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WEAVING CONTINUOUS K-FRAMES IN HILBERT SPACES

Abstract. In this paper, we introduce and study weaving continuous K-frames in Hilbert spaces. We first introduce a useful result for the production of these frames and then examine them under the influence of a bounded operator. Due to the basic and useful applications of different types of frames in restoring some deleted information on data transfer issues, we give at the end of the paper some conditions of setting the frame under the removal of some members of the measure space and we show that this is related to the discrete K-frames.

Key words: *continuous-frames, continuous K-frames, weaving continuous K-frames*

2020 Mathematical Subject Classification: *30D40*

1. Introduction. The concept of frames in Hilbert spaces was first introduced by Duffin and Schaeffer ([10]) in 1952 to study some profound issues in nonharmonic Fourier series. Frames play so significant role in both pure and applied mathematics that are considered as the fundamental research area in mathematics, computer science, and quantum information. Besides their former application, they are also applied in some other fields, such as signal processing, image processing, data compression, and sampling theory.

The special importance of frames is related to the types of their generalizations, which are mainly in seven areas:
1) Continuous frames (or briefly c-frames) ([1], [11]), which are introduced to the measure spaces.
2) Generalization frames (or briefly g-frames) ([22]), which are introduced to the boundary operators between Hilbert spaces.
3) Fusion frames, ([7]) for subspaces of Hilbert spaces.
4) Frames for operators or K-frames have been introduced in [14] to study

the nature of atomic systems for a separable Hilbert space with respect to a bounded linear operator K.
5) Controlled frames ([5]), which are introduced to improve the numerical efficiency of interactive algorithms for inverting the frame operator on Hilbert spaces.
6) Weaving frames ([6]), which are motivated by distributed signal processing.
7) Combining the above mentioned (as [2], [3], [11], [15], [17], [19], [20], [21], [23], [24]).

In this paper, we introduce the concept of weaving continuous K-frames in Hilbert spaces. After introducing these frames, we introduce the necessary and sufficient conditions for their production. Then, by affecting the bounded operator on these frames and imposing some conditions, we try to maintain the existing frame and, in some cases, we see that we have to make the main space smaller. Finally, by removing a subset of the measure space, we introduce the conditions for creating those frames for the new measure space and introduce its relationship with the discrete frames.

Throughout this paper, H is a Hilbert space, (X, μ) is a measure space with positive and σ-finite measure μ, and $\mathcal{B}(H, K)$ is the set of all bounded linear operators from H into K.

First, we review some topics and results of the operator theory. Suppose that $U \in \mathcal{B}(H_1, H_2)$ is an operator on the Hilbert space H_1 into the Hilbert space H_2. The pseudo-inverse of U is denoted by $U^\dagger \in \mathcal{B}(H_2, H_1)$ and is defined by $UU^\dagger x = x$ for each $x \in \mathcal{R}(U)$. In the following result, (Lemma A.7.2 [8]) we present some properties of the operator $U^\dagger$:

Lemma 1. *[8]*

1) $(U^*)^\dagger = (U^\dagger)^*$.
2) *The orthogonal projection of H_2 onto $\mathcal{R}(U)$ is given by $UU^\dagger$.*
3) *The orthogonal projection of H_1 onto $\mathcal{R}(U^\dagger)$ is given by $U^\dagger U$.*
4) $\ker U^\dagger = \mathcal{R}(U)^\perp$ *and* $\mathcal{R}(U^\dagger) = \ker U^\perp$.
5) *On $\mathcal{R}(U)$, we have $U^\dagger = U^*(UU^*)^{-1}$.*

2. Continuous frames. The space $\mathscr{L}^2(X)$ is the class of all measurable mappings $F\colon X \to \mathbb{C}$, such that for each $x \in X$:

$$\int_X \|F(x)\|^2 \, d\mu(x) < \infty.$$

It is a Hilbert space with the inner product defined by

$$\langle F, G\rangle = \int_X \langle F(x), G(x)\rangle \, d\mu(x) \ , \qquad F, G \in \mathscr{L}^2(X).$$

