i\varphi_1} = -e^{i\varphi_2}$. Let us construct f_0. By Λ_2 denote the sequence of all pairs $-\lambda_k, n_k$, such that $(\lambda_k, n_k) \in \Lambda_2$. Let $\Lambda_3 = -\Lambda_2 \cup \Lambda_1$. Then $\Xi(\Lambda_3) = \{e^{i\varphi_1}\}$ and $\overline{n}(\Lambda_3) < +\infty$. By Theorem 2.3 in [12], for any $\varepsilon > 0$ and $\delta \in (0,1)$ there exist $\gamma > 0$, Λ^0, and a strictly increasing unbounded sequence of positive integers $\{\tau_s\}_{s=1}^{\infty}$, such that

$$\tau_{s+1} \leqslant (1+\delta)\tau_s, \quad s \geqslant 1, \quad \Lambda \subseteq \Lambda_3 \subseteq \Lambda^0,$$

$$|\ln|f_0(re^{i\varphi})| - \pi\gamma r|\sin(\varphi+\varphi_1)|| \leqslant \varepsilon r,$$

$$re^{i\varphi} \in (\mathbb{C} \setminus (\Gamma(\delta,\varphi_1) \cup B(0,\tau_1))) \cup \left(\bigcup_{j=1}^{\infty} S(0,\tau_s)\right),$$

where $f_0(z) = f(z,\Lambda^0)$. Let $T = I = [0, 2\pi\overline{a}]$, $a = \gamma e^{i(\varphi_1+\pi/2)}$. Further reasoning is completely similar to the that carried out in the first case. The lemma is proved. $\square$

Let $\varphi \in \mathbb{R}$ and

$$S_\Lambda(\varphi) = \min_{\{\lambda_{k(j)}\}} \lim_{\delta\to 0} \varliminf_{j\to\infty} \frac{\ln|q_\Lambda^{k(j)}(\lambda_{k(j)},\delta)|}{|\lambda_{k(j)}|},$$

where the minimum is taken over all subsequences $\{\lambda_{k(j)}\}$ of the sequence $\{\lambda_k\}$, such that $\lambda_{k(j)}/|\lambda_{k(j)}| \to e^{-i\varphi}$, $j \to \infty$.

The sequence $\Lambda = \{\lambda_k, n_k\}$ is called almost real if $\Xi(\Lambda) = \{1\}$ and $\operatorname{Re}\lambda_k > 0$, $k \geqslant 1$. Finally, we formulate and proof the main result of this work.

Theorem 1. *Let $\Lambda = \{\lambda_k, n_k\}$, D be a convex domain and the system $\mathcal{E}(\Lambda)$ be not complete in $H(D)$. The following statements are equivalent:*

1) *Each function* $g \in W(\Lambda, D)$ *is represented by the series* (1) *converging uniformly on compact sets in* $D_0 = \Pi(H_D(\varphi_1), \varphi_1) \cap \Pi(H_D(\varphi_2), \varphi_2)$;
2) $\Xi(\Lambda) \subseteq \overline{J(D)}$, $\partial J(D) \subseteq \{e^{i\varphi_1}, e^{i\varphi_2}\}$, $S_\Lambda > -\infty$ *and* $S_\Lambda(\varphi) = 0$, $\varphi \in \partial J(D) \setminus J(D)$.

Proof. Assume that 1) holds. In particular, it means that each function $g \in W(\Lambda, D)$ analytically continues to the domain D_0 and it is approximated by linear combinations of elements of the system $\mathcal{E}(\Lambda)$ in D_0. Then, by Lemma 1 in [18], the following embedding is true:

$$D_0 \subseteq \{z \in \mathbb{C} : \operatorname{Re}(ze^{-i\varphi}) < H_D(\varphi), e^{-i\varphi} \in \Xi(\Lambda)\}.$$

It follows that $\Xi(\Lambda) \subseteq \overline{J(D)}$ and $\partial J(D) \subseteq \{e^{i\varphi_1}, e^{i\varphi_2}\}$.

Assume that $S_\Lambda = -\infty$. Then, by Theorem 4.3 in [12], there exist numbers $\{d_{k,n}\}$ and k_s, $1 = k_1 < k_2 < \dots$ such that the series

$$\sum_{s=1}^{\infty} \left(\sum_{k=k_s}^{k_{s+1}-1} \sum_{n=0}^{n_k-1} d_{k,n} z^n e^{\lambda_k z} \right)$$

converges uniformly on compact sets in the plane and the series (1) diverges at each point of the plane. Let g be the sum of this converging series. Then $g \in W(\Lambda, \mathbb{C}) \subseteq W(\Lambda, D)$. According to the statement 1, we have

$$g(z) = \sum_{k=1,\, n=0}^{\infty,\, n_k-1} b_{k,n} z^n e^{\lambda_k z}, \quad z \in D_0,$$

and the series converges uniformly on compact sets in D_0. By Lemma 1 and Lemma 2, it follows that $d_{k,n} = b_{k,n}$, $k \geqslant 1$, $n = \overline{0, n_k - 1}$. We get a contradiction. Thus, we have $S_\Lambda > -\infty$.