We denote the vector space of all equivalence classes of almost everywhere finite-valued measurable functions on X by $\mathscr{L}^0(X)$. Let $F\colon X \to H$ be a weakly measurable (i. e., for all $h \in H$, the mapping $x \mapsto \langle F(x), h\rangle$ is measurable). We define the vector valued integral as follows:

$$\int_X \cdot F \, d\mu \colon \mathscr{L}^2(X) \to H,$$

$$\langle \int_X GF \, d\mu, h\rangle = \int_X G(x)\langle F(x), h\rangle d\mu(x), \qquad h \in H.$$

We can show that if $F\colon X \to H$ is a weakly measurable, then for each $G \in \mathscr{L}^2(X)$, the value of $\int_X GF \, d\mu$ exists in H if and only if $\langle F, h\rangle \in \mathscr{L}^2(X)$ for each $h \in H$ (see [18]).

Definition 1. *Let $F\colon X \to H$ be a weakly measurable and $K \in \mathcal{B}(H)$. Then the map F is called a continuous K-frame (or briefly c-K-frame) for H with respect to X, if there exist $0 < A \leqslant B < \infty$, such that for each $h \in H$:*

$$A\|K^*h\|^2 \leqslant \int_X |\langle h, F(x)\rangle|^2 \, d\mu(x) \leqslant B\|h\|^2. \tag{1}$$

The numbers A and B are called *frame bounds*. If $K = Id_H$, then F is called *c-frame*. We say that F is a *tight* c-K-frame when

$$\int_X |\langle h, F(x)\rangle|^2 \, d\mu(x) = A\|K^*h\|^2,$$

and F is called the Parseval c-K-frame when $A = 1$. If only the right-hand side of (1) holds, we say F is a c-Bessel mapping with the bound B. When μ is the counting measure and $X = \mathbb{N}$, then F becomes an ordinary K-frame (for more details about K-frames, we refer to [14], [4]).

Example 7. Assume that $H = \ell^2(\mathbb{N})$, $X = \mathbb{R}$ and μ is the Lebesgue measure. Let $a > 0$ be a constant and define

$$K\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \qquad K\delta_i = \delta_{i+1},$$

and

$$F\colon X \longrightarrow H,$$
$$F(x) = \begin{cases} a\delta_i, & i \leqslant x < i+1, \\ 0, & x < 1, \end{cases}$$

where $\{\delta_i\}_{i=1}^{\infty}$ is an orthonormal base for $\ell^2(\mathbb{N})$. Since

$$K^*\delta_1 = 0\,, \qquad K^*\delta_i = \delta_i\,,\ i \geqslant 2,$$

therefore, for every $h \in H$ we have

$$\int\limits_{-\infty}^{\infty} |\langle h, F(x)\rangle|^2\, d\mu(x) = a^2\|h\|^2 \geqslant a^2\|K^*h\|^2.$$

So, F is a c-K-frame for H with bounds $A = B = a^2$.

Example 8. Assume that $H = \mathbb{R}^3$ with the standard orthonormal base $\{e_1, e_2, e_3\}$, $X = \mathbb{R}$ and μ is the Lebesgue measure. Define

$$K\colon \mathbb{R}^3 \to \mathbb{R}^3,$$
$$Ke_1 = e_1,\ \ Ke_2 = e_1,\ \ Ke_3 = 0,$$

and

$$F\colon \mathbb{R} \longrightarrow \mathbb{R}^3,$$
$$F(x) = \exp\Big(-\frac{x^2}{2}, 0, 0\Big).$$

It is easy to check that $K \in \mathcal{B}(H)$ and, also,

$$K^*e_1 = e_1 + e_2,\ \ K^*e_2 = K^*e_3 = 0.$$

Now, let $h = (h_1, h_2, h_3) \in \mathbb{R}^3$ be an arbitrary element. Then $\|K^*h\|^2 = 2h_1^2$, and so

$$\int\limits_{-\infty}^{\infty} |\langle h, F(x)\rangle|^2\, d\mu(x) = h_1^2 \int\limits_{-\infty}^{\infty} \exp(-x^2)\, dx = \frac{\sqrt{\pi}}{2}\|K^*h\|^2.$$

So, F is a tight c-K-frame for $\mathbb{R}^3$ with the bound $A = B = \frac{\sqrt{\pi}}{2}$.

In the following, we will define the synthesis, analysis, and frame operators. If F is a c-Bessel mapping, then the synthesis and analysis operators are defined by

$$\begin{aligned} T_F \colon \mathscr{L}^2(X) &\longrightarrow H, \\ \langle T_F(G), h\rangle &= \int\limits_X G(x)\langle F(x), h\rangle \, d\mu(x), \end{aligned}$$

and

$$\begin{aligned} T_F^* : H &\longrightarrow \mathscr{L}^2(X), \\ T_F^* h &= \langle h, F\rangle. \end{aligned}$$

For the synthesis operator, we can write, using the notation of the vector-valued integral:

$$T_F(G) = \int\limits_X GF \, d\mu, \qquad G \in \mathscr{L}^2(X).$$

Therefore, the frame operator $S_F := T_F T_F^*$ is given by

$$S_F h = \int\limits_X \langle h, F\rangle F \, d\mu.$$

Now, when F is a c-frame for H with the frame bounds A and B, we get

$$AId_H \leqslant S_F \leqslant BId_H.$$

Hence, S_F is a positive, self-adjoint, and invertible operator. For more details about c-frames, we refer to [12]. This property does not hold for c-K-frames. Indeed, the frame operator of a c-K-frame is not invertible in general, but when K is a closed range, then S_F on $\mathcal{R}_K$ is invertible and for each $h \in \mathcal{R}_K$ we have ([25])

$$B^{-1}\|h\|^2 \leqslant \langle (S_F|_{\mathcal{R}_K})^{-1}h, h\rangle \leqslant A^{-1}\|K^\dagger\|^2\|h\|^2.$$

In the following, we can construct c-K-frames with the help of a c-frame.

Lemma 2. [16] *Let $F : X \longrightarrow H$ be a c-frame for H and $K \in \mathcal{B}(H)$. Then KF is a c-K-frame for H.*

Example 9. [8] Suppose that $\psi \in \mathscr{L}^2(\mathbb{R})$, such that

$$C_\psi := \int_{-\infty}^{+\infty} \frac{|\psi(\gamma)|}{\gamma} d\gamma < \infty.$$

For each $x \in \mathbb{R}$, define

$$F\colon \mathbb{R} - \{0\} \times \mathbb{R} \longrightarrow \mathscr{L}^2(\mathbb{R})$$
$$F(a,b)(x) = (T_b D_a)(x) = \frac{1}{\sqrt{a}} \psi\Big(\frac{x-b}{a}\Big),$$

where, T_b and D_a are operators on $\mathscr{L}^2(\mathbb{R})$ defined by

$$T_b\colon \mathscr{L}^2(\mathbb{R}) \longrightarrow \mathscr{L}^2(\mathbb{R}), \qquad D_a\colon \mathscr{L}^2(\mathbb{R}) \longrightarrow \mathscr{L}^2(\mathbb{R}),$$
$$(T_b f)(x) = f(x-b), \qquad (D_a f)(x) = \frac{1}{\sqrt{a}} f\Big(\frac{x}{a}\Big).$$

Via Propositions 11.1.1 and 11.1.2 in [8], we can get

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \langle f, F(a,b)\rangle \langle F(a,b), g\rangle \frac{dadb}{a^2} = C_\psi (f,g), \quad \forall f,g \in \mathscr{L}^2(\mathbb{R}).$$

Thus, F is a c-frame for $\mathscr{L}^2(\mathbb{R})$ with respect to $(\mathbb{R} - \{0\} \times \mathbb{R}, \mu)$, where $\mu = \dfrac{dadb}{a^2}$. So, by Lemma 2, if $K \in \mathcal{B}(H)$ is a given bounded operator, then KF is a c-K-frame for $\mathscr{L}^2(\mathbb{R})$.