Let $\varphi \in \partial J(D) \setminus J(D)$ and the sequence $\{\lambda_{k(j)}\}$ be such that $\lambda_{k(j)}/|\lambda_{k(j)}| \to e^{-i\varphi}$, $j \to \infty$. Let $\Lambda_0 = \{e^{i\varphi}\lambda_{k(j)}, n_{k(j)}\}$. Then $\Xi(\Lambda_0) = \{1\}$. Therefore, there exists a number m, such that $\Lambda_{0,0} = \{e^{i\varphi}\lambda_{k(j)}, n_{k(j)}\}_{j=m}^{\infty}$ is an almost real sequence. Let $\Lambda_{0,1} = \{\lambda_{k(j)}, n_{k(j)}\}_{j=m}^{\infty}$. By 1), each function $g \in W(\Lambda, D_0) \subseteq W(\Lambda, D)$ is represented by the series (1), which converges uniformly on compact sets in D_0. Therefore, each function $g_0 \in W(\Lambda, e^{-i\varphi}D_0)$ has the same property, but now on compact sets in the domain $D_1 = e^{-i\varphi}D_0$.

Since $\varphi \in \partial J(D) \setminus J(D)$, $1 \in \partial J(D_1)$ and $H_{D_1}(\varphi) < +\infty$. Therefore, by Theorem 3.8 in [11], the equality $S_{\Lambda_{0,0}} = 0$ holds. It follows that

$S_{\Lambda_{0,1}} = 0$. And, finally, according to the definition of the value $S_\Lambda(\varphi)$, we have: $S_\Lambda(\varphi) = 0$.

Assume that 2) holds and $g \in W(\Lambda, D)$. If $D = \mathbb{C}$, then, according to the corollary of Theorem 9.5 in [9], statement 1) holds. Let D be the half-plane. Since the system $\mathcal{E}(\Lambda)$ is not complete in $H(D)$, $\overline{n}(\Lambda) < +\infty$. Thus, statement 1) holds, by Theorem 4.4 in [12].

Let the domain D be distinct from the plane and the half-plane. Then $\partial J(D) = \{e^{i\varphi_1}, e^{i\varphi_2}\}$ is a two-element set. By Theorem 3.4 in [12], there exist the sequences Λ_1 and Λ_2, such that $\Lambda = \Lambda_1 \cup \Lambda_2$, $\Xi(\Lambda_2) \subset S(0,1) \setminus \operatorname{int} J(D)$, and the representation $g = g_1 + g_2$ holds; here $g_1 \in W(\Lambda_1, \mathbb{C})$ and $g_2 \in W(\Lambda_2, D)$.

Since $S_\Lambda > -\infty$, the function g_1 is represented by the series (1), according to the corollary of Theorem 9.5 in [9]; this series converges uniformly on compact sets in the plane.

Since $\Xi(\Lambda) \subseteq \overline{J(D)}$, $\Xi(\Lambda_2) \subseteq \partial J(D)$. If $\Xi(\Lambda_2) = \emptyset$, then, by Lemma 3.3 in [12], the function g_2 is an exponential polynomial, which is a special case of the series (1). Let $\Xi(\Lambda_2) = \{e^{i\varphi_j}\}$. Then Λ_2 can be represented as $\Lambda_2 = \Lambda_{2,1} \cup \Lambda_{2,2}$, where $e^{i\varphi_j}\Lambda_{2,1}$ is an almost real sequence, and $\Lambda_{2,2}$ consists of a finite number of elements. By Lemma 3.3 in [12], the representation $g_2 = g_{2,1} + g_{2,2}$ holds, where $g_{2,2}$ is an exponential polynomial and $g_{2,1} \in W(\Lambda_{2,1}, D)$. If $H_D(\varphi_j) = +\infty$, then, by Theorem 6 in [13], the function $g_{2,1}$ is represented by the series (1), which converges uniformly on compact sets in the plane. If $H_D(\varphi_j) < +\infty$, then, by Theorem 3.8 in [11], the function $g_{2,1}$ is represented by the series (1), which converges uniformly on compact sets in the half-plane $\Pi(H_D(\varphi_j), \varphi_j)$.

Finally, let $\Xi(\Lambda_2) = \{e^{i\varphi_1}, e^{i\varphi_2}\}$. Then Λ_2 can be represented as $\Lambda_2 = \Lambda_{2,1} \cup \Lambda_{2,2} \cup \Lambda_{2,3}$, where $e^{i\varphi_j}\Lambda_{2,j}$, $j = 1, 2$, is an almost real sequence, and $\Lambda_{2,3}$ consists of finite number of elements. As above, applying Lemma 3.3 in [12], Theorem 6 in [13], and Theorem 3.8 in [11], we obtain the statement 1). The theorem is proved. □

Remark 1. *Special cases of Theorem 1 are the corollary of Theorem 9.5 in [9], Theorem 4.4 in [12], Theorem 6 in [13], and Theorem 3.8 in [11].*

Remark 2. *From the results of the works [5–8], it follows that in the case when $\Xi(\Lambda)$ does not lie in $\overline{J(D)}$, the conditions of the fundamental principle cannot be formulated just in terms of the condensation index S_Λ. In this case, it is necessary to take into account the relationship between the maximum angular density of the sequence Λ, the multiplicities of the points λ_k, and the function of a length of an arc of the domain D.*

Thus, Theorem 1 describes all cases of mutual arrangement of Λ and D, in which the conditions of the fundamental principle can be formulated just in terms of the condensation index S_Λ.

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Institute of Mathematics,
Ufa Federal Research Center, RAS,
112 Chernyshevsky str., Ufa 450008, Russia
E-mail: kriolesya2006@yandex.ru

Baskir State University
32 Z. Validi St., Ufa 450076, Russia
E-mail: kriolesya2006@yandex.ru