3. c-K-woven frame. In this section, we aim to introduce weaving c-K-frames (or c-K-woven frame) and study some results about them. Throughout the paper, by partition of a measure space (Ω, μ) we mean a partition of Ω into disjoint measurable sets. For each $m > 1$, where $m \in \mathbb{N}$, we define $[m] := \{1, 2, \ldots, m\}$ and $[m]^c = \{m+1, m+2, \ldots\}$.

Definition 2. *Let $\mathscr{F} := \{F_i\}_{i \in [m]}$ be a family of c-K-frames for H with respect to the measure μ. We say $\mathscr{F}$ is a woven frame when there exist $0 < A \leqslant B < \infty$, such that for every partition $\{\sigma_i\}_{i\in[m]}$ of X, the family $\cup_{i\in[m]}\{F_i\}$ is a c-K-frame for H with bounds A and B.*

Via Proposition 3.4 in [23], the upper bound of the woven frame is evident; indeed, if

$$\{F_i\}_{h\in[m]}$$

is a c-Bessel sequence for H with bounds B_i, then for each partition $\{\sigma_i\}_{i\in[m]}$ of X, the family $\cup_{i\in[m]}\{F_i\}$ is a c-Bessel sequence for H with the bound $\sum_{h\in[m]} B_i$.

The following result presents the necessary and sufficient condition to construct c-K-woven frame.

Theorem 1. *Let $F, G\colon X \to H$ be two c-K-frames for H. The following assertions are equivalent:*

1) *F and G are two c-K-woven frames.*
2) *There is a number $\alpha > 0$, such that for each measurable subset $\sigma \subset X$, there exists a bounded operator*

$$\Gamma_\sigma\colon \mathscr{L}^2_\sigma(X) \longrightarrow H,$$
$$\langle \Gamma_\sigma \varphi, h\rangle = \int\limits_{\sigma} \varphi(x)\langle F(x), h\rangle\, d\mu(x) + \int\limits_{\sigma^c} \varphi(x)\langle G(x), h\rangle\, d\mu(x),$$

such that $\alpha K K^* \leqslant \Gamma_\sigma \Gamma_\sigma^*$, where

$$\mathscr{L}^2_\sigma(X) = \Big\{\varphi \in \mathscr{L}^2(X)\,, \quad \varphi = F\big|_\sigma \cup G\big|_{\sigma^c}\Big\}.$$

Proof. (1) ⇒ (2). Suppose that A is a minimum of lower bounds of two frames F and G. Consider $\alpha := A$ and for any measurable subset $\sigma \subset X$, assume that $\Gamma_\sigma = T_\sigma$, where T_σ is the synthesis operator of $F\big|_\sigma \cup G\big|_{\sigma^c}$. Now, for each $\varphi \in \mathscr{L}^2_\sigma(X)$ and $h \in H$, we have

$$\langle \Gamma_\sigma \varphi, h\rangle = \langle T_\sigma \varphi, h\rangle = \int\limits_{\sigma} \varphi(x)\langle F(x), h\rangle\, d\mu(x) + \int\limits_{\sigma^c} \varphi(x)\langle G(x), h\rangle\, d\mu(x),$$

and, also,

$$\alpha\|K^* h\|^2 \leqslant \|T_\sigma^* h\|^2 = \|\Gamma_\sigma^* h\|^2.$$

Therefore, $\alpha K K^* \leqslant \Gamma_\sigma \Gamma_\sigma^*$.

(2) ⇒ (1). The upper bound is obvious. Suppose that $\sigma \subset X$ is a measurable subset, $\varphi \in \mathscr{L}^2_\sigma(X)$ and $h \in H$ are arbitrary. We get

$$\langle \Gamma_\sigma^* h, \varphi\rangle = \overline{\langle \Gamma_\sigma \varphi, h\rangle} =$$

$$= \int_{\sigma} \langle h, F(x)\rangle \overline{\varphi(x)}\, d\mu(x) + \int_{\sigma^c} \langle h, G(x)\rangle \overline{\varphi(x)}\, d\mu(x) =$$
$$= \Big\langle \langle h, F\rangle\big|_{\sigma} \cup \langle h, G\rangle\big|_{\sigma^c}, \varphi \Big\rangle.$$

Thus,

$$\Gamma_\sigma^* h = \langle h, F\rangle\big|_{\sigma} \cup \langle h, G\rangle\big|_{\sigma^c}.$$

Therefore,

$$\alpha\|K^*h\|^2 = \langle \alpha K K^* h, h\rangle \leqslant \langle \alpha \Gamma_\sigma \Gamma_\sigma^* h, h\rangle = \|\Gamma_\sigma^* h\|^2 =$$
$$= \int_{\sigma} |\langle h, F(x)\rangle|^2\, d\mu(x) + \int_{\sigma^c} |\langle h, G(x)\rangle|^2\, d\mu(x).$$

So, we conclude that α is a lower bound for F and G. □

Example 10. Let $H = \ell^2(\mathbb{N})$, $X = \mathbb{R}$ and μ is the Lebesgue measure. For each $a, b > 0$, define

$$K\colon H \to H,$$
$$K\delta_i = \delta_{i+1}$$

and

$$F, G\colon X \longrightarrow H,$$
$$F(x) = \begin{cases} a\delta_i, & i \leqslant x < i+1, \\ 0, & x < 1 \end{cases}, \qquad G(x) = \begin{cases} b\delta_i, & i \leqslant x < i+1, \\ 0, & x < 1, \end{cases}$$

where $\{\delta_i\}_{i=1}^{\infty}$ is the orthonormal base for $\ell^2(\mathbb{N})$. Via Example 7, F and G are two c-K-frames for H with similar bounds, respectively, a^2 and b^2. Consider $\sigma \subset \mathbb{R}$ to be a Lebesgue-measurable subset and let

$$\Gamma_\sigma\colon \mathscr{L}^2(\mathbb{R}) \longrightarrow H,$$
$$\Gamma_\sigma \varphi = \Big\{ \int_i^{i+1} a\varphi(x)\, dx \Big\}_{i\in\sigma} \cup \Big\{ \int_i^{i+1} b\varphi(x)\, dx \Big\}_{i\in\sigma^c}.$$

For every $\varphi \in \mathscr{L}^2(\mathbb{R})$, we have

$$\|\Gamma_\sigma \varphi\|^2 = a^2 \sum_{i\in\sigma} \int_i^{i+1} |\varphi(x)|^2 + b^2 \sum_{i\in\sigma^c} \int_i^{i+1} |\varphi(x)|^2 \leqslant \max\{a^2, b^2\} \|\varphi\|_2^2.$$

So, Γ_σ is bounded. For each $h \in H$, we can calculate that

$$\|\Gamma_\sigma^* h\|^2 \geqslant \min\{a^2, b^2\}\|h\|^2 \geqslant \min\{a^2, b^2\}\|K^* h\|^2.$$

Thus, by Theorem 1, F and G are c-K-woven frames for H.

In the next results, we construct a c-K-woven frame with a bounded operator.

Theorem 2. *Let $\{F_i : X \to H\}_{i\in[m]}$ be a c-K-woven frame for H with bounds A and B, also $U \in \mathcal{B}(H)$ be a closed range, such that $KU^\dagger = U^\dagger K$. Then $\{UF_i\}_{i\in[m]}$ is a c-K-woven frame for $\mathcal{R}(U)$ with bounds $A\|U^\dagger\|^{-2}$ and $B\|U\|^2$.*

Proof. Suppose that $i \in [m]$ and consider

$$\begin{aligned} &\phi\colon X \longrightarrow \mathbb{C}, \\ &\phi(x) = \langle h, F_i(x)\rangle \qquad (h \in H). \end{aligned}$$

So, the mapping ϕ is measurable for each h. Thus, the mapping

$$x \to \langle U^* h, F_i(x)\rangle$$

is measurable and, so, the operator UF_i is weakly measurable.

For each $h \in \mathcal{R}(U)$, we can write

$$\begin{aligned} A\|K^* h\|^2 &= A\|K^*(U^\dagger)^* U^* h\|^2 = \\ &= A\|(U^\dagger)^* K^* U^* h\|^2 \leqslant \\ &\leqslant A\|U^\dagger\|^2 \|K^* U^* h\|^2 \leqslant \\ &\leqslant \|U^\dagger\|^2 \sum_{i\in[m]} \int_X |\langle U^* h, F_i(x)\rangle|^2 \, d\mu(x) = \\ &= \|U^\dagger\|^2 \sum_{i\in[m]} \int_X |\langle h, UF_i(x)\rangle|^2 \, d\mu(x). \end{aligned}$$

For the upper bound, we have

$$\sum_{i\in[m]} \int_X |\langle h, UF_i(x)\rangle|^2 \, d\mu(x) \leqslant B\|U^* h\|^2 \leqslant B\|U\|^2 \|h\|^2.$$

□

Theorem 3. *Let $\{F_i : X \to H\}_{i\in[m]}$ be a c-K-woven frame for H with bounds A and B, also let $U \in \mathcal{B}(H)$ be such that $\mathcal{R}(U^*) \subseteq \mathcal{R}(K)$. Then $\{UF_i\}_{i\in[m]}$ is a c-K-woven frame for H if and only if there exists $\delta > 0$, such that for every $h \in H$ we have*

$$\|U^*h\| \geqslant \delta\|K^*h\|.$$

Proof. First, assume that $\{UF_i\}_{i\in[m]}$ is a c-K-woven frame H with the lower bounds A. For any $h \in H$, we get

$$\begin{aligned} A\|K^*h\|^2 &\leqslant \sum_{i\in[m]} \int_X |\langle Uh, F_i(x)\rangle|^2 \, d\mu(x) = \\ &= \sum_{i\in[m]} \int_X |\langle U^*h, F_i(x)\rangle|^2 \, d\mu(x) \leqslant \\ &\leqslant B\|U^*h\|^2. \end{aligned}$$

Therefore, $\|U^*h\| \geqslant \sqrt{\frac{A}{B}}\,\|K^*h\|$. For the opposite, if $h \in H$ is arbitrary, then

$$\|U^*h\| = \|(K^\dagger)^*K^*U^*h\| \leqslant \|K^\dagger\|\,\|K^*U^*h\|.$$

Now, consider $\{\sigma_i\}_{i\in[m]} \subset X$; we can write

$$\begin{aligned} A\delta^2\|K^\dagger\|^{-2}\|K^*h\|^2 &\leqslant A\|K^\dagger\|^{-2}\|U^*h\|^2 \leqslant A\|K^*U^*h\|^2 \leqslant \\ &\leqslant \sum_{i\in[m]} \int_{\sigma_i} |\langle U^*h, F_i(x)\rangle|^2 \, d\mu(x) = \\ &= \sum_{i\in[m]} \int_{\sigma_i} |\langle h, UF_i(x)\rangle|^2 \, d\mu(x) \leqslant \\ &\leqslant B\|U\|^2\|h\|^2. \end{aligned}$$

So, $\{UF_i\}_{i\in[m]}$ is a c-K-woven frame for H with bounds $A\delta^2\|K^\dagger\|^{-2}$and $B\|U\|^2$ □

The following theorem is an extension of Proposition 2.10 in [9] for the case of continuous K-frames:

Theorem 4. *Let H have closed range and $\{F_i\}_{i\in[m]}$ be a family of c-K-frames for H. The following assertions are equivalent:*

1) $\{F_i\}_{i\in[m]}$ *is a c-K-woven frame for H.*
2) *For each $U \in \mathcal{B}(H)$, such that UK is well-defined, the family $\{UF_i\}_{i\in[m]}$ is a c-UK-woven frame for H.*

Proof. (1) $\Rightarrow$ (2). Suppose that A and B are frame bounds of $\{F_i\}_{i\in[m]}$. For each partition $\{\sigma_i\}_{i\in[m]} \subset X$ and $h \in H$, we have

$$\sum_{i\in[m]} \int_{\sigma_i} |\langle h,UF_i(x)\rangle|^2 \, d\mu(x) = \sum_{i\in[m]} \int_{\sigma_i} |\langle U^*h,F_i(x)\rangle|^2 \, d\mu(x) \leqslant B\|U\|^2\|h\|^2.$$

Similarly, for the lower bound, we can write:

$$\sum_{i\in[m]} \int_{\sigma_i} |\langle h,UF_i(x)\rangle|^2 \, d\mu(x) = \sum_{i\in[m]} \int_{\sigma_i} |\langle U^*h,F_i(x)\rangle|^2 \, d\mu(x) \geqslant A\|(UK)^*h\|^2.$$

For the opposite, consider $U := id_H$ and the proof is evident. $\square$

In the next results, our aim is to delete a measurable subset of the measure space X and generate a new c-K-woven frame.

Theorem 5. *Let K have closed range and $\{F_i\colon X \to H\}_{i\in[m]}$ be a c-K-woven frame for H with bounds A and B. If Y is a measurable subset of X and*

$$C := \sum_{i\in[m]} \int_{Y} \|F_i(x)\|^2 \, d\mu(x) < A\|K^\dagger\|^{-2},$$

then $\{F_i\colon X \setminus Y \to H\}_{i\in[m]}$ is a c-K-woven frame for $\mathcal{R}(K)$ with bounds $(A - C\|K^\dagger\|^2)$ and B.

Proof. The upper bound is evident. Assume that $\{\sigma_i\}_{i\in[m]} \subset X \setminus Y$ is an arbitrarily partition, so

$$\{\tau_i\}_{i\in[m]} := \{\sigma_i\}_{i\in[m]} \cup Y$$

is a partition for X. For each $h \in \mathcal{R}(K)$, we have

$$\sum_{i\in[m]} \int_{\sigma_i} |\langle h, F_i(x)\rangle|^2 \, d\mu(x) = \sum_{i\in[m]} \int_{\tau_i} |\langle h, F_i(x)\rangle|^2 \, d\mu(x) -$$

$$- \sum_{i\in[m]} \int_{Y} |\langle h, F_i(x)\rangle|^2 \, d\mu(x) \geqslant$$

$$\geqslant A\|K^*h\|^2 - \|h\|^2 \sum_{i\in[m]} \int_Y \|F_i(x)\|^2 \, d\mu(x) \geqslant$$
$$\geqslant (A - C\|K^\dagger\|^2)\|K^*h\|^2.$$

□

Theorem 6. *Let K have closed range, every singleton subset of X be non-zero measurable, and $\{F_i \colon X \to H\}_{i\in[m]}$ be a tight c-K-woven frame for H with the bound A. If, for any $x_0 \in X$, the set $\{F_i(x_0)\}_{i\in[m]}$ is a K-frame for $\mathcal{R}(K)$ with the upper bound B, where $A > B\mu(\{x_0\})\|K^\dagger\|^2$, then $\{F_i \colon X \setminus \{x_0\} \to H\}_{i\in[m]}$ is a c-K-woven frame for $\mathcal{R}(K)$.*

Proof. Consider $\{\sigma_i\}_{i\in[m]}$ to be a partition for $X \setminus \{x_0\}$. Then $\{\tau_i\}_{i\in[m]} := \{\sigma_i\}_{i\in[m]} \cup \{x_0\}$ is a partition for X. Now, for each $h \in \mathcal{R}(K)$ we can get

$$\sum_{i\in[m]} \int_{\sigma_i} |\langle h, F_i(x)\rangle|^2 \, d\mu(x) = \sum_{i\in[m]} \int_{\tau_i} |\langle h, F_i(x)\rangle|^2 \, d\mu(x) -$$
$$- \sum_{i\in[m]} \mu(\{x_0\})|\langle h, F_i(x_0)\rangle|^2 \geqslant$$
$$\geqslant A\|K^*h\|^2 - B\mu(\{x_0\})\|h\|^2 \geqslant$$
$$\geqslant (A - B\mu(\{x_0\})\|K^\dagger\|^2)\|K^*h\|^2.$$

On the other hand, we have

$$\sum_{i\in[m]} \int_{\sigma_i} |\langle h, F_i(x)\rangle|^2 \, d\mu(x) \leqslant \sum_{i\in[m]} \int_{\tau_i} |\langle h, F_i(x)\rangle|^2 \, d\mu(x) \leqslant A\|K\|^2\|h\|^2.$$

This completes the proof. □

In the next result, the converse of Theorem 6 can be constructed with a slight change.

Theorem 7. *Use the notation of Theorem 6. If, for some $x_0 \in X$, the family $\{F_i : X \setminus \{x_0\} \to H\}_{i\in[m]}$ is a c-K-woven frame for $\mathcal{R}(K)$ with the upper bound B', such that $A > B'\|K^\dagger\|^2$, then $\{F_i(x_0)\}_{i\in[m]}$ is a K-frame for $\mathcal{R}(K)$.*

Proof. Suppose C and B' be the frame bounds of $\{F_i \colon X \setminus \{x_0\} \to H\}_{i\in[m]}$. With the same assumptions as in Theorem 6, we can write

$$C\|K^*h\|^2 \leqslant \sum_{i\in[m]} \int_{\sigma_i} |\langle h, F_i(x)\rangle|^2 \, d\mu(x) =$$

$$= \sum_{i\in[m]} \int_{\tau_i} |\langle h, F_i(x)\rangle|^2 \, d\mu(x) - \sum_{i\in[m]} \mu(\{x_0\})|\langle h, F_i(x_0)\rangle|^2 =$$
$$= A\|K^*h\|^2 - \sum_{i\in[m]} \mu(\{x_0\})|\langle h, F_i(x_0)\rangle|^2.$$

Therefore,

$$0 < C \leqslant A - \frac{\mu(\{x_0\})}{\|K\|^2 \, \|h\|^2} \sum_{i\in[m]} |\langle h, F_i(x_0)\rangle|^2,$$

or

$$\sum_{i\in[m]} |\langle h, F_i(x_0)\rangle|^2 \leqslant \frac{A\|K\|^2}{\mu(\{x_0\})} \|h\|^2.$$

Hence, $\{F_i(x_0)\}_{i\in[m]}$ is a Bessel sequence for $\mathcal{R}(K)$. Since

$$A\|K^*h\|^2 - \sum_{i\in[m]} \mu(\{x_0\})|\langle h, F_i(x_0)\rangle|^2 = \sum_{i\in[m]} \int_{\tau_i} |\langle h, F_i(x)\rangle|^2 \, d\mu(x) -$$
$$- \sum_{i\in[m]} \mu(\{x_0\})|\langle h, F_i(x_0)\rangle|^2 =$$
$$= \sum_{i\in[m]} \int_{\sigma_i} |\langle h, F_i(x)\rangle|^2 \, d\mu(x) \leqslant$$
$$\leqslant B'\|h\|^2 \leqslant B'\|K^\dagger\|^2 \|K^*h\|^2,$$

we get

$$\frac{(A - B'\|K^\dagger\|^2)}{\mu(\{x_0\})} \|K^*h\|^2 \leqslant \sum_{i\in[m]} |\langle h, F_i(x_0)\rangle|^2.$$

The proof is completed. □

Summary. In this note, we connected three concepts of the frame theory: continuous frames, K-frames, and weaving frames. Mainly, we studied the effects of weaving applied to the continuous K-frames.

